Lecture 6: Noetherian rings & modules, II.

1) Finitely generated algebras.
2) Properties of Noetherian modules.
3) Artinian modules & rings.
4) What's next?

References: [AM], Chapter 6; Chapter 7, introduction.

1) Finitely generated algebras.

We proceed to a generalization of the Hilbert basis Thm.

Definition: Let B be an A-algebra. Then B is finitely generated (as an A-algebra) if $\exists\, b_1, \dots b_k \in B$ s.t. $\forall b \in B\ \exists\, F \in A[x_1, \dots x_k]$ s.t. $b = F(b_1, \dots b_k)$

Hence $\varphi: A[x_1, \dots x_k] \longrightarrow B$, $F \mapsto F(b_1, \dots b_k)$, is surjective. So B is fin gen'd A-algebra $\Leftrightarrow \exists\, k \mid B \simeq$ a ring quotient of $A[x_1, \dots x_k]$

Corollary: Let A be Noetherian & B be a finitely generated A-algebra. Then B is a Noetherian ring.

Proof: Use Hilbert's Thm k times to see that $A[x_1, \dots x_k]$ is Noetherian. Let $I \subset B$ be ideal, need to show it's fin gen'd $J := \varphi^{-1}(I) \subset A[x_1, \dots x_k]$ is ideal so $J = (F_1, \dots F_\ell)$. But then $I = \varphi(J) = (\varphi(F_1), \dots \varphi(F_\ell))$ is finitely generated $\square$

Since fields & $\mathbb{Z}$ are Noetherian rings, any finitely generated algebra over those are Noetherian.

In fact, as we will see later, many constructions (e.g. localization) produce Noetherian rings from Noetherian rings. This is why Noetherian rings are so wide-spread.

2) Further properties of Noetherian modules.

Let A be a ring (may not be Noetherian) & M be A-module.

The following result compares the property of being Noetherian for M & its subs & quotients.

Proposition: let $N \subset M$ be a submodule. TFAE

(1) M is Noetherian

(2) Both N, M/N are Noetherian.

Proof: (1) $\Rightarrow$ (2): M is Noetherian $\Rightarrow$ N is Noeth'n (tautology)

Check M/N is Noetherian by verifying that $\forall$ AC of submod's of M/N terminates. Let $\pi: M \twoheadrightarrow M/N$, $m \mapsto m+N$.

Let $(\underline{N}_i)_{i\geq 0}$ be an AC of submodules in M/N, $N_i = \pi^{-1}(\underline{N}_i)$

$\underline{N}_i \subset \underline{N}_{i+1} \Rightarrow N_i \subset N_{i+1}$ so $(N_i)_{i\geq 0}$ form an AC of submodules of M, it must terminate: $\exists\, k\geq 0 \mid N_j = N_k$ $\forall j \geq k$. But $\underline{N}_i = \pi(N_i)$ so $\underline{N}_j = \pi(N_j) = \pi(N_k) = \underline{N}_k$. So $(\underline{N}_i)_{i\geq 0}$ terminates.

(2) $\Rightarrow$ (1): Have $(N_i)_{i\geq 0}$ is an AC of submodules in M. Want to show it terminates. Then $(N_i \cap N)_{i\geq 0}$ is AC in N &

$(\pi(N_i))_{i \geq 0}$ is AC in M/N. We know that both terminate $\Rightarrow$
$\exists\, k \geq 0$ s.t. $N_j \cap N = N_k \cap N$ & $\pi(N_j) = \pi(N_k)$ $\forall j \geq k$.

Want to check: $N_j = N_k$ (so (N_i) terminates):

$n \in N_j \rightsquigarrow \pi(n) \in \pi(N_j) = \pi(N_k)$ so $\exists\, n' \in N_k \mid \pi(n') = \pi(n)$
$\Leftrightarrow \pi(n-n') = 0 \Leftrightarrow n-n' \in N$. But $n, n' \in N_j$ (b/c $n' \in N_k \subset N_j$) $\Rightarrow$
$n-n' \in N_j \Rightarrow n-n' \in N \cap N_j = N \cap N_k \Rightarrow n = n' + (n-n') \in N_k$ b/c
both summands are in N_k. This shows $N_j = N_k$. $\square$

We now proceed to characterizing Noetherian modules over Noetherian rings. In general, Noetherian $\Rightarrow$ fin. gen'd. But, when A is Noetherian, we also have $\Leftarrow$.

