Lecture 4: rings, ideals & modules, IV.

1) Constructions w. modules.
2) Submodules & quotient modules.
3) Finitely generated modules.

References: [AM], Chapter 2, Sections 2,3,5.

1) Constructions with modules (& homomorphisms).

1.1) Direct sums & products.

M_1, M_2 A-modules $\rightsquigarrow$
$M_1 \oplus M_2$ (direct sum) $= M_1 \times M_2$ (direct product) $=$ product $M_1 \times M_2$ as abelian groups w. $a(m_1, m_2) := (am_1, am_2)$.

More generally, for a set I (possibly infinite) & modules $M_i, i\in I$, define direct product $\prod_{i\in I} M_i = \{(m_i)_{i\in I} \mid m_i \in M_i\}$ w. componentwise operations.

Direct sum: $\bigoplus_{i\in I} M_i = \{(m_i)_{i\in I} \mid \text{only fin. many } m_i \neq 0\}$

Have A-module inclusion:

$$\bigoplus_{i\in I} M_i \hookrightarrow \prod_{i\in I} M_i$$

which is an isomorphism $\Leftrightarrow$ I is finite.

1.2) Hom module: let M, N be A-modules

As a set $\operatorname{Hom}_A(M,N) := \{A\text{-linear maps } M \to N\}$

Claim: $\operatorname{Hom}_A(M,N)$ has a natural A-module structure.

Need to define addition & multipl'n by elements of A.

$\varphi, \varphi' \in \mathrm{Hom}_A(M,N)$, $a \in A$

$[\varphi+\varphi'](m) := \varphi(m)+\varphi'(m) \in N$

$[a\varphi](m) := a\varphi(m) \quad \in N$

Lemma: 1) $\varphi+\varphi'$, $a\varphi$ are A-linear maps.

2) The operations $+, \cdot$ turn $\mathrm{Hom}_A(M,N)$ into A-module.

Partial proof: $[a\varphi](bm) = b[a\varphi](m)$ - part of linearity for $a\varphi$.

$[a\varphi](bm) = a(\varphi(bm)) = ab\,\varphi(m) = [ab=ba] = b(a\varphi(m)) = b[a\varphi](m)$.

Rest of proof is an exercise. $\square$

Example: 1) Let $M=A$. Then $\mathrm{Hom}_A(A,N) \xrightarrow{\sim} N$ as A-modules.

Exercise: Prove that maps $\mathrm{Hom}_A(A,N) \to N$, $\varphi \mapsto \varphi(1)$ & $N \to \mathrm{Hom}_A(A,N)$, $n \mapsto \varphi_n : \varphi_n(a) = an$ are mutually inverse.

2) $\mathrm{Hom}_A(M_1 \oplus M_2, N) \xrightarrow{\sim} \mathrm{Hom}_A(M_1,N) \times \mathrm{Hom}_A(M_2,N)$

$\varphi \mapsto (\varphi|_{M_1}, \varphi|_{M_2})$, where $M_1 \hookrightarrow M_1 \oplus M_2$ via $m_1 \mapsto (m_1, 0)$, and similarly for M_2.

Inverse maps $(\varphi_1, \varphi_2) \in \mathrm{Hom}_A(M_1,N) \times \mathrm{Hom}_A(M_2,N)$ goes to $\varphi: M_1 \oplus M_2 \to N$ given by $\varphi(m_1,m_2) := \varphi_1(m_1)+\varphi_2(m_2)$

Exercise: Prove that these two maps are mutually inverse A-module isomorphisms.

3) There is a direct analog of this example for $M_1 \oplus \ldots \oplus M_k$.

E.g. $\mathrm{Hom}_A(A^{\oplus k}, N) \xrightarrow{\sim} \mathrm{Hom}_A(A,N)^{\times k} \xrightarrow{\sim} N^{\times k}$

$\psi \longmapsto (\psi(e_1), \dots \psi(e_k))$

where $e_i = (0, \dots 1, \dots 0)$ (1 in the ith place). The inverse map is given by $\underline{n} = (n_1, \dots n_k) \mapsto \psi_{\underline{n}} : (a_1, \dots a_k) \mapsto \sum a_i n_i$.

Rem: Example 2 further generalizes to infinite direct sums:

$$\mathrm{Hom}_A\left(\bigoplus_{i\in I} M_i, N\right) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_A(M_i, N)$$

The proof is similar to the above and is left as exercise.

2) Sub & quotient modules.

2.1) Submodules: Let A be a comm've ring.

Definition: let M be an A-module; a submodule in M is an abelian subgroup $N \subset M$ s.t. $a \in A, n \in N \Rightarrow an \in N$.

Rem: N has a natural A-module str're.

Examples: 0) $\{0\}, M \subset M$ are submodules.

1) A is a field (so module = vector space): Submodule = subspace.

2) $A = \mathbb{Z}$ (so module = abelian group): Submodule = subgroup.

3) $A = \mathbb{F}[x]$ ($\mathbb{F}$ is a field). A module M = $\mathbb{F}$-vector space w. operator $X: M \to M$. A submodule $N \subset M$ - subspace s.t. $X(N) \subseteq N$. Conversely, every X-stable subspace is a submodule b/c $f(x)m = f(X)m$ & $X(N) \subseteq N \Rightarrow f(X)(N) \subset N$.

4) A is any ring, $M = A$: submodule = ideal.

2.2) Constructions w. submodules.

1) $\varphi: M \to N$ A-module homom'm: $\ker\varphi \subset M$ & $\operatorname{im}\varphi \subset N$ are submodules, left as exercise.

