Bonus lecture B1: Connections to Algebraic geometry III.

1) Algebra homomorphisms vs polynomial maps
2) Category of affine varieties/schemes.
3) More general varieties/schemes.

Refs: [V], Sec. 9.6; [E], Sec 1.9

1) Algebra homomorphisms vs polynomial maps.

Let $\mathbb{F}$ be algebraically closed field. Let X be an algebraic subset of $\mathbb{F}^n$ (the set of solutions to a system of polynomial equations). Recall that to X we assign the ideal $I(X) = \{f \in \mathbb{F}[x_1,\dots x_n] \mid f|_X = 0\}$ & the algebra $\mathbb{F}[X] = \mathbb{F}[x_1,\dots x_n]/I(X)$ whose elements can be interpreted as polynomial functions on X.

The natural projection $\mathbb{F}[x_1,\dots x_n] \twoheadrightarrow \mathbb{F}[X]$ is the restriction to X. Set $\bar{x}_i := x_i|_X$, note that $\bar{x}_1,\dots \bar{x}_n$ generate the algebra $\mathbb{F}[X]$.

1.1) Polynomial maps.

Definition: Let $X \subset \mathbb{F}^n$, $Y \subset \mathbb{F}^m$ be algebraic subsets. A map $\varphi: X \to Y$ is called polynomial if $\exists\, f_1,\dots f_m \in \mathbb{F}[X]$ s.t. $\varphi(\alpha) = (f_1(\alpha),\dots f_m(\alpha))$ $\forall \alpha \in X$ (in particular, $(f_1(\alpha),\dots, f_m(\alpha)) \in Y$).

Rem: polynomial map $X \to \mathbb{F}$ = polynomial function on X.

Exercise: the composition of polynomial maps is polynomial.

In particular, let $\varphi: X \to Y$ be a polynomial map & $g \in \mathbb{F}[Y]$, i.e. $g: Y \to \mathbb{F}$. Consider the composed polynomial map $g \circ \varphi: X \to \mathbb{F}$. When viewed as an element of $\mathbb{F}[X]$, $g\circ\varphi$ will be denoted by $\varphi^*(g)$ and called the pullback from g (under φ).

Lemma: 1) $\varphi^*: \mathbb{F}[Y] \to \mathbb{F}[X]$ is algebra homomorphism.

2) On the generators $\bar{y}_j (:= y_j|_Y)$, $\varphi^*(\bar{y}_j) = f_j$

Proof:

1) is exercise, compare to Problem 5 in HW3.

2): $\varphi^*(\bar{y}_j)(\alpha) = \bar{y}_j(\varphi(\alpha)) = f_j(\alpha) \Rightarrow \varphi^*(\bar{y}_j) = f_j$. $\square$

Examples: 1) The inclusion map $\iota: X \hookrightarrow \mathbb{F}^n$ is polynomial, $\iota^*: \mathbb{F}[x_1, \dots x_n] \twoheadrightarrow \mathbb{F}[X]$ is the restriction map. More generally, if $X \subset Y \subset \mathbb{F}^n$ are algebraic subsets, then the inclusion map $\iota: X \hookrightarrow Y$ is polynomial & $\iota^*: \mathbb{F}[Y] \to \mathbb{F}[X]$ is $g \mapsto g|_X$.

2) $X = \mathbb{F}$, $Y = V(y_1^2 - y_2^3) \subset \mathbb{F}^2$, $\varphi: X \to Y$, $x \mapsto (x^3, x^2)$ is a polynomial map. The ideal $(y_1^2 - y_2^3)$ is prime (Problem 3 in HW6) hence radical, so $\mathbb{F}[Y] = \mathbb{F}[y_1, y_2]/(y_1^2 - y_2^3)$. By 2) of Lemma, $\varphi^*(\bar{y}_1) = x^3$, $\varphi^*(\bar{y}_2) = x^2$; this determines φ^* uniquely because $\bar{y}_1, \bar{y}_2$ generate the algebra $\mathbb{F}[Y]$.

