Lecture 24: Connections to Algebraic geometry, II.

1) Prime ideals & irreducibility.
2) Geometric significance of localization.

Refs: [V], Sec 9.6; [E], Intro to Sec 2, Sec 3.8.

Small modification to Sec 2.2 on 12/7.

1) Prime ideals & irreducibility.

Reminder on prime ideals: A is commutative ring, $I \subset A$ ideal. Say I is prime (Lec 3, Sect 1) if one of equiv't conditions hold:

1) A/I is domain
2) $a_1, a_2 \notin I \Rightarrow a_1 a_2 \notin I$.
3) if $I_1, I_2 \subset A$ are ideals & $I_1 I_2 \subset I \Rightarrow I_1$ or $I_2 \subset I$.

Thx to 2), prime $\Rightarrow$ radical: $a^n \in I \Rightarrow a$ or $a^{n-1} \in I \Rightarrow a \in I$.

Let $\mathbb{F}$ be an algebraically closed field so that {radical ideals in $\mathbb{F}[x_1,\dots,x_n]$} $\xrightarrow{\sim}$ {algebraic subsets of $\mathbb{F}^n$}, Sec 2.2 of Lec 23.

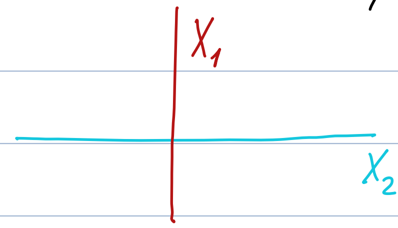
Question: find a geometric characterization of algebraic subsets in $\mathbb{F}^n$ corresponding to prime ideals.

1.1) Irreducible algebraic subsets.

Definition: an alg. subset X in $\mathbb{F}^n$ is called

- irreducible: if X cannot be represented as $X_1 \cup X_2$, where $X_i \subsetneq X$ is algebraic.
- reducible, else.

Example: Set $X = V(x_1x_2) \subset \mathbb{F}^2$. It's reducible: $X = X_1 \cup X_2$, where $X_1 = V(x_1)$, $X_2 = V(x_2)$



Proposition: TFAE

(a) X is irreducible.

(b) $I(X) \left(= \{f \in \mathbb{F}[x_1, \dots x_n] \mid f|_X = 0\}\right)$ is prime.

(c) $\mathbb{F}[X] \ (= \mathbb{F}[x_1 \dots x_n]/I(X))$ is a domain.

Proof: (b) $\Leftrightarrow$ (c): see the reminder above.

(a) $\Rightarrow$ (b): assume that $I(X)$ isn't prime, i.e. $\exists f_i \in \mathbb{F}[x_1 \dots x_n] \setminus I(X)$ s.t. $f_1 f_2 \in I(X)$; $X_i := \{\alpha \in X \mid f_i(\alpha) = 0\}$, $i = 1, 2$. Then $X_i \subsetneq X$ (properly b/c $f_i \notin I(X)$, i.e. $f_i|_X \neq 0$), is an algebraic subset & $X_1 \cup X_2 = \{\alpha \in X \mid (f_1 f_2)(\alpha) = 0\} = [f_1 f_2 \in I(X)] \supset X$. Contradiction w. X being irreducible.

(b) $\Rightarrow$ (a): assume X is reducible: $X = X_1 \cup X_2$ w. $X_i \subsetneq X$ alg'c subset, define $I_i := I(X_i) \supsetneq I(X)$ ($\supsetneq$ is b) of Cor in Sec 2.2 of Lec 23). By Lemma there, $I(X) = I_1 \cap I_2$, so $I(X) \supset I_1 I_2$. Since $I(X)$ is prime $\Rightarrow$ say $I(X) \supset I_1 \Leftrightarrow$ [by the same Corollary] $X \subset V(I_1) = X_1$. Contradiction w. $X_1 \subsetneq X$. $\square$

Examples: 1) $\mathbb{F}^n$ is irreducible b/c $\mathbb{F}[\mathbb{F}^n] = \mathbb{F}[x_1 \dots x_n]$ is domain.

