Bonus lecture B3: Connections to Algebraic Geometry, III.

1) Dedekind domains in Algebraic geometry.
2) Class groups in Algebraic geometry.

1) Dedekind domains in Algebraic geometry.

As was mentioned in Lectures 25 & 26 there is an important class of domains coming from Algebraic geometry: algebras of functions on (irreducible) smooth algebraic curves. We elaborate on what this means in this section

1.1) Dimensions in Algebraic Geometry.

Let $X \subset \mathbb{F}^n$ be an algebraic subset (a.k.a. affine algebraic variety). Here we discuss what one means by the dimension of X. By Sec 1.2 in Lec 24, $X = \bigcup_{i=1}^{k} X_i$, the union of irreducible components. It's natural to assume that $\dim X = \max_k \dim X_k$ and so it's enough to deal w. irreducible X.

Recall that in the C^∞ setting the dimension of a manifold M is the maximal number of (functionally) independent functions on M. We can apply the same logic in the algebraic situation

Definition: For irreducible X, define $\dim X$ as the transcendence degree of $\operatorname{Frac} \mathbb{F}[X]$ as a field extension of $\mathbb{F}$.

A sanity check - for $X=\mathbb{F}^n$, the field $\operatorname{Frac} \mathbb{F}[X]$ is the field of rational functions $\mathbb{F}(x_1, \dots x_n)$ of transcendence deg $= n$).

The observation that any nonconstant element in $\mathbb{F}[X]$ can be included into a transcendence basis (recall $\mathbb{F}$ is algebraically closed) - plus some work - leads to the following result.

Proposition: Let X be an irreducible affine variety. Let $f \in \mathbb{F}[X]$ be a non-constant element. Then every irreducible component of the subvariety of zeroes of f in X has dimension $\dim X - 1$.

Corollary: for irreducible X TFAE:

a) $\dim X = 1$

b) Every nonzero prime ideal in $\mathbb{F}[X]$ is maximal.

Sketch of proof: (a)$\Rightarrow$(b): Let $\mathfrak{p} \subset \mathbb{F}[X]$, $\mathfrak{p} \neq \{0\}$ be a prime ideal & $f \in \mathfrak{p} \setminus \{0\}$. Let $V(\mathfrak{p}) \subset V(f) \subset X$ be the sets of zeroes of $\mathfrak{p}$ & f in X. Then every irreducible component of $V(f)$ has dimension 0, i.e. is a point. Since $V(\mathfrak{p})$ is irreducible, it is contained in one of these components, so is a point. But $I \mapsto V(I)$ gives a bijection between the radical ideals in $\mathbb{F}[X]$ & algebraic subsets of X (this follows from Corollary in Sec 2.2 of Lec 23). Hence if $V(\mathfrak{p}) = \{\alpha\}$, then $\mathfrak{p} = \{g \in \mathbb{F}[X] \mid g(\alpha) = 0\}$. By Corollary in Sec 1.2 of Lec 23, $\mathfrak{p}$ is maximal.

(b) $\Rightarrow$ (a): Analogously to the previous argument, we see that every proper irreducible algebraic subset of X is a point. Now $\dim X = 1$ follows from Proposition. □

Of course, the varieties of dim 1 are called curves.

1.2) Normality & smoothness.

Now our question is when, for an irreducible affine curve X, the domain $\mathbb{F}[X]$ is normal. For this we need to discuss the tangent spaces in Algebraic geometry.

Let's first recall one definition of the tangent space in the C^∞-setting. Let M be a C^∞-manifold & $m \in M$. Then the tangent space $T_m M$ is identified with the space of $\mathbb{R}$-linear maps $\partial: C^\infty(M) \longrightarrow \mathbb{R}$ satisfying the following version of the Leibniz identity:

(1) $$\partial(fg) = (\partial f)\, g(m) + f(m)\, \partial g.$$

The same construction makes sense in the algebraic setup. Take an affine variety X & $\alpha \in X$. Then the tangent space $T_\alpha X$ is <u>defined</u> as the space of $\mathbb{F}$-linear maps satisfying the direct analog of (1).

Exercise: Let $\mathfrak{m}$ be the maximal ideal $\{f \in \mathbb{F}[X] \mid f(\alpha)=0\}$. Then any $\partial \in T_\alpha X$ sends 1 & $\mathfrak{m}^2$ to 0 giving rise to a map $T_\alpha X \longrightarrow (\mathfrak{m}/\mathfrak{m}^2)^*$. Show that this map is an isomorphism.

