Bonus lecture B2: Connections to Algebraic geometry IV

1) Vector bundles on C^∞-manifolds.

2) Modules, algebro-geometrically.

1) Vector bundles on C^∞-manifolds.

Vector bundles are of crucial importance for Differential geometry. In this section we sketch what they are.

Let M be a C^∞-manifold (e.g. a submanifold in some $\mathbb{R}^n$). On the most basic level, a vector bundle is an assignment of a vector space to each point of M, the vector space is supposed to "depend smoothly on the point, in particular, to have the same dimension (if M is connected). To make this into an actual definition, we need to formalize the notion of fibers depending on a point "in a C^∞-way." To see how this should be done, consider the most classical example: the tangent bundle. It assigns the tangent space, $T_m M$, to each $m \in M$. If we have local coordinates, $x_1, \dots x_n$, on some neighborhood $U \subset M$ of m, then $T_m M$ gets a basis $\frac{\partial}{\partial x_i}\big|_u$, $i = 1,..n$. This allows us to trivialize the tangent bundle over U. Note that a different choice of coordinate functions, $x_1', .. x_n'$, on U give a different basis $\frac{\partial}{\partial x_i'}\big|_u$, the two bases are related via the Jacobian matrix $\left(\frac{\partial x_i}{\partial x_j'}\right)$.

Now we are ready to define the notion of a vector bundle.

Let E be another manifold w. a C^∞-map $\pi: E \to M$. We say that E (or, more precisely, (E, π)) is a vector bundle of rank n if $\exists$ cover $X = \bigcup_{\alpha \in I} U_\alpha$ by open subsets and a collection of isomorphisms

(1) $$\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

s.t. $\forall\, \alpha, \beta \in I \ \exists\ \varphi_{\alpha\beta} \in GL_n(C^\infty(U_\alpha \cap U_\beta))$ (invertible matrix) s.t.

(2) $$\varphi_\alpha \circ \varphi_\beta^{-1}((u,x)) = (u, \varphi_{\alpha\beta}(u)x) \quad \forall\, u \in U_\alpha \cap U_\beta,\ x \in \mathbb{R}^n.$$

Note that the matrices $\varphi_{\alpha\beta}$ satisfy the following conditions: $\varphi_{\alpha\alpha} = Id$, $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$ & $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta}\varphi_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

In fact given such a collection of matrices (called the transition functions) we can reconstruct the vector bundle uniquely – by gluing the open subsets: $U_\alpha \times \mathbb{R}^n$ & $U_\beta \times \mathbb{R}^n$ are glued along $(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ using (2).

In particular, we can equip $TM = \{(m, \xi) \mid m \in M, \xi \in T_m M\}$ with the structure of a vector bundle by using the Jacobian matrices for the transition functions.

The next important construction is that of module of sections. By a section of $\pi : E \to M$ we mean a C^∞-map $\sigma : M \to E$ s.t. $\pi \circ \sigma = id_M$. It turns out the set of such maps has a natural str're of a module over $C^\infty(M)$, the algebra of C^∞-functions. For example, let's explain how to define $f\sigma$ for $f \in C^\infty(M)$ and a section σ. It's enough to define $(f\sigma)|_{U_\alpha}$'s and show they agree on intersections. We set

$$(f\sigma)|_{U_\alpha} = \varphi_\alpha^{-1} f(\varphi_\alpha \circ \sigma)$$

Here $\varphi_\alpha \circ \sigma$ is a map $U_\alpha \to U_\alpha \times \mathbb{R}^n$ of the form $u \mapsto (u, g_\alpha(u))$ – it's a section of the projection $U_\alpha \times \mathbb{R}^n \to U_\alpha$, where g_α is a

C^∞-map $U_\alpha \to \mathbb{R}^n$. So, by $f(\varphi_\alpha \circ \sigma)$ we mean the map $U_\alpha \to U_\alpha \times \mathbb{R}^n$ given by $u \mapsto (u, fg_\alpha(u))$. And then $(f\sigma)|_{U_\alpha}$ is indeed a section of $U_\alpha \to \pi^{-1}(U_\alpha)$.

Important exercise:

- Show that $f\sigma$ is well-defined: the restrictions of $f\sigma$ to U_α and to U_β agree on $U_\alpha \cap U_\beta$ (hint: use (2))
- Define $\sigma_1 + \sigma_2$ for two sections σ_1, σ_2
- Show that these operations equip the set of sections w. a $C^\infty(M)$-module structure. This set will be denoted by $\Gamma(M, E)$

In fact, using the same construction, we see that the set $\Gamma(U, E)$ of sections of $\pi^{-1}(U) \to U$ is a $C^\infty(U)$-module. The module structures on $\Gamma(U, E)$ for various E are compatible in a suitable sense, making the collection of $\Gamma(U, E)$ into a "sheaf" – an important notion that we are not going to define. One can actually recover E from knowing the sheaf, but we will skip this as well. Finally, note that if E trivializes over U: $\pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$, then $\Gamma(U, E)$ is a free rank n module over $C^\infty(U)$.