1.2) Main result.

The following is the main result of this section.

Theorem: $\varphi \mapsto \varphi^*$ defines a bijection between:

(I) $\{$polynomial maps $\varphi: X \to Y\}$

(II) $\{$algebra homomorphisms $\mathbb{F}[Y] \to \mathbb{F}[X]\}$.

Proof: Given $\varphi = (f_1, \dots f_m)$, φ^* is the unique algebra homomorphism $\mathbb{F}[Y] \to \mathbb{F}[X]$ s.t. $\varphi^*(\bar{y}_j) = f_j$. We'll use this observation to construct the inverse map (II) $\to$ (I).

Given an algebra homomorphism $\tau: \mathbb{F}[Y] \to \mathbb{F}[X]$ define $\psi_\tau: X \to \mathbb{F}^m$ by $\psi_\tau := (\tau(\bar{y}_1), \dots, \tau(\bar{y}_m))$. We claim $\operatorname{im} \psi_\tau \subset Y$ (so that ψ_τ can be viewed as a polynomial map $X \to Y$). We have $\operatorname{im}\psi_\tau \subset Y \Leftrightarrow G(\operatorname{im}\psi_\tau) = 0 \ \forall G \in I(Y) \Leftrightarrow$

$$G(\tau(\bar{y}_1), \dots \tau(\bar{y}_m)) = 0 \qquad (*)$$

Note that $G(\bar{y}_1, \dots, \bar{y}_m) = 0$ b/c $G \in I(Y)$ & τ preserves polynomial relations b/c it's an algebra homomorphism. $(*)$ follows.

So we have maps $\varphi \mapsto \varphi^*: (I) \leftrightarrows (II): \tau \mapsto \psi_\tau$. We have $\psi_\tau^*(\bar{y}_i) = [i\text{th coordinate in } \tau] = \tau(\bar{y}_i) \Rightarrow \psi_\tau^* = \tau$ b/c $\bar{y}_i$'s generate. On the other hand, $\psi_{\varphi^*} = (\varphi^*(\bar{y}_1), \dots \varphi^*(\bar{y}_m)) = (\varphi_1, \dots, \varphi_m) = \varphi$. So these maps are mutually inverse, finishing the proof. $\square$

2) Category of affine varieties/schemes.

2.1) Affine varieties:

Similarly to Problem 5 in HW3, we see that:

- for algebraic subsets $X \subset \mathbb{F}^n, Y \subset \mathbb{F}^m, Z \subset \mathbb{F}^k$ & polynomial maps $\varphi: X \to Y, \psi: Y \to Z$, we have $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$

• $(id_X)^* = id_{\mathbb{F}[X]}$.

Exercise: prove that, in the notation of the proof of Thm $\psi_{\tau\tau'} = \psi_{\tau'}\psi_\tau$ & $\psi_{id} = id$.

So far we have discussed algebraic subsets as subsets of $\mathbb{F}^n$. A natural question (motivated, for example, by a similar issue with C^∞-manifolds: embedded into $\mathbb{R}^n$ vs defined abstractly) is: can we define "algebraic subsets irrespective of embedding" a.k.a. affine varieties.

Our first observation is that on the level of algebras the inclusion $i: X \hookrightarrow \mathbb{F}^n$ corresponds to the natural surjective map $\mathbb{F}[x_1, \dots x_n] \twoheadrightarrow \mathbb{F}[X]$, in other words, to the identification $\mathbb{F}[X] = \mathbb{F}[x_1,\dots,x_n]/I(X)$ (which is how the algebra $\mathbb{F}[X]$ was defined to start with). So to "forget" the inclusion should mean to forget the identification $\mathbb{F}[X] = \mathbb{F}[x_1, \dots x_n]/I(X)$.

The language of Category theory allows to do this.

Consider categories:

1) $\mathcal{C}$: $Ob(\mathcal{C}) = \{$fin. generated $\mathbb{F}$-algebras A w/o nonzero nilpotents: $a \in A$, $a^n = 0$ for some n but $a \neq 0\}$.

