Lecture 9: Localization I.

1) Localization of rings, cont'd
2) Localization of modules.

Refs: [AM], Sec 3 (up to "Local properties")

BONUS: Localization in non-commutative rings.

1.0) Reminder.

Reminder: A is commutative ring, $S \subset A$ multiplicative subset ($1 \in S$; $s, t \in S \Rightarrow st \in S$)

$\rightsquigarrow$ equivalence relation $\sim$ on $A \times S$: $(a,s) \sim (b,t) \Leftrightarrow \exists\, u \in S \mid usb = uat$.

$A[S^{-1}]$ = set of equiv. classes, $\frac{a}{s}$, that are added & multiplied as fractions.

$\iota: A \to A[S^{-1}]$, $a \mapsto \frac{a}{1}$, ring homomorphism.

Remarks: 1) if S contains no zero divisors, then the description of $\sim$ simplifies: $ta = sb$. But, in general, the latter equality fails to give an equivalence relation.

2) We view $A[S^{-1}]$ as an A-algebra via ι.

1.1) Further examples

1) A is a domain, $S = A \setminus \{0\}$ is multiplicative; $A[S^{-1}]$ is a field: $(\frac{a}{b})^{-1} = \frac{b}{a}$ if $a, b \in A \setminus \{0\}$. It's called the fraction field of

A and is denoted by $\mathrm{Frac}(A)$. For example, $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$.
One can check that $\ker \iota = \{0\}$, so $A \hookrightarrow \mathrm{Frac}(A)$.

Exercise: Let A be general & $S = \{\text{all non-zero divisors in } A\}$. Then
i) S is multiplicative.
ii) every non-zero divisor in $A[S^{-1}]$ is invertible.

2) Let $f \in A \rightsquigarrow S := \{f^n \mid n \geq 0\}$ is multiplicative. The resulting localization is denoted by $A[f^{-1}]$. We'll give an alternative characterization of this ring a bit later.

1.2) Universal property of $A[S^{-1}]$

Let A be a commutative ring. Recall the ring homomorphism $\iota: A \to A[S^{-1}]$, $a \mapsto \frac{a}{1}$. Note that $\iota(s) = \frac{s}{1}$ is invertible in $A[S^{-1}]$.

Proposition: Let $\varphi: A \to B$ be a ring homomorphism s.t. $\varphi(s) \in B$ is invertible $\forall\, s \in S$. Then the following hold:

1) $\exists!$ ring homom'm $\varphi': A[S^{-1}] \to B$ that makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[d, "\iota"'] \arrow[dr, "\varphi"] & \\ A[S^{-1}] \arrow[r, dashed, "\varphi'"] & B \end{tikzcd}$$

2) φ' is given by $\varphi'\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1}$.

Sketch of proof:

Existence: need to show that formula in 2) indeed gives a well-defined ring homomorphism.

Well-defined: need to check $\frac{a}{s}=\frac{b}{t} \Rightarrow \varphi(a)\varphi(s)^{-1}=\varphi(b)\varphi(t)^{-1}$.
Indeed: $\frac{a}{s}=\frac{b}{t} \Leftrightarrow \exists\, u\in S$ s.t $uta=usb \Rightarrow \varphi(u)\varphi(t)\varphi(a)$ $=\varphi(u)\varphi(s)\varphi(b)$. But $\varphi(u), \varphi(t), \varphi(s)$ are invertible. It follows that $\varphi(a)\varphi(s)^{-1}=\varphi(b)\varphi(t)^{-1}$. So φ' is well-defined.

Exercise – on addition & multiplication of fractions. Check that φ' is a ring homomorphism.

Uniqueness: φ' makes diagram comm've $\Leftrightarrow \varphi'(\frac{a}{1})=\varphi(a)\ \forall a\in A$
$\Rightarrow \varphi'(\frac{s}{1})=\varphi(s)$ -invertible $\Rightarrow \varphi'(\frac{1}{s})=\varphi(s)^{-1} \Rightarrow$
$\varphi'(\frac{a}{s})=\varphi'(\frac{a}{1})\varphi'(\frac{1}{s})=\varphi(a)\varphi(s)^{-1}$ $\square$

Corollary: Let A be a domain, $S\subset A$ be multiplicative subset s.t $0\notin S$. Then $A[S^{-1}]\simeq\{\frac{a}{s}\in Frac(A) \mid a\in A, s\in S\}$, a ring isom'm

