Lecture 6: Noetherian rings & modules, II.

1) Finitely generated algebras.
2) Properties of Noetherian modules.
3) Artinian module & rings.
4) What’s next?

References: [AM], Chapter 6; Chapter 7, introduction.

1) Finitely generated algebras.

We proceed to a generalization of the Hilbert basis Thm.

**Definition:** Let $B$ be an $A$-algebra. Then $B$ is finitely generated (as an $A$-algebra) if $\exists b_1, \ldots, b_k \in B$ s.t. $\forall b \in B \exists F \in A[x_1, \ldots, x_n] \text{ s.t. } b = F(b_1, \ldots, b_k)$

Hence $\Phi: A[x_1, \ldots, x_n] \to B, F \mapsto F(b_1, \ldots, b_k)$, is surjective. So $B$ is fin. gen’d $A$-algebra $\iff \exists k \mid B \simeq$ a ring quotient of $A[x_1, \ldots, x_n]$

**Corollary:** Let $A$ be Noetherian & $B$ be a finitely generated $A$-algebra. Then $B$ is a Noetherian ring.

**Proof:** Use Hilbert’s Thm $k$ times to see that $A[x_1, \ldots, x_n]$ is Noetherian. Let $I \subseteq B$ be ideal, need to show it’s fin. gen’d

$J := \Phi^{-1}(I) \subseteq A[x_1, \ldots, x_n]$ is ideal so $J = (F_1, \ldots, F_n)$. But then $I = \Phi(J) = (\Phi(F_1), \ldots, \Phi(F_n))$ is finitely generated. $\square$
Since fields & $\mathbb{Z}$ are Noetherian rings, any finitely generated algebra over those are Noetherian.

In fact, as we will see later, many constructions (e.g. localization) produce Noetherian rings from Noetherian rings. This is why Noetherian rings are so widespread.

2) Further properties of Noetherian modules

Let $A$ be a ring (may not be Noetherian) & $M$ be $A$-module. The following result compares the property of being Noetherian for $M$ & its subs & quotients.

Proposition: let $N \subseteq M$ be a submodule. TFAE

1. $M$ is Noetherian
2. Both $N, M/N$ are Noetherian.

Proof: (1) $\Rightarrow$ (2): $M$ is Noetherian $\Rightarrow$ $N$ is Noetherian (tautology)

Check $M/N$ is Noetherian by verifying that $\forall$ AC of submods of $M/N$ terminates. Let $\varphi: M \twoheadrightarrow M/N, m \mapsto m+N$.

Let $(N_i)_{i \in \mathbb{N}}$ be an AC of submodules in $M/N$, $N_i = \varphi^{-1}(N_i)$

$N_i \subseteq N_{i+1}$ $\Rightarrow$ $N_i \subseteq N_{i+1}$ so $(N_i)_{i \in \mathbb{N}}$ form an AC of submodules of $M$; it must terminate: $\exists K \geq 0$ $| N_i = N_K \forall j \geq K$. But $N_i = \varphi^{-1}(N_i)$ so $N_i = \varphi^{-1}(N_i) = \varphi^{-1}(N_K) = N_K$. So $(N_i)_{i \in \mathbb{N}}$ terminates.

(2) $\Rightarrow$ (1): Have $(N_i)_{i \in \mathbb{N}}$ is an AC of submodules in $M$. Want to show it terminates. Then $(N_i \cap N)_{i \in \mathbb{N}}$ is AC in $N$ &
$(\mathfrak{p}(N_i))_{i>0}$ is AC in $M/N$. We know that both terminate $\Rightarrow$

$\exists k>0$ s.t. $N_j \cap N = N_k \cap N$ & $\mathfrak{p}(N_j) = \mathfrak{p}(N_k)$ $\forall j \neq k$.

Want to check: $N_j = N_k$ (so $(N_i)$ terminates):

\[ n \in N_j \rightarrow \mathfrak{p}(n) \in \mathfrak{p}(N_j) = \mathfrak{p}(N_k) \rightarrow \exists n' \in N_k \mid \mathfrak{p}(n') = \mathfrak{p}(n) \]

$\iff \mathfrak{p}(n-n') = 0 \iff n-n' \in N$. But $n,n' \in N_j$ (by $n' \in N_k < N_j$) $\Rightarrow$

\[ n-n' \in N_j \rightarrow n-n' \in N \cap N_j = N_j N_k \Rightarrow n = n' + (n-n') \in N_k \]

both summands are in $N_k$. This shows $N_j = N_k$. $\square$

We now proceed to characterizing Noetherian modules over Noetherian rings. In general, Noetherian $\Rightarrow$ fin.gen'd. But, when $A$ is Noetherian, we also have $\Leftarrow$.