One can show that $\dim T_\alpha X \geqslant \dim X$ for all $\alpha \in X$. An example when $\dim T_\alpha X$ is provided by $X = \{(x,y) \mid y^2 - x^3 = 0\}$ (the "cusp") & $\alpha = (0,0)$. Here $\mathfrak{m} = (\bar{x}, \bar{y})$, where $\bar{x}, \bar{y}$ are the images of x, y in $\mathbb{F}[X] = \mathbb{F}[x,y]/(y^2 - x^3)$ & $\mathfrak{m}^2 = (\bar{x}^2, \bar{x}\bar{y}, \bar{y}^2)$. We claim that $\bar{x}, \bar{y}$ are linearly independent modulo $\mathfrak{m}^2$, equivalently, the images of x, y in $\mathbb{F}[x,y]/J$, where $J = (x^2, xy, y^2) + (x^2 - y^3) = (x^2, xy, y^2)$ are linearly independent - which is manifest. From here we deduce $\dim T_\alpha X = 2$, while X is a curve (so $\dim X = 1$).

Definition: We say that $\alpha \in X$ is **smooth** if $\dim T_\alpha X = \dim X$ (and **singular** otherwise). We say that irreducible X is **smooth** (and $\mathbb{F}[X]$ is **regular**) if all points of X are smooth.

This generalizes to reducible X (one needs to be a bit careful with the definition when there are components of different dimension). Any point lying in more than one component turns out to be singular.

For $\mathbb{F}=\mathbb{C}$, the smooth points are those, where a neighborhood of $\alpha \in X$ looks like a small disc around $0 \in \mathbb{C}^n$. So, the locus of smooth points in X is a complex manifold).

In particular, we can talk about smooth irreducible curves.

Fact: $\mathbb{F}[X]$ is a Dedekind domain iff X is a smooth irreducible curve.

In general regular implies normal, but not vice versa, the first example is $\mathbb{F}[X]$, where $X=\{(x,y,z)\in\mathbb{F}^3 \mid x^2+y^2+z^2=0\}$.

2) Class groups in Algebraic geometry.

Let X be a smooth irreducible algebraic curve. We want to better understand the class group $Cl(X):=Cl(\mathbb{F}[X])$. In fact, we'll see that to get a meaningful and nice answer we need to replace X with its "projective" closure.

2.1) Projective closure

Let $X\subset\mathbb{F}^n$ be an algebraic subset. Let $x_1, \dots x_n$ denote the standard coordinates on $\mathbb{F}^n$. We embed $\mathbb{F}^n$ as the coordinate chart $x_0 \neq 0$ in the projective space $\mathbb{P}^n$ (see Sec 3.2 in Lec B2). We define $\overline{X}$, the projective closure of X, to be the minimal algebraic subset of $\mathbb{P}^n$ containing X. Equivalently, $\overline{X}$ is the common locus of zeroes of the homogeneous polynomials $f(x_0, x_1, \dots x_n)$ s.t. $f(1, x_1, \dots x_n) \in I(X)$.

Now assume that X is a smooth projective curve. The variety $\overline{X}$ can be singular. However, for a suitable choice of generators of $\mathbb{F}[X]$ (that gives rise to an embedding of X into $\mathbb{F}^m$ as an algebraic subset), $\overline{X}$ is smooth. The variety $\overline{X}$ up to an isomorphism depends only on $\operatorname{Frac}(\mathbb{F}[X])$. Note that $\operatorname{Frac}\mathbb{F}[X]$ is naturally identified with $\mathbb{F}[U]$ for every Zariski open affine $U \subset \overline{X}$. This field is called the field of rational functions on $\overline{X}$ & is denoted by $\mathbb{F}(\overline{X})$.

2.2) The divisor of a function

To nonzero $f \in \mathbb{F}(\overline{X})$ & $\alpha \in \overline{X}$ we can assign the order of f at α, to be denoted by $\operatorname{ord}_\alpha(f)$. This is done as follows.

Let U be a Zariski open affine neighborhood of α in $\overline{X}$ (so that $\overline{X} \setminus U$ is finite; if $\overline{X} \subset \mathbb{P}^n$, for U we can take the intersection of $\overline{X}$ with a standard coordinate chart containing α. Then U is a smooth affine curve & $\mathbb{F}[U]$ is a

Dedekind domain. Recall that the maximal ideals in $\mathbb{F}[U]$ are in bijection w. U ($\alpha \leftrightarrow \mathfrak{m}_\alpha$), see Corollary in Sec 1.2 of Lec 23. The unique factorization theorem from Sec 2 of Lec 25 (see also Sec 1.0 of Lec 26) extends to nonzero fractional ideals: every such ideal in $\mathbb{F}[U]$ uniquely factorizes into the product of integer powers of maximal ideals. For $ord_\alpha(f)$ we take the power of $\mathfrak{m}_\alpha$ in the decomposition of $\mathbb{F}[U]f$ (equal to 0 if $\mathfrak{m}_\alpha$ doesn't appear). For example, if $f \in \mathbb{F}[U]$, then $ord_\alpha(f) \geq 0$ & $ord_\alpha(f) > 0$ means that $f(\alpha) = 0$. In fact, $ord_\alpha(f)$ is interpreted as the order of zero/pole of f at α.