2) Let $f \in \mathbb{F}[x_1, \dots x_n]/(f)$. Decompose $f = f_1^{n_1} \dots f_k^{n_k}$, where f_i's are irreducible. Then $V(f) \subset \mathbb{F}^n$ is irreducible $\Leftrightarrow k = 1$.

1.2) Irreducible components.

Theorem: Let X be an algebraic subset in $\mathbb{F}^n$. Then

a) $\exists$ irreducible algebraic subsets $X_1,\dots X_k$ s.t. $X=\bigcup_{i=1}^k X_i$.

b) For $X_1,\dots X_k$ we can take maximal (w.r.t. inclusion) irreducible algebraic subsets contained in X.

Note, that (b) recovers $X_1,\dots X_k$ uniquely.

Def'n: These $X_1,\dots X_k$ (from b)) are called irreducible components of X.

Example: Irreducible components of $V(x_1x_2)$ are $V(x_1)$ & $V(x_2)$. More generally, for $f=f_1^{n_1}\dots f_k^{n_k}$, the irreducible components of $V(f)$ are $V(f_1),\dots, V(f_k)$.

Proof of Theorem:

a) Assume the contrary: $\exists$ $X \neq$ finite union of irreducibles $\Leftrightarrow$ the set $\mathfrak{A}$ of all such X's is $\neq \varnothing$. $\rightsquigarrow$ nonempty set $\{I(X) \mid X\in\mathfrak{A}\}$. Since $\mathbb{F}[x_1,\dots x_n]$ is Noetherian, every nonempty set of ideals has maximal (w.r.t $\subset$) element. Pick $X'\in\mathfrak{A}$ s.t. $I(X')$ is maximal in $\{I(X)\mid X\in\mathfrak{A}\}$ $\Longleftrightarrow$ X' is minimal in $\mathfrak{A}$ w.r.t. $\subset$. But X' is reducible b/c $X'\in\mathfrak{A}$ $\Leftrightarrow$ $X'=X^1\cup X^2$ w. $X^i\subsetneq X'$ $\Rightarrow$ [X' is min'l in $\mathfrak{A}$] $X^i\notin\mathfrak{A}$ $\rightsquigarrow$ $X^i=\bigcup_j X^i_j$ (finite unions of irreducibles) $\rightsquigarrow$ $X'=\bigcup_j X^1_j \cup \bigcup_j X^2_j$ – contradicts $X'\in\mathfrak{A}$.

b) $X = \bigcup_{i=1}^{k} X_i$, where assume that none of X_i's is contained in another. Need to show: X_i is max'l irreducible (exercise) & if $Y \subset X$ max'l irreducible $\Rightarrow Y = X_i$ (for autom. unique i). To prove this, we observe

$$Y = \bigcup_{i=1}^{k} (Y \cap X_i);\ \text{since } Y \text{ is irreducible} \Rightarrow Y = Y \cap X_i \text{ for}$$

some $i \Rightarrow Y \subset X_i$, but since Y is maximal, $Y = X_i$. $\square$

Corollary (alg'c formulation of Thm): Let $I \subset \mathbb{F}[x_1, \dots x_n]$ be radical ideal. Then $I = \bigcap_{i=1}^{k} I_i$, where I_i is prime; and we can recover I_i's uniquely if we assume they are minimal (w.r.t $\subseteq$) w. $I \subset I_i$.

Remark: the same statement is true if $\mathbb{F}[x_1, \dots, x_n]$ w. arbitrary Noetherian ring (exercise). There's a suitable generalization to arbitrary ideals: primary decomposition, [AM], Ch. 4 & 7.1.

2) Geometric significance of localization

2.1) Localizing one element.

Let $X \subset \mathbb{F}^n$ be an algebraic subset & $f \in \mathbb{F}[X]$. We want to find a geometric interpretation of the localization $\mathbb{F}[X][f^{-1}]$.