Morphisms: homomorphisms of algebras.

2) $\tilde{\mathcal{C}}$: $Ob(\tilde{\mathcal{C}}) = \{$algebras of the form $\mathbb{F}[x_1, \dots x_n]/I$ for some n & radical ideal $I \subset \mathbb{F}[x_1,\dots,x_n]\}$

So objects of $\tilde{\mathcal{C}}$ are labelled by pairs (n, I). Morphisms in $\tilde{\mathcal{C}}$ are algebra homomorphisms.

3) $\tilde{\mathcal{D}}$: $Ob(\tilde{\mathcal{D}}) = \{$algebraic subsets in some $\mathbb{F}^n\}$

Morphisms: polynomial maps.

By what was explained in the beginning of this section we have functors:

$\mathcal{F}: \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}^{opp}: X \mapsto \mathbb{F}[x_1,\dots,x_n]/I(X), \varphi \mapsto \varphi^*$.

$\mathcal{G}: \tilde{\mathcal{C}}^{opp} \to \tilde{\mathcal{D}}: A = \mathbb{F}[x_1,\dots x_n]/I \mapsto V(I), \tau \mapsto \psi_\tau$.

Crucially important exercise: $\mathcal{F}$ & $\mathcal{G}$ are mutually inverse.

Note that there is an obvious functor $\tilde{\mathcal{C}} \to \mathcal{C}$, it is full and essentially surjective so is a category equivalence, see Bonus to Lec 13.

Definition: The category of affine varieties (over $\mathbb{F}$) is $\mathcal{D} := \mathcal{C}^{opp}$.

The objects in $\mathcal{D}$ can be thought of as algebraic subsets "irrespective" of embedding into $\mathbb{F}^n$. This is because in $\mathcal{C}$ compared to $\tilde{\mathcal{C}}$ we no longer view our algebras as given by $\mathbb{F}[x_1,\dots,x_n]/I$.

The morphisms in $\mathcal{D}$ are still polynomial maps.

Example: Let $X_1 = \mathbb{F}$ & $X_2 = V(x_2 - x_1^2) \subset \mathbb{F}^2$. They have isomorphic algebras of functions: polynomials in one variable but different embeddings into $\mathbb{F}^n$'s. From the point of view of algebraic geometry they behave in the same way so can be viewed as the same variety.

While this definition of an affine variety looks like cheating, we can talk, among other things, about

- Algebra of polynomial functions $\mathbb{F}[X]$ of an affine variety X (X viewed as an object of $\mathcal{C}$)
- Points of X: algebra homomorphisms $\mathbb{F}[X] \to \mathbb{F}$ (compare w. Corollary in Sec 1.2 of Lec 23).
- The Zariski topology on X.

Etc.

Remark: We can generalize the definition of an affine variety to include more general algebras (removing the conditions that our algebras have no nonzero nilpotent elements / are finitely generated) and even general commutative rings: we can define the categories of affine schemes as the opposite category of the category of affine rings. This is useful for various purposes, which are studied in courses on scheme-theoretic algebraic geometry and go beyond the purpose of this introduction

3) More general varieties/schemes.

3.1) What is an algebraic variety?

We've discussed affine (algebraic) varieties. Now we are going to address the question in the title.

A common approach to constructing geometric objects is to "glue" them from simpler objects. For example, C^∞-manifolds are glued from balls in Euclidian spaces: $M = \bigcup_\alpha D_\alpha$, where $D_\alpha \xrightarrow[\sim]{\varphi_\alpha} \{v \in \mathbb{R}^n \mid \|v\| < 1\}$. The condition is, roughly, that for all α, β in the index set, the images of $D_\alpha \cap D_\beta$ under $\varphi_\alpha, \varphi_\beta$ are open subsets in $\{v \in \mathbb{R}^n \mid \|v\| < 1\}$ and the resulting composition

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(D_\alpha \cap D_\beta) \xrightarrow[\sim]{\varphi_\alpha^{-1}} D_\alpha \cap D_\beta \xrightarrow[\sim]{\varphi_\beta} \varphi_\beta(D_\alpha \cap D_\beta)$$

is C^∞ (which makes sense b/c this is a map between open subsets in $\mathbb{R}^n$). Thanks to this definition it makes to speak about various C^∞-objects, e.g. C^∞-maps $M \to N$.