Example: For the tangent bundle TM its sections on U, $\Gamma(U, TM)$ is nothing else but the vector fields on U.

Remark: Note that $\Gamma(M, E)$ may fail to be free over $C^\infty(M)$.

An example is provided by $M = \mathbb{S}^2$ (the 2-sphere) & $E = TM$. The claim that $\Gamma(M, TM)$ is not free over $C^\infty(M)$ can be deduced from the heggehog theorem (every vector field on $\mathbb{S}^2$ vanishes at some point)

To finish, let's explain why we care about vector bundles and their sections:

- Vector fields are fundamentally important in the study of manifolds.
- So are differential forms – sections of exterior powers $\Lambda^k T^*M$ – for example, they are crucial for the general Stokes theorem
- Sections of $S^2(T^*M)$ (symmetric forms) are important for the (pseudo) Riemmanian geometry & Relativity theory, while certain sections of $\Lambda^2 T^*M$ (symplectic forms) are important for Symplectic geometry & Classical Mechanics.

2) Modules, algebro-geometrically.

$\mathbb{F}$ is alg. closed field, X is an alg'c subset in $\mathbb{F}^n$ (i.e. an affine alg'c variety), $A = \mathbb{F}[X]$

Q: How to think about A-modules geometrically?

Quick A: General modules give some "singular version" of vector bundles ("singular" refers to fibers of different dimensions, basically). Locally free modules give a complete algebraic analog of vector bundles. We'll elaborate on this below.

2.1) Fibers

Given an A-module M and a point $\alpha\in X$, we want to assign a vector space $M(\alpha)$. Recall (Sec 1.2 of Lec 23) that

$$\begin{array}{ccl} X & \xrightarrow{\sim} & \{\text{max. ideals in } A\} \\ \psi & & \quad\psi \\ \alpha & \longmapsto & \mathfrak{m}_\alpha := \{f\in A \mid f(\alpha)=0\};\ A/\mathfrak{m}_\alpha \xrightarrow{\sim} \mathbb{F}. \end{array}$$

Definition: For an A-module M, its fiber at α is

$M(\alpha) := M/\mathfrak{m}_\alpha M$, an $\mathbb{F}$-vector space

Rem: if M is fin. gen'd $\Rightarrow \dim M(\alpha)<\infty\ \forall\alpha$

So: from M we get a collection of vector spaces indexed by pts. of X, a precursor of the definition of a vector bundle. However, these fibers may have different dimensions for different points α.

Examples: 1) $M=A^{\oplus n} \Rightarrow M(\alpha)=\mathbb{F}^n$.

2) $M=A/I$, where $I\subset A$ is an ideal.

$M(\alpha) = (A/I)/\mathfrak{m}_\alpha(A/I) = A/(I+\mathfrak{m}_\alpha)$.

If $\mathfrak{m}_\alpha\supset I \Rightarrow I+\mathfrak{m}_\alpha=\mathfrak{m}_\alpha \Rightarrow M(\alpha)=\mathbb{F}$.

$\Leftrightarrow \alpha\in V(I)$

If $\mathfrak{m}_\alpha\not\supset I \Rightarrow I+\mathfrak{m}_\alpha = A \Rightarrow M(\alpha)=\{0\}$.

2.2) Localization of modules vs fibers.

$X \subset \mathbb{F}^n$ alg. subset, $A = \mathbb{F}[X]$, $f \in A \setminus \{0\}$. Then the localization $A[f^{-1}]$ is the algebra $\mathbb{F}[X_f]$, where
$X_f := \{\alpha \in X \mid f(\alpha) \neq 0\}$ (Section 2.1 of Lecture 24)

Let M be A-module $\rightsquigarrow$ the $A[f^{-1}]$-module $M[f^{-1}]$. Now to $\alpha \in X_f$ we can assign two fibers $M(\alpha)$, $M[f^{-1}](\alpha) = M[f^{-1}]/M[f^{-1}]\mathfrak{m}'_\alpha$, where $\mathfrak{m}'_\alpha \subset \mathbb{F}[X_f]$ is the maximal ideal corresponding to α.

Proposition: $\forall\, \alpha \in X_f$ have natural isomorphism $M(\alpha) \xrightarrow{\sim} M[f^{-1}](\alpha)$.

