Lecture 4: rings, ideals & modules, IV.

1) Constructions w. modules.
2) Submodules & quotient modules.
3) Finitely generated modules.

References: [AM], Chapter 2, Sections 2,3,5.

1) Constructions with modules (& homomorphisms).

1.1) Direct sums & products.

\[ M_1, M_2 \text{ A-modules \rightarrow} \]
\[ M_1 \oplus M_2 \text{ (direct sum)} = M_1 \times M_2 \text{ (direct product)} = \text{ product} \]
\[ M_1 \times M_2 \text{ as abelian groups w. a } (m_1, m_2) := (am_1, am_2). \]

More generally, for a set \( I \) (possibly infinite) & modules \( M_i, i \in I \), define direct product \( \prod_{i \in I} M_i = \{ (m_i)_{i \in I} | m_i \in M_i \} \) w. componentwise operations.

Direct sum: \( \bigoplus_{i \in I} M_i = \{ (m_i)_{i \in I} | \text{only fin. many } m_i \neq 0 \} \)

Have \( A \)-module inclusion:

\[ \bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i \]

which is an isomorphism \( \iff \) \( I \) is finite.

1.2) \( \text{Hom module: } \) let \( M,N \) be \( A \)-modules

As a set \( \text{Hom}_A(M,N) := \{ A \text{-linear maps } M \rightarrow N \} \)

Claim: \( \text{Hom}_A(M,N) \) has a natural \( A \)-module structure.
Need to define addition & multiplication by elements of $A$.

\[ \psi, \psi' \in \text{Hom}_A(M, N), \ a \in A \]

\[ [\psi + \psi'] (m) = \psi(m) + \psi'(m) \in N \]

\[ [a \psi] (m) = a \psi(m) \in N \]

**Lemma:** 1) $\psi + \psi'$, $a \psi$ are $A$-linear maps.

2) The operations $+$, $\cdot$ turn $\text{Hom}_A(M, N)$ into $A$-module.

**Partial proof:**
\[ [a \psi](6m) = 6[a \psi](m) \] - part of linearity for $a \psi$.
\[ [a \psi](6m) = a(y(6m)) = ab \psi(m) = [ab - ba] = 6(a \psi(m)) = 6 \text{lay}(m) \]

Rest of proof is an exercise. $\square$

**Example:** 1) Let $M = A$. Then $\text{Hom}_A(A, N) \rightarrow N$ as $A$-modules.

**Exercise:** Prove that maps $\text{Hom}_A(A, N) \rightarrow N$, $\psi \mapsto \psi(1)$ & $N \rightarrow \text{Hom}_A(A, N)$, $n \mapsto \psi_n : \psi_n(x) = an$ are mutually inverse.

2) $\text{Hom}_A(M_1 \oplus M_2, N) \rightarrow \text{Hom}_A(M_1, N) \times \text{Hom}_A(M_2, N)$

\[ \psi \mapsto (\psi|_{M_1}, \psi|_{M_2}), \ \text{where } M_2 \rightarrow M_1 \oplus M_2 \]

via $m \rightarrow (m, 0)$, and similarly for $M_2$.

Inverse maps $(\psi_1, \psi_2) \in \text{Hom}_A(M_1, N) \times \text{Hom}_A(M_2, N)$ go to

\[ \varphi : M_1 \oplus M_2 \rightarrow N \text{ given by } \varphi(m, m_2) = \psi_1(m) + \psi_2(m) \]

**Exercise:** Prove that these two maps are mutually inverse $A$-module isomorphisms.

3) There is a direct analog of this example for $M_1 \oplus \ldots \oplus M_n$. 2
\[ \text{E.g. } \text{Hom}_A(A^{\oplus k}, N) \xrightarrow{\psi} \text{Hom}_A(A, N)^{\times k} \xrightarrow{\psi} N^{\times k} \]

where \( \psi = (e_1, \ldots, e_k) \) (1 in the \( i \)th place). The inverse map is given by \( \psi^{-1} : (q_1, \ldots, q_k) \rightarrow \sum q_i n_i \).

Rem: Example 2 further generalizes to infinite direct sums:
\[ \text{Hom}_A(\bigoplus_{i \in I} M_i, N) \xrightarrow{\psi} \prod_{i \in I} \text{Hom}_A(M_i, N) \]

The proof is similar to the above and is left as exercise.

2) Submodules & quotient modules.

2.1) Submodules: Let \( A \) be a commutative ring.

Definition: let \( M \) be an \( A \)-module; a submodule in \( M \) is an abelian subgroup \( N \subset M \) s.t. \( a \in A, n \in N \Rightarrow an \in N \).

Rem: \( N \) has a natural \( A \)-module structure.

Examples:
1) \( \{0\} \subset M \subset M \) are submodules.
2) \( A \) is a field (so module = vector space): Submodule = subspace.
3) \( A = \mathbb{Z} \) (so module = abelian group): Submodule = subgroup.
4) \( A = \mathbb{F}[x] \) (\( \mathbb{F} \) is a field). A module \( M = \mathbb{F} \)-vector space w. operator \( X : M \rightarrow M \). A submodule \( N \subset M \) - subspace s.t. \( X(N) \subseteq N \). Conversely, every \( X \)-stable subspace is a submodule.
5) \( A \) is any ring, \( M = A \): submodule = ideal.
2.2) Constructions w. submodules.

1) $\psi: M \to N$ an $A$-module homomorphism: ker $\psi \subseteq M$ & $\im \psi \subseteq N$ are submodules, left as exercise.