Similarly, it makes sense to speak about complex analytic manifolds: we use balls in $\mathbb{C}^n$ and require that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is complex analytic (you might have studied that for $n=1$ — in which case the resulting objects appear when you study analytic continuation of holomorphic functions).

Something like this happens for algebraic varieties too. The building blocks are affine algebraic varieties and they are glued together using polynomial isomorphisms: if the variety of interest is reasonable ("separated" in a suitable sense) the intersection of two open affine subvarieties is again affine so we can just use what we have in this lecture.

We can define the notion of a polynomial map (a.k.a. morphism): $\varphi: X \to Y$ is a morphism if we can cover $X = \bigcup U_i$, $Y = \bigcup V_j$ w. open affine varieties s.t. $\forall i \ \exists j \mid \varphi(U_i) \subset V_j$ & $\varphi: U_i \to V_j$ is a polynomial map of affine varieties.

3.2) Projective varieties and graded algebras.

Here comes one of the most important example of the construction sketched above: projective varieties.

We start with $\mathbb{F}^{n+1}$ (viewed as a vector space). The projective space $\mathbb{P}^n (= \mathbb{P}(\mathbb{F}^{n+1}))$ as a set consists of 1-dimensional subspaces in $\mathbb{F}^{n+1}$. In other words, it consists of equivalence classes $[x_0 : \ldots : x_n]$ w. $(x_0, \ldots x_n) \in \mathbb{F}^{n+1} \setminus \{0\}$, where equivalent means proportional. Let us explain how gluing works.

Let $U_i = \{[x_0 : \ldots : x_n] \mid x_i \neq 0\}$, $i = 0, \ldots n$. Then the map $U_i \xrightarrow[\sim]{\varphi_i} \mathbb{F}^n: [x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i}\right)$ is a bijection that will be used to identify U_i w. $\mathbb{F}^n$. Note that $\varphi_i(U_i \cap U_j)$ is given by non-vanishing of a single coordinate so is an affine variety (a principal open subset in $\mathbb{F}^n$). And one can show that

$$\varphi_j \circ \varphi_i^{-1}: \ \varphi_i(U_i \cap U_j) \xrightarrow{\sim} \varphi_j(U_i \cap U_j)$$

is a polynomial isomorphism.

Example: let $n=2$. Let $y_0 = \frac{x_1}{x_0}$ & $y_1 = \frac{x_0}{x_1}$ be coordinates on $\varphi_0(U_0) \simeq \mathbb{F}$ & $\varphi_1(U_1) \simeq \mathbb{F}$. Then $\varphi_i(U_0 \cap U_1)$ is given by $y_i \neq 0$ & $\varphi_1 \circ \varphi_0^{-1}$ sends y_0 to y_0^{-1}, which is a polynomial isomorphism as we have inverted y_0.

So $\mathbb{P}^n$ is an algebraic variety in the sense of Sec 3.1.

One can generalize this construction. Let $F_1, \dots F_k \in \mathbb{F}[x_0, \dots x_n]$ be homogeneous polynomials of degree >0. If F_i vanishes at a nonzero point in $\mathbb{F}^{n+1}$, then it also vanishes on the line between this point & 0. So it makes sense to speak about the zero locus of F_i in $\mathbb{P}^n$ (note that F_i is NOT a function $\mathbb{P}^n \longrightarrow \mathbb{F}$). This gives rise to the zero locus $V(F_1, \dots F_k)$ and hence to the notion of an algebraic subset of $\mathbb{P}^n$.