Proof: Note that $\mathfrak{m}'_\alpha = \{\frac{g}{f^k} \mid g(\alpha) = 0\}$. We can view $M[f^{-1}]$ as an A-module - via the homomorphism $\iota: A \to A[f^{-1}]$, $g \mapsto \frac{g}{1}$. Since $\iota(\mathfrak{m}_\alpha) \subset \mathfrak{m}'_\alpha$, the ideal $\mathfrak{m}_\alpha$ acts by 0 on $M[f^{-1}](\alpha)$. So the homomorphism $\iota_M: M \to M[f^{-1}]$ descends to

(3) $$M(\alpha) \to M[f^{-1}](\alpha),\ m + \mathfrak{m}_\alpha M \mapsto \frac{m}{1} + \mathfrak{m}'_\alpha M[f^{-1}].$$

We need to show (3) is injective & surjective.

Injectivity: we need to show that $\frac{m}{1} \in \mathfrak{m}'_\alpha M[f^{-1}] \Rightarrow m \in \mathfrak{m}_\alpha M$.
The inclusion $\frac{m}{1} \in \mathfrak{m}'_\alpha M[f^{-1}]$ is equivalent to:
$$f^\ell \Big(m - \sum_{i=1}^k g_i m_i\Big) = 0 \text{ for some } \ell > 0,\ g_i \in \mathfrak{m}_\alpha,\ m_i \in M.$$
From $f - f(\alpha) \in \mathfrak{m}$ we get $f^\ell - f(\alpha)^\ell \in \mathfrak{m}_\alpha$. So
$f(\alpha)^\ell m = -(f^\ell - f(\alpha)^\ell)m - \sum_{i=1}^k g_i m_i \in \mathfrak{m}_\alpha M$. Since $f(\alpha) \neq 0$, we are done.

Surjectivity: We need to show that $\forall\, m\in M,\ \ell\in\mathbb{Z}_{\geq 0}\ \exists\, \tilde{m}\in M$ s.t. $\frac{m}{f^\ell}-\frac{\tilde m}{1}\in \mathfrak{m}_\alpha' M[f^{-1}]$. Just take $\tilde m = f(\alpha)^{-\ell}m$. $\square$

An informal way to think about the claim of the proposition: the collection of fibers of $M[f^{-1}]$ is just the restriction of the collection of fibers of M from X to X_f.

2.3) Projective modules as vector bundles.

Let X be an affine variety. Set $A:=\mathbb{F}[X]$ and let M be a finitely generated projective (equiv., locally free) module. By Problem 4 in HW3, $\exists\, f_1,\dots f_k\in A$ w. $(f_1,\dots f_k)=A$ s.t. $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$-module. The condition $(f_1,\dots f_k)=A$ geometrically means that $X=\bigcup_{i=1}^k X_{f_i}$. Proposition in Sec 2.2 suggests that $M[f_i^{-1}]$ can be viewed as the "restriction" of M to X_{f_i}. This gives another explanation to the term "locally-free module."

We view M as an algebraic analog of the module of sections of a vector bundle. For example the corresponding affine variety E is recovered as follows.

Exercise: Prove that $M^*:=\mathrm{Hom}_A(M,A)$ is also a finitely generated projective A-module.

We can form the symmetric algebra $S_A(M^*)$.

Exercise: 1) Prove that $S_A(M^*)[f_i^{-1}] \simeq S_{A[f_i^{-1}]}(M[f_i^{-1}])$, hence isomorphic to the algebra of polynomials w. coefficients in $A[f_i^{-1}]$.

2) Prove that $S_A(M^*)$ contains no nonzero nilpotent elements.

Let E denote the affine variety corresponding to $S_A(M^*)$. The natural homomorphism $A \to S_A(M^*)$ gives rise to a morphism of varieties $\pi: E \to X$. Then $\pi^{-1}(X_{f_i})$ is the affine variety corresponding to $S_A(M)[f_i^{-1}]$ – as this is precisely the locus in E, where the function $f_i \circ \pi$ is nonzero.

Assume now, for simplicity, that $M[f_i^{-1}] \simeq A[f_i]^{\oplus n}$, where n is the same for all i (true, for example, when X is irreducible, or, more generally, connected in Zariski topology). Then by 1) of the previous exercise, $\pi^{-1}(X_{f_i}) \xrightarrow{\sim} X_{f_i} \times \mathbb{A}^n$ (in fact, intertwining the maps to X_{f_i}). This is an analog of (1) from Section 1. One can also get the direct analog of (2) there: it comes from the natural isomorphisms $M[f_i^{-1}][f_j^{-1}] \simeq M[(f_if_j)^{-1}] \simeq M[f_j^{-1}][f_i^{-1}]$ & the identifications $M[f_i^{-1}] \simeq A[f_i^{-1}]^{\oplus n}$, $M[f_j^{-1}] \simeq A[f_j^{-1}]^{\oplus n}$.

We finish with the following nice fact (whose proof based on the Nakayama lemma we skip).

Thm: Suppose X is irreducible (or, more generally, connected). Let M be a finitely generated $A = \mathbb{F}[X]$-module. If $\dim M(\alpha)$ is the same $\forall \alpha \in X$, then M is locally free.