Let $f_1, \dots, f_m$ be generators of $I(X)$. Then Exercise 2 in Sec 1.2 of Lec 9 tells us that

$$\mathbb{F}[X][f^{-1}] \simeq \mathbb{F}[X][t]/(tf-1) = \mathbb{F}[x_1, \dots x_n, t]/(f_1, \dots, f_m, tf-1).$$

Exercise: Show that if A is an algebra w/o nonzero nilpotent elements, then any localization of A has no nonzero nilpotent elements.

It follows that the ideal $(f_1, \dots, f_m, tf-1)$ is radical. The corresponding algebraic subset of $\mathbb{F}^{n+1}$ is
$\{(\alpha_1, \dots, \alpha_n, z) \in \mathbb{F}^{n+1} \mid f_i(\alpha_1, \dots \alpha_n) = 0 \ \forall i = 1, \dots m;\ z f(\alpha_1, \dots \alpha_n) = 1\}$

The projection $\mathbb{F}^{n+1} \to \mathbb{F}^n$ forgetting the z coordinate identifies this algebraic subset w $\{\alpha \in X \mid f(\alpha) \neq 0\}$. Denote this subset by X_f. We note that it's <u>not</u> an algebraic subset of $\mathbb{F}^n$ in our terminology. This subset of X is called a **principal open subset**.

Here's an explanation of the terminology.

Definition: • a subset $Y \subset X$ is called **Zariski closed** if it's an algebraic subset of $\mathbb{F}^n$.
• A subset $U \subset X$ is **Zariski open** if $X \setminus U$ is Zariski closed.

Example: $X_f \subset X$ is Zariski open.

Exercise: Any Zariski open subset of X is the union of, in fact, finitely many, principal open subsets.

Remark: Zariski open/closed subsets are open/closed subsets in a topology (called Zariski topology). Principal open subsets form a "base of topology."

2.2) Localization at the complement of a maximal ideal.

Let $X \subset \mathbb{F}^n$ be algebraic subset, $A := \mathbb{F}[X]$, $\mathfrak{m} \subset A$ a maximal ideal. Recall that we write $A_{\mathfrak{m}}$ for $A[(A \setminus \mathfrak{m})^{-1}]$.

Note that $A_{\mathfrak{m}}$ is not finitely generated (in general) so is not the algebra of functions of an algebraic subset. It still has a geom'c meaning that we are going to discuss now.

For simplicity, assume X is irreducible $\Leftrightarrow A = \mathbb{F}[X]$ is domain $\rightsquigarrow$ fraction field $\mathrm{Frac}(A) = \{\frac{f}{g} \mid g \neq 0\}$, every localization of A is contained in $\mathrm{Frac}(A)$ as a subring, Corollary in Sec 1.2 of Lec 9.

By Corollary in Sec 1.2 of Lec 23, the maximal ideals of A are in bijection w. X: $\mathfrak{m} \leftrightarrow \alpha$ w. $\mathfrak{m} = \{f \in A \mid f(\alpha) = 0\}$. Then
$A_{\mathfrak{m}} = \{\frac{f}{g} \mid g(\alpha) \neq 0\} = \bigcup_{g \mid g(\alpha) \neq 0} A[g^{-1}] = \bigcup_{g \mid g(\alpha) \neq 0} \mathbb{F}[X_g]$ (recall that X_g is an algebraic subset (in $\mathbb{F}^{n+1}$) so it makes sense to speak about its algebra of functions. Section 2.1 shows that this algebra is $A[g^{-1}]$.

Conclusion:

Every element of $A_{\mathfrak{m}}$ is a function on a Zariski open subset containing α, but which subset we choose depends on this element.

Remark: When X is reducible, the conclusion still holds but $A_{\mathfrak{m}} = \bigcup_g \mathbb{F}[X_g]$ makes no sense b/c $\mathbb{F}[X_g] = A[g^{-1}]$ are not subrings in a fixed ring (in general). To fix this one replaces the union w. the "direct limit."

Remark (on terminology): Recall (Sec 2 of Lec 10) that a commutative ring B is called local if it has unique maximal ideal.

For example, $A_{\mathfrak{m}}$ is local. The discussion above gives a geometric justification to this terminology: this algebra controls what happens locally (in Zariski topology) near $\alpha \in X$.