Proof: The inclusion $\varphi: A\hookrightarrow Frac(A)$ satisfies $\varphi(s)$ is invertible. Consider the resulting ring homomorphism $\varphi': A[S^{-1}]\to Frac(A)$. Its image coincides w. $\{\frac{a}{s} \mid s\in S\}$. It remains to show φ' is injective: $\varphi(\frac{a}{s})=0 \Leftrightarrow \varphi(a)\varphi(s)^{-1}=0 \Leftrightarrow \varphi(a)=0 \Leftrightarrow a=0 \Leftrightarrow \frac{a}{s}=0$ $\square$

Exercise 1: Let $f_1, \dots f_k \in A$ & $S = \{f_1^{n_1} \dots f_k^{n_k} \mid n_i \geq 0\}$. Then we have a ring isomorphism $A[S^{-1}] = A[\underline{S}^{-1}]$ w. $\underline{S} = \{(f_1 \dots f_k)^n \mid n \geq 0\}$

Now we describe $A[f^{-1}]$, which is the shorthand for $A[\{f^n \mid n \geq 0\}^{-1}]$.

Exercise 2: Show that $A[f^{-1}] \simeq A[t]/(tf-1)$ as follows:

1) Let φ be the composition $A \hookrightarrow A[t] \twoheadrightarrow A[t]/(tf-1)$.
Then $\varphi(f)$ is invertible $\rightsquigarrow \varphi' : A[f^{-1}] \longrightarrow A[t]/(tf-1)$.

2) Consider the homomorphism $A[t] \longrightarrow A[f^{-1}]$, $F(t) \mapsto F(\frac{1}{f})$.
Show that it factors through the unique

$$\varphi'' : A[t]/(tf-1) \longrightarrow A[f^{-1}]$$

3) Show that φ' and φ'' are mutually inverse.

2) Localization of modules.

2.1) Definition: A, S as before. Let M be an A-module.
Define it's localization $M[S^{-1}]$ as the set of equivalence classes $M \times S/\sim$ w. $\sim$ defined by:

$$(*) \quad (m,s) \sim (n,t) \overset{def}{\Longleftrightarrow} \exists\, u \in S \mid utm = usn$$

Equiv. class of (m,s) will be denoted by $\frac{m}{s}$.

Proposition: $M[S^{-1}]$ has a natural $A[S^{-1}]$-module str're (w. addition of fractions) & $A[S^{-1}] \times M[S^{-1}] \to M[S^{-1}]$ given by $\frac{a}{s} \frac{m}{t} := \frac{am}{st}$.

Proof: for the same price as the ring structure on $A[S^{-1}]$. $\square$

2.2) Properties of $M[S^{-1}]$

The ring homomorphism $\iota : A \to A[S^{-1}]$ gives an A-module structure on $M[S^{-1}]$: $a\frac{m}{s} = \frac{am}{s}$. The map $M \xrightarrow{\iota_M} M[S^{-1}]$, $m \mapsto \frac{m}{1}$, is A-module homomorphism ($\iota = \iota_A : A \to A[S^{-1}]$ is a special case).

Proposition:

1) $\ker \iota_M = \{m \in M \mid \exists\, u \in S \text{ s.t. } um = 0\}$. In particular, ι is injective iff $um = 0 \Rightarrow m = 0$ (S acts by non-zero divisors on M).

2) $\operatorname{im} \iota_M$ generates $M[S^{-1}]$. In particular, $M[S^{-1}] = \{0\} \Leftrightarrow \iota_M = 0 \Leftrightarrow \ker \iota_M = M \overset{[(i)]}{\Leftrightarrow} \forall\, m \in M\ \exists\, u \in S$ w. $um = 0$.

3) Universal property of ι_M: let N be $A[S^{-1}]$-module and $\xi \in \operatorname{Hom}_A(M, N)$. Then $\exists!\, \tilde{\xi} \in \operatorname{Hom}_{A[S^{-1}]}(M[S^{-1}], N)$ making the following commutative:

$$\begin{tikzcd} M \arrow[d, "\iota_M"'] \arrow[dr, "\xi"] & \\ M[S^{-1}] \arrow[r, dashed] & N \end{tikzcd}$$

Proof:

1) $\ker \iota_M = \{m \in M \mid (m,1) \sim (0,1) \Leftrightarrow \exists\, u \in S \mid um = 0\}$. The other claims in 1) follow.

2) $\mathrm{Span}_{A[S^{-1}]}\, \mathrm{im}\, \iota_M \ni \frac{1}{s}\cdot\frac{m}{1} = \frac{m}{s}$ so coincides w. $M[S^{-1}]$. The remaining claims in 2) follow.