**Corollary:** Let $A$ be Noetherian. Then $\Rightarrow$ fin. gen'd $A$-module $M$

is Noetherian.

**Proof:**

By Sec 3.1 of Lec 4, $M$ is a quotient of $A^\oplus k$. By (1)$\Rightarrow$(2)
of Proposition, it's enough to show $A^\oplus k$ is Noetherian. Since $A$
is Noetherian, it's enough to check that the direct sum of

2 Noetherian modules, say $M_1,M_2$, is Noetherian - then we'll be
done by induction. Note that we have inclusion $M_1 \hookrightarrow M_1 \oplus M_2:

M_1 \rightarrow (M_1,0)$ & projection $M_1 \oplus M_2 \rightarrow M_2$, $(m_1,m_2) \mapsto m_2$ whose kernel is the
image of $M_2$ so $(M_1 \oplus M_2)/M_1 \sim M_2$. We use (2)$\Rightarrow$(1) of Proposi-
tion to conclude $M_1 \oplus M_2$ is Noetherian. $\square$
3) Artinian modules & rings.

3.1) Definition of Artinian modules

 definition: let \( M \) be an \( A \)-module. A descending chain (DC) of submodules is \((N_i)_{i \geq 0}\) s.t. \( N_k \supseteq N_{k+1} \) for \( k > 0 \).

Definition: \( M \) is an Artinian \( A \)-module if \\

3.2) Basic properties.

The first result (together with its proof) is analogous to \\
Proposition in Sec 1 of Lec 5).

Proposition 1: For \( A \)-module \( M \) TFAE:

1) \( M \) is Artinian

2) If nonempty set of submodules of \( M \) has a minimal elt (w.r.t. \( \subseteq \))
Proposition 2: \( M \) is \( A \)-module, \( N \subseteq M \) is an \( A \)-submodule.

TFAE: 1) \( M \) is Artinian.

2) Both \( N \) \& \( M/N \) are Artinian.

Proofs: repeat those in North'n case (exercise)

3.3) Artinian rings.

Definition: A ring \( A \) is Artinian if it's Artinian as \( A \)-module.

Examples: 1) Any field is Artinian.

2) Let \( F \) be a field, \( A \) be an \( F \)-algebra s.t. \( \dim_F A < \infty \). Then \( A \) is Artinian ring (b/c \( A \)-submodule is a subspace).

3) \( A = \mathbb{Z}/n\mathbb{Z} \) is Artinian (b/c it's a finite set so every \( DC \) of subsets terminates)

4) Every nonzero el't \( a \) of Artinian ring is either invertible or zero-divisor. Indeed, let \( a \in A \) be noninvertible \& non zero divisor. \( (a) \supseteq (a^2) \supseteq (a^3) \supseteq ... \) a DC of ideals. It terminates \( (a^k) = (a^{k+1}) \) \( \Rightarrow \exists \ b \in A \ s.t. a^k = ba^{k+1} \iff (1-ab)a^k = 0 \) \( \iff a \ is \ zero \ divisor \ or \ 1 = ab. \)

In particular, every Artinian domain is a field.

Thm: Every Artinian ring is Noetherian.

For proof, see [AM], Prop 8.1 - Thm 8.5 (comments: nilradical = \( \mathfrak{f} \mathfrak{o} \) = \( \bigcap \) all prime ideals by Prop 1.8, Jacobson radical = \( \bigcap \) all max ideals).
3.9) Finite length modules.

This motivates us to consider modules that are both Noetherian (AC condition) & Artinian (DC condition) so satisfy ("AC/DC" condition). They admit an equivalent characterization.