2) $m_1, \dots, m_k \in M \rightsquigarrow \operatorname{Span}_A(m_1, \dots, m_k) := \{\sum_{i=1}^{k} a_i m_i \mid a_i \in A\}$ - this is special case of image: $\underline{m} = (m_1, \dots, m_k) \rightsquigarrow \varphi_{\underline{m}}: A^{\oplus k} \to M$ (see example 3 in Sec 1.2) Then $\operatorname{Span}_A(m_1, \dots, m_k) = \operatorname{im}\varphi_{\underline{m}}$. Note also that this generalizes the ideal generated by a given collection of elements, Sec 3.1 of Lec 1. More generally, for index set I & $m_i \in M$ $(i \in I) \rightsquigarrow \operatorname{Span}_A(m_i \mid i \in I) = \{$finite A-linear combinations of m_i's$\}$.

3) Sums & intersections: $M_1, M_2 \subset M$ submodules $M_1 \cap M_2$, $M_1 + M_2 = \{m_1 + m_2 \mid m_i \in M_i\}$ - submodules.

4) Product w. ideal: $N \subset M$ submodule, $I \subset A$ ideal $IN := \{\sum_{i=1}^{\ell} a_i n_i \mid a_i \in I, n_i \in N\}$ - submodule, exercise. (compare to product of ideals in 1.1 of Lecture 2).

2.3) Quotient modules:

M is A-module, $N \subset M$ submodule $\rightsquigarrow$ abelian group $M/N = \{m + N \mid m \in M\}$ & abelian group homom'm $\pi: M \twoheadrightarrow M/N$, $\pi(m) := m + N$. Then M/N has a natural A-module str're. The following is analogous to Proposition in Sec 3.2 of Lecture 1.

Proposition: 1) The map $A \times (M/N) \to M/N$, $(a, m+N) \mapsto am+N$ is well-defined ($am+N$ only depends on $m+N$ & not on m itself) and equips M/N w. A-module structure.

2) This module str're is unique s.t. $\pi: M \to M/N$ is a module homomorphism.

3) (Universal property of M/N & $\pi: M \twoheadrightarrow M/N$) Let $\varphi: M \to M'$ be A-module homom'm s.t. $N \subset \ker\varphi$. Then $\exists!$ module homom'm $\underline{\varphi}: M/N \to M'$ s.t. the following diagram is commutative

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ M/N \arrow[r, dashed, "\underline{\varphi}"'] & M' \end{tikzcd}$$

$\underline{\varphi}$ is given by: $\underline{\varphi}(m+N) := \varphi(m)$

Proof: exercise.

Remarks:

1) Let $I \subset A$ be ideal $\rightsquigarrow$ submodule $IM \subset M \rightsquigarrow$ quotient M/IM is an A/I-module via $(a+I)(m+IM) = am+IM$ (compare to Obs'n II in Sec 2.2 of Lec 3). For example, if $I = \mathfrak{m}$ is maximal $\Rightarrow A/\mathfrak{m}$ is a field so $M/\mathfrak{m}M$ is vector space over $A/\mathfrak{m}$.

This gives a way to reduce the study of modules over rings to study of vector spaces over fields.

2) We have standard "isomorphism theorems":

- for $\varphi: M \to N$, A-module homom'm, then $M/\ker\varphi \xrightarrow{\sim} \operatorname{im}\varphi$ (A-module isomorphism).
- for submodules $K \subset N \subset M$, have $(M/K)/(N/K) \xrightarrow{\sim} M/N$.
- for submodules $N_1, N_2 \subset M$, have $N_1/N_1\cap N_2 \xrightarrow{\sim} (N_1+N_2)/N_2$

The reason is that the standard abelian group isomorphisms are also module isomorphisms.

3) There are bijections between:

$$\begin{tikzcd} \{\text{submodules } L \subset M \mid N \subset L\} \arrow[d, bend right, "L \mapsto \pi(L) = L/N"'] \\ \{\text{submodules } \underline{L} \subset M/N\} \arrow[u, bend right, "\underline{L} \mapsto \pi^{-1}(\underline{L})"'] \end{tikzcd}$$

We have seen a similar claim for ideals in 3.2 of Lecture 1.

3) Finitely generated & free modules.

3.1) Finitely generated modules

Definition: • Elements $m_i \in M$ $(i \in I)$ are **generators** (a.k.a. **spanning set**) of M if $M = \operatorname{Span}_A(m_i \mid i \in I)$, i.e. $\forall m \in M$ is A-linear combination of finite number of m_i's.

• M is **finitely generated** if it has a finite spanning set.

Remarks: 1) $A^{\oplus I}$ is finitely generated $\Leftrightarrow$ I is finite.

2) If M is fin. generated, then so is M/N $\forall N \subset M$:
$M = \mathrm{Span}_A(m_1, \ldots m_k) \Rightarrow M/N = \mathrm{Span}_A(\pi(m_1), \ldots, \pi(m_k))$.

3) $\underline{m} = (m_1, \ldots m_k) \rightsquigarrow \psi_{\underline{m}} : A^{\oplus k} \longrightarrow M$, $\psi_{\underline{m}}(a_1, \ldots a_k) = \sum_{i=1}^{k} a_i m_i$.
$M = \mathrm{Span}_A(m_1, \ldots m_k) \Leftrightarrow \psi_{\underline{m}}$ is surj've $\Rightarrow M \xrightarrow{\sim} A^{\oplus k}/\ker \psi_{\underline{m}}$
So: fin. gen'd modules = quotients of $A^{\oplus k}$ for some $k \in \mathbb{Z}_{\geq 0}$.