Exercise: $V(F_1, \dots F_k) \cap U_i$ is an algebraic subset in $U_i \xrightarrow{\sim} \mathbb{F}^n$.

So $V(F_1, \dots F_k)$ is an algebraic variety, varieties of that kind are called projective.

Here's a reason why we care about them. Let $\mathbb{F} = \mathbb{C}$. So $\mathbb{C}^n$ has the usual topology. And so does $\mathbb{P}^n$ with U_i's being open subsets.

Important exercise: $\mathbb{P}^n$ is compact – in the usual topology.

And so, every $V(F_1, \dots F_k)$ is compact. In Geometry & Topology we like compact spaces more than noncompact as they behave better in many ways. And while not all compact (in the usual topology) algebraic varieties are projective, the projective ones are nicest.

Now we discuss a connection between projective varieties & graded algebras. The vanishing locus of $V(F_1, \dots F_k)$ depends only on $(F_1, \dots F_k)$, a homogeneous ideal.

Exercise: If $I \subset \mathbb{F}[x_0, \dots x_n]$ is a homogeneous ideal, then so is its radical.

In fact, $V(F_1, \dots F_k)$ only depends on $\sqrt{(F_1, \dots F_k)}$, similarly to the affine case. This gives rise to a bijection between

- Algebraic subsets of $\mathbb{P}^n$
- and radical homogeneous ideals in $\mathbb{F}[x_0, \dots x_n]$ not containing 1.

Exercise: What ideal corresponds to $\emptyset$?

So starting from an algebraic subset in $\mathbb{P}^n$ we get a fin gen'd graded algebra w/o nilpotent elements, the quotient of $\mathbb{F}[x_0, \dots x_n]$ by the corresponding ideal. Note that the elements of this algebra are not functions on the initial algebraic subset of $\mathbb{P}^n$.

Conversely, let $A = \bigoplus_{i=0}^{\infty} A_i$ be a fin. gen'd graded $\mathbb{F}$-algebra w/o nilpotents s.t. $A_0 = \mathbb{F}$. From this algebra we can construct a projective variety. Namely, if A is generated by A_1 ($\Leftrightarrow$ A is a graded quotient of $\mathbb{F}[x_0, \dots x_n]$), then we consider the algebraic subset of $\mathbb{P}^n$ defined by the kernel of $\mathbb{F}[x_0, \dots x_n] \twoheadrightarrow A$, which is a homogeneous ideal.

In general – if A isn't generated by A_1 – we have the following:

Exercise: $\exists\, d > 0$ s.t. $A_{(d)} := \bigoplus_{i=0}^{\infty} A_{di}$ is generated by A_d.

A fun fact: the projective variety we get is independent of the choice of d up to an isomorphism.

Example: Take $A = \mathbb{F}[x_0, x_1]$ (w. usual grading). It gives rise to the projective line $\mathbb{P}^1$. Now consider $A_{(2)}$. It's generated by $y_0 := x_0^2$, $y_1 := x_0 x_1$, $y_2 := x_1^2$. The relations between the elements y_0, y_1, y_2 are generated by $y_0 y_2 - y_1^2$. The corresponding algebraic subset is $\{[y_0 : y_1 : y_2] \mid y_0 y_2 = y_1^2\} \subset \mathbb{P}^2$. Denote it by X.

We are going to construct two mutually inverse polynomial maps between $\mathbb{P}^1$ & X. Let $\varphi : \mathbb{P}^1 \to X$ be given by $[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_1^2]$. Now we define $\psi : X \to \mathbb{P}^1$:

$$\psi([y_0:y_1:y_2]) = \begin{cases} [y_0:y_1], \text{ if } y_2 \neq 0 \\ [y_1:y_2], \text{ if } y_0 \neq 0. \end{cases}$$

Exercise: Check φ, ψ are well-defined & mutually inverse maps. Furthermore, check that φ, ψ are morphisms (in the sense explained in the end of Section 3.1).

A connection with projective varieties is one of the reasons to care about graded algebras.