Corollary: Let A be Noetherian. Then $\forall$ fin. gen'd A-module M is Noetherian.

Proof:

By Sec 3.1 of Lec 4, M is a quotient of $A^{\oplus k}$. By (1)$\Rightarrow$(2) of Proposition, it's enough to show $A^{\oplus k}$ is Noetherian. Since A is Noetherian, it's enough to check that the direct sum of 2 Noetherian modules, say M_1, M_2, is Noetherian - then we'll be done by induction. Note that we have inclusion $M_1 \hookrightarrow M_1 \oplus M_2$: $m_1 \mapsto (m_1, 0)$ & projection $M_1 \oplus M_2 \twoheadrightarrow M_2$, $(m_1, m_2) \mapsto m_2$ whose kernel is the image of M_1 so $(M_1 \oplus M_2)/M_1 \xrightarrow{\sim} M_2$. We use (2)$\Rightarrow$(1) of Proposition to conclude $M_1 \oplus M_2$ is Noetherian. $\square$

3) Artinian modules & rings.

3.1) Definition of Artinian modules

Noetherian $\Leftrightarrow$ satisfies AC condition.

Definition: Let M be A-module. A descending chain (DC) of submodules is $(N_i)_{i\geqslant 0}$ s.t. $N_k \supseteq N_{k+1}$ $\forall k\geqslant 0$.

Definition: M is an Artinian A-module if $\forall$ DC of submodules terminates (DC condition)

Example: $A=\mathbb{F}$ (a field). Claim: Artinian $\Leftrightarrow$ finite dim'l.

$\Leftarrow$: is clear b/c dimensions decrease in DC's.

$\Rightarrow$: let $\dim M=\infty \Leftrightarrow \exists$ lin. indep. vectors $m_i\in M$, $i\geqslant 0$.

Define $M_j=\operatorname{Span}_{\mathbb{F}}(m_i \mid i\geqslant j)$ – a DC of subspaces that doesn't terminate.

3.2) Basic properties.

The first result (together with its proof) is analogous to Proposition in Sec 1 of Lec 5)

Proposition 1: For A-module M TFAE:

1) M is Artinian

2) $\forall$ nonempty set of submodules of M has a min'l el't (w.r.t. $\subset$)

Proposition 2: M is A-module, $N \subset M$ is an A-submodule.
TFAE: 1) M is Artinian.
2) Both N & M/N are Artinian.
Proofs: repeat those in Noeth'n case (exercise).

3.3) Artinian rings.

Definition: A ring A is Artinian if it's Artinian as A-module.

Examples: 1) Any field is Artinian.
2) Let $\mathbb{F}$ be a field, A be an $\mathbb{F}$-algebra s.t. $\dim_{\mathbb{F}} A < \infty$. Then A is Artinian ring (b/c A-submodule is a subspace).
3) $A = \mathbb{Z}/n\mathbb{Z}$ is Artinian (b/c it's a finite set so every DC of subsets terminates)
4) Every nonzero el't a of Artinian ring is either invertible or zero-divisor. Indeed, let $a \in A$ be noninvertible & non zero divisor.
$(a) \supseteq (a^2) \supseteq (a^3) \supseteq \dots$ a DC of ideals. It terminates
$(a^k) = (a^{k+1}) \Rightarrow \exists\, b \in A$ s.t. $a^k = ba^{k+1} \Leftrightarrow (1-ab)a^k = 0$
$\Leftrightarrow$ a is zero divisor or $1 = ab$.
In particular, every Artinian domain is a field.

Thm: Every Artinian ring is Noetherian.
For proof, see [AM], Prop 8.1 - Thm 8.5 (comments: nilradical $= \sqrt{0} =$
$= \bigcap$ all prime ideals by Prop. 1.8, Jacobson radical $= \bigcap$ all max. ideals).

3.4) Finite length modules.

Thm motivates us to consider modules that are both Noetherian (AC condition) & Artinian (DC condition) so satisfy ("AC/DC" condition). They admit an equivalent characterization.