To make this interpretation more explicit, consider the case of $\mathbb{F} = \mathbb{C}$. Here $\overline{X}$ is a compact 1-dimensional complex manifold, in particular, every $\alpha \in \overline{X}$ has a neighborhood (in the "usual" topology) that can be identified w. $\{z \in \mathbb{C} \mid |z| < 1\}$, where $\alpha \in \overline{X}$ corresponds to 0. The restriction of f to this neighborhood is meromorphic, and $ord_\alpha(f)$ is the order of zero/pole of f at $z=0$.

Exercise: $\{\alpha \mid ord_\alpha(f) \neq 0\}$ is finite.

Def'n: The divisor, $div(f)$, of f is $\sum_{\alpha \in \overline{X}} ord_\alpha(f)\alpha \in \mathbb{Z}^{\oplus \overline{X}}$.

$\mathbb{7}$

Facts: 1) $\mathrm{div}(f)=0 \Leftrightarrow f$ is constant (in the complex analytic setting, this follows from the maximum principle)

2) $\sum_{\alpha\in\bar{X}} \mathrm{ord}_\alpha(f)=0$.

2.3) Structure of $Cl(\bar{X})$.

Note that for an open affine subset U, we have $FI(\mathbb{F}[U])\simeq \mathbb{Z}^{\oplus U}$ (exercise). So the analog of FI for $\bar{X}$ is $\mathbb{Z}^{\oplus\bar{X}}$. And the analog of PFI is $\{\mathrm{div}(f) \mid f\in \mathrm{Frac}(\mathbb{F}[U])\setminus\{0\}\}$ (the fields $\mathrm{Frac}(\mathbb{F}[U])$ are naturally identified for different choices of U).

We set

(1) $$Cl(\bar{X})=\mathbb{Z}^{\oplus\bar{X}}/\{\mathrm{div}(f)\}.$$

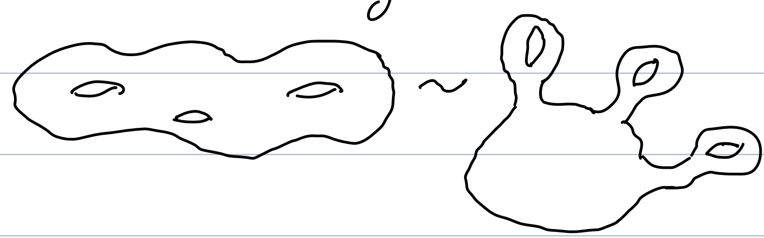
With this definition, for an open affine $U\subset\bar{X}$ we have

(2) $$Cl(\mathbb{F}[U])=\mathbb{Z}^{\oplus\bar{X}}/(\{\mathrm{div}(f)\}+\mathbb{Z}^{\oplus\bar{X}\setminus U}).$$

Let's explain a few basic things about the structure of $Cl(\bar{X})$. First of all consider the "degree map" $\mathbb{Z}^{\oplus\bar{X}}\mapsto\mathbb{Z}$, the group homomorphism, sending $\forall\, \alpha\in X$ (a basis element in $\mathbb{Z}^{\oplus\bar{X}}$) to 1. By Fact 2 in Section 2.2, this homomorphism descends to a group homomorphism $\deg: Cl(\bar{X})\longrightarrow\mathbb{Z}$. This homomorphism admits a right inverse depending on the choice of a point $\alpha\in\bar{X}$, it sends $n\in\mathbb{Z}$ to the class of $n\alpha$ in $Cl(\bar{X})$. So $Cl(\bar{X})\simeq\mathbb{Z}\oplus\ker(\deg)$.

We write $Cl^0(\bar{X})$ for ker(deg). It remains to describe the structure of $Cl^0(\bar{X})$. The description is easier and more classical when $\mathbb{F}=\mathbb{C}$, which is what we are going to assume from now on.

Note that $\bar{X}$ can be viewed as a 2-dimensional C^∞-manifold. It comes from a complex manifold, hence is orientable. It's projective, hence compact. Compact orientable surfaces are classified up to homeomorphism (or diffeomorphism) by the number of handles known as the genus. Here's a cartoon for a genus 3 surface:



Example: Consider a curve $\bar{X}$ in $\mathbb{P}^2$, zeroes of a homogeneous polynomial $F(x,y,z)$ of degree d. The condition that $\bar{X}$ is smooth is equivalent to the claim that $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