2) $m_1, \ldots, m_k \in M \mapsto \text{Span}_A(m_1, \ldots, m_k) = \{ \sum_{i=1}^{k} a_i m_i | a_i \in A \} -$ this is a special case of image: $m = (m_1, \ldots, m_k) \mapsto \psi_m: A^k \to M$ (see example 3 in Sec 1.2). Then $\text{Span}_A(m_1, \ldots, m_k) = \im \psi_m$.
Note also that this generalizes the ideal generated by a given collection of elements, Sec 3.1 of Lec 1. More generally, for index set $I$ & $m_i \in M$ ($i \in I$) $\mapsto \text{Span}_A(m_i | i \in I) = \{ \text{finite } A\text{-linear combinations of } m_i \text{'s} \}$.

3) Sums & intersections: $M_1, M_2 \subseteq M$ submodules
$M_1 \cap M_2, M_1 + M_2 = \{ m_1 + m_2 | m_1 \in M_1, m_2 \in M_2 \} -$ submodules.

4) Product w. ideal: $N \subseteq M$ submodule, $I \subseteq A$ ideal
$IN = \{ \sum_{i=1}^{k} a_i n_i | a_i \in I, n_i \in N \} -$ submodule, exercise. (compare to product of ideals in 1.1 of Lecture 2).

2.3) Quotient modules: $M$ is an $A$-module, $N \subseteq M$ submodule
$\leadsto$ abelian group $M/N = \{ m+N | m \in M \}$ & abelian group homomorphism $\delta: M \to M/N$, $\delta(m) = m+N$. Then $M/N$ has a natural $A$-module structure. The following is analogous to Proposition in Sec 3.2 of Lecture 1.
Proposition: 1) The map \( A \times (M/N) \to M/N, (a, m+N) \mapsto am+N \) is well-defined \((am+N\) only depends on \(m+N\) \& not on \(m\) itself\) and equips \(M/N\) w. \(A\)-module structure.

2) This module str\'e is unique s.t. \( \phi: M \to M/N \) is a module homomorphism.

3) (Universal property of \( M/N \& \phi: M \to M/N \)) Let \( \psi: M \to M' \) be \(A\)-module homomorphism s.t. \( N \leq \ker \psi \). Then there \( \exists! \) module homomorphism \( \bar{\psi}: M/N \to M' \) s.t. the following diagram is commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{\psi} & M' \\
\downarrow{\phi} & & \downarrow{\bar{\psi}} \\
M/N & \xrightarrow{\bar{\psi}} & M'
\end{array}
\]

\( \bar{\psi} \) is given by:

\( \bar{\psi}(m+N) = \psi(m) \)

Proof: exercise.

Remarks:

1) Let \( I \subset A \) be ideal \( \Rightarrow \) submodule \( IM < M \to \) quotient \( M/IM \) is an \( A/I\)-module via \((a+I)(m+IM) = am+IM\) (compare to Obs.:n II in Sec 2.2 of Lec 3). For example, if \( I = \mathfrak{m} \) is maximal \( \Rightarrow A/\mathfrak{m} \) is a field so \( M/\mathfrak{m}M \) is vector space over \( A/\mathfrak{m} \).

This gives a way to reduce the study of modules over rings to study of vector spaces over fields.
2) We have standard isomorphism theorems:
- for \(\varphi: M \rightarrow N\), \(A\)-module homomorphism, then \(M/\ker \varphi \cong \text{im} \varphi\) (\(A\)-module isomorphism).
- for submodules \(K \triangleleft N \subseteq M\), have \((M/K)/(N/K) \cong M/N\)
- for submodules \(N_1, N_2 \subseteq M\), have \(N_1/M \cap N_2 \cong (N_1+N_2)/M\).

The reason is that the standard abelian group isomorphisms are also module isomorphisms.

3) There are bijections between:

\[
\begin{align*}
\{\text{submodules } L \subseteq M \mid N \subseteq L \} & \xrightarrow{\cong} \{\text{submodules } L \subseteq M/N \} \\
L \mapsto \overline{\pi}(L) = L/N \\
\{\text{submodules } L \subseteq M/N \} & \xrightarrow{\cong} \text{Sub}(L).
\end{align*}
\]

We have seen a similar claim for ideals in 3.2 of Lecture 1.

3) Finitely generated \& free modules.
3.1) Finitely generated modules

Definition: Elements \(m_i \in M \ (i \in I)\) are generators (a.k.a. spanning set) of \(M\) if \(M = \text{Span}(m_i \mid i \in I)\), i.e. \(\forall m \in M\) is an \(A\)-linear combination of finite number of \(m_i\)‘s.

\(\cdot\) \(M\) is finitely generated if it has a finite spanning set.

Remarks: 1) \(A^{\oplus I}\) is finitely generated \(\iff I\) is finite.
2) If \( M \) is fin generated, then so is \( M/N \nsubseteq \text{CM} \):
\[
M = \text{Span}_A (m_1, \ldots, m_k) \Rightarrow M/N = \text{Span}_A (\pi(m_1), \ldots, \pi(m_k)).
\]

3) \( m = (m_1, \ldots, m_k) \mapsto \psi_m : A^{\oplus k} \rightarrow M \), \( \psi_m (a_1, \ldots, a_k) = \sum_{i=1}^k a_i m_i \).
\( M = \text{Span}_A (m_1, \ldots, m_k) \iff \psi_m \) is surj \( \Rightarrow M \rightarrow A^{\oplus k}/\ker \psi_m \)
So: fin gen d modules = quotients of \( A^{\oplus k} \) for some \( k \in \mathbb{Z}_{>0} \).