Definition: Let M be an A-module.

i) Say that M is simple if $\{0\} \neq M$ are the only two submodules of M.

ii) Let M be arbitrary. By a filtration (by submodules) on M we mean $\{0\} = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_k = M$ (finite AC of submodules).

iii) A Jordan-Hölder (JH) filtr'n is a filtr'n $\{0\} = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_k = M$ s.t. M_i / M_{i-1} is simple $\forall i$ (so a JH filtr'n is "tightest possible")

iv) M has finite length if a JH filtr'n exists.

Example: 1) When $A = \mathbb{F}$ is a field, an A-module M is simple $\Leftrightarrow \dim_{\mathbb{F}} M = 1$.

2) Let $A = \mathbb{Z}$ & consider the A-module $M = \mathbb{Z}/4\mathbb{Z}$. It's JH filtration is $M_0 = \{0\}$, $M_1 = 2\mathbb{Z}/4\mathbb{Z}$, $M_2 = M$.

Proposition: For an A-module M TFAE:

1) M is Artinian & Noetherian.

2) M has finite length.

Proof: 2) $\Rightarrow$ 1): M has fin length $\rightsquigarrow$ JH filtr'n $\{0\}=M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_k = M$. We prove by induction on i that M_i is Artinian & Noetherian.

Base: $i=1$: M_1 is simple $\Rightarrow$ Artinian & Noetherian.

Step: $i-1 \rightsquigarrow i$: M_{i-1} is Art'n & Noeth'n, so is M_i/M_{i-1} b/c it's simple. $\Rightarrow$ by Prop in Sec 1 M_i is Noetherian & by Prop 2 in 2.1, M_i is Artinian.

Use this for $i=k \rightsquigarrow M_i = M$ is Artinian & Noeth'n. So 2) $\Rightarrow$ 1).

1 $\Rightarrow$ 2): M is Artinian & Noetherian. Want to produce a JH filtr'n. By induction: $M_0=\{0\}$.

Suppose we've constr'd $M_i \subset M$. Need M_{i+1}.

Note: M/M_i is Artinian & therefore $\forall$ nonempty set of submodules has a min. el't. Assume $M_i \neq M$. Consider the set of all nonzero submodules of M/M_i. It's $\neq \emptyset$ so has a min'l element, N. This N must be simple. Now take M_{i+1} to be the preimage of N under $M \twoheadrightarrow M/M_i$. So $M_{i+1}/M_i \simeq N$, simple.

We've got is an AC $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots$, it must terminate b/c M is Noeth'n. By constr'n it can only terminate at $M_i = M$. So we've got a JH filtration $\square$

Exercise: We can classify simple modules as follows: a map

$\mathfrak{m} \mapsto A/\mathfrak{m}$ defines a bijection between the set of maximal ideals in A and the set of simple A-modules (up to isomorphism).

4) What's next?: classification questions.

Motivation: for a field $\mathbb{F}$, we can completely classify finite dimensional $\mathbb{F}$-vector spaces: $\forall$ such V $\exists\, k \in \mathbb{Z}_{\geq 0}$ s.t $V \simeq \mathbb{F}^{\oplus k}$; this k is uniquely recovered from V: $k = \dim V$.

Q: Can we classify finitely gen'd modules over a ring?

A: Yes, but only in very rare - yet important - cases. We can do so for domains such as $\mathbb{Z}$ & $\mathbb{F}[x]$ but not for many more complicated domains - for example $\mathbb{Z}[x]$ is already hopeless.

Here's the class of rings that we need.

Definition: A ring A is a principal ideal domain (PID) if it's a domain (no zero divisors) & every ideal is principal ($=(a)$ for some $a \in A$).

Examples: • $\mathbb{Z}$, $\mathbb{F}[x]$ ($\mathbb{F}$ is field) are PID's; $\forall$ "Euclidian domain" ($\approx$ can divide w. remainder) are PID's, e.g. $\mathbb{Z}[i]$.

Non-examples: $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[x]$, $\mathbb{F}[x,y]$ are not PID:

$(2, 1+\sqrt{-5}) \subset \mathbb{Z}[\sqrt{-5}]$, $(2,x) \subset \mathbb{Z}[x]$, $(x,y) \subset \mathbb{F}[x,y]$ - not principal.