3) We should have
$$\tilde{\xi}\left(\frac{m}{s}\right) = \left[\frac{1}{s}\in A[S^{-1}]\right] = \frac{1}{s}\tilde{\xi}\left(\frac{m}{1}\right) = \frac{1}{s}\tilde{\xi}(\iota_M(m)) = \frac{1}{s}\xi(m).$$

which recovers $\tilde{\xi}$ uniquely assuming it's well-defined & $A[S^{-1}]$-linear (these two checks give the existence)

Well-defined: $\frac{m}{s} = \frac{n}{t} \Leftrightarrow utm = usn \Rightarrow ut\xi(m) = us\xi(n) \Rightarrow$ [u,t,s are invertible on N] $\frac{1}{s}\xi(m) = \frac{1}{t}\xi(n)$. ✓

$A[S^{-1}]$-linear: exercise. □

We apply 3) to produce, from an A-module homomorphism $\varphi: M_1 \to M_2$, an $A[S^{-1}]$-linear map $\varphi[S^{-1}]: M_1[S^{-1}] \to M_2[S^{-1}]$, take $\xi := \iota_{M_2}\circ\varphi: M_1 \to M_2[S^{-1}]$, and set $\varphi[S^{-1}] := \tilde{\xi}$, explicitly
$\varphi[S^{-1}]\left(\frac{m_1}{s}\right) = \frac{\varphi(m_1)}{s}$, $\forall\, m_1 \in M_1,\ s\in S$.

Important exercise: Check that

0) $\mathrm{id}_M[S^{-1}] = \mathrm{id}_{M[S^{-1}]}$.

1) For $\varphi_1: M_1 \to M_2, \varphi_2: M_2 \to M_3$, have $(\varphi_2\circ\varphi_1)[S^{-1}] = \varphi_2[S^{-1}]\circ\varphi_1[S^{-1}]$.

2) For $\varphi, \varphi': M_1 \to M_2$, have $(\varphi+\varphi')[S^{-1}] = \varphi[S^{-1}] + \varphi'[S^{-1}]$.

BONUS: Localization in noncommutative rings.

When we define the ring structure on A_S it's important that the elements of S commute w. all elements of A. Otherwise, assume for simplicity that all elements of S are invertible. We are trying to multiply right fractions as^{-1} and bt^{-1} and get a right fraction. We get $as^{-1}bt^{-1}$ - and we are stuck...

How to do localization in noncommutative rings was discovered by Ore (who was a faculty at Yale 1927-1968)

Let S be a subset of a (noncommutative) ring A such that $0 \notin S$, $1 \in S$; $s,t \in S \Rightarrow st \in S$ as before. There are so called Ore conditions that guarantee that there is a localization A_S consisting of right, equivalently, of left fractions. Namely if S doesn't contain zero divisors we need to require:

(O1) $\forall\, a \in A, s \in S\ \exists\, b \in A, t \in S$ s.t. $ta = bs$ (think, $as^{-1} = t^{-1}b$).

+ its mirror analog (left $\leftrightarrow$ right)

When S contains zero divisors we also should require:

(O2) if $sa = 0$ for $a \in A, s \in S$, then $\exists\, t \in S$ w. $at = 0$

- and its mirror condition.

In fact, (O2) allows to reduce to the case when there are no zero divisors in S: $J := \{a \in A \mid \exists\, s \in S \text{ s.t. } sa = 0\}$ is a two-sided ideal thx to (O2) + its mirror, so we replace A w. A/J, and S with its image in A/J. So we can just assume there are no zero divisors in S & (O1) and its mirror.

The we can define the set A_S of equivalence classes in A_S: $(a,s) \sim (a',s')$ (think $as^{-1} = a's'^{-1}$): we find b,t w $ta = bs$ (think $as^{-1} = t^{-1}b$) and declare $(a,s) \sim (a',s')$ if $ta' = bs'$.

Here we already see that everything becomes more painful: even to see that this doesn't depend on the choice of b,t requires a check. And there's more of this. Eventually, one gets the localization A_S consisting of right fractions (equivalently left) fractions) w. natural ring structure. It has a universal property similar to what we have in the commutative case.

Checking the Ore conditions is hard. And they are not always satisfied. For example, they aren't satisfied when $A = \mathbb{F}\langle x,y\rangle$ is a free $\mathbb{F}$-algebra & $S = A \setminus \{0\}$.

Still, they are satisfied in a number of examples. Namely, recall that A is prime if for any two-sided ideals I, J we have $IJ = \{0\} \Rightarrow I \neq \{0\}$ or $J \neq \{0\}$. We say A is Noetherian if all left & right ideals are finitely generated.

Theorem (Goldie) Let A be a prime Noetherian ring. Then the set S of all non-zero divisors in A satisfies the Ore conditions. The localization $A[S^{-1}]$ is of the form $\mathrm{Mat}_n(D)$, where $n > 0$ & D is a skew-field (a.k.a. divison ring).

In particular, A has no zero divisors $\Leftrightarrow n = 1$.