**Definition:** Let $M$ be an $A$-module.

i) Say that $M$ is simple if $\{0\} \neq M$ are the only two submodules of $M$.

ii) Let $M$ be arbitrary. By a **filtration** (by submodule) on $M$ we mean $\{0\} = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$ (finite AC of submodule).

iii) A **Jordan–Hölder (JH) filtration** is a filtration $\{0\} = M_0 \not\subset M_1 \not\subset M_2 \not\subset \ldots \not\subset M_k = M$ s.t. $M_i/M_{i-1}$ is simple $\forall i$ (so a JH filtration is "tightest possible")

iv) $M$ has finite length if a JH filtration exists.

**Example:** 1) When $A = \mathbb{F}$ is a field, an $A$-module $M$ is simple $\iff \dim_{\mathbb{F}} M = 1$.

2) Let $A = \mathbb{Z}$ & consider the $A$-module $M = \mathbb{Z}/4\mathbb{Z}$. It is JH filtration is $M_0 = \{0\}, M_1 = 2\mathbb{Z}/4\mathbb{Z}, M_2 = M$.

**Proposition:** For an $A$-module $M$ TFAE:

1) $M$ is Artinian & Noetherian.

2) $M$ has finite length.
Proof: 2) \(\Rightarrow\) 1): \(M\) has fin length \(\Rightarrow\) JH filtration

\(0 \Rightarrow M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_k = M\). We prove by induction on \(i\) that \(M_i\) is Artinian & Noetherian

Base: \(i=1\): \(M_1\) is simple \(\Rightarrow\) Artinian & Noetherian

Step: \(i-1 \Rightarrow i\): \(M_{i-1}\) is Artin & Noetherian, so \(M_i/M_{i-1}\) is simple. \(\Rightarrow\) by Prop in Sec 1, \(M_i\) is Noetherian & by Prop 2 in 2.1, \(M_i\) is Artinian.

Use this for \(i=k\): 1) \(\Rightarrow\) 2)

1) \(\Rightarrow\) 2): \(M\) is Artinian & Noetherian. Want to produce a JH filtration. By induction: \(M_0 = \{0\}\).

Suppose we've constr'd \(M_i \subseteq M\). Need \(M_{i+1}\).

Note: \(M/M_i\) is Artinian & therefore has a nonempty set of submodules, has a min el't. Assume \(M_i \neq M\). Consider the set of all nonzero submodules of \(M/M_i\). It's \(\neq \emptyset\) so has a min el't, \(N\) This \(N\) must be simple. Now take \(M_{i+1}\) to be the preimage of \(N\) under \(M \Rightarrow M/M_i\).

So \(M_{i+1}/M_i \cong N\), simple.

We've got is an AC \(M_0 \supseteq M_1 \supseteq M_2 \ldots\), it must terminate b/c \(M\) is Noetherian. By constrn it can only terminate at \(M_i = M\). So we've got a JH filtration  \(\Box\)

Exercise: We can classify simple modules as follows: a map
In $\mathbb{m} \mapsto A/\mathbb{m}$ defines a bijection between the set of maximal ideals in $A$ and the set of simple $A$-modules (up to isomorphism).

4) What's next?: classification questions.

Motivation: for a field $\mathbb{F}$, we can completely classify finite dimensional $\mathbb{F}$-vector spaces: $\forall$ such $V \ni \exists \ k \geq 0$, s.t.
$V \cong \mathbb{F}^{\oplus k}$; this $k$ is uniquely recovered from $V$: $k = \dim V$.

Q: Can we classify finitely gen'd modules over a ring?

A: Yes, but only in very rare - yet important - cases. We can do so for domains such as $\mathbb{Z}$ & $\mathbb{F}[x]$ but not for many more complicated domains - for example $\mathbb{Z}[x]$ is already hopeless.

Here's the class of rings that we need.

Definition: A ring $A$ is a principal ideal domain (PID) if it's a domain (no zero divisors) & every ideal is principal ($=(a)$ for some $a \in \mathbb{A}$).

Example: $\mathbb{Z}$, $\mathbb{F}[x]$ (if $\mathbb{F}$ is field) are PID's; $*$ Euclidean domain $*$ (can divide w/ remainder) are PID's, e.g. $\mathbb{Z}[i]$.

Non-examples: $\mathbb{Z}[(1+\sqrt{5})/2, x, y]$ are not PID: $(2, 1+\sqrt{5})$ $(2, x)$ $(x, y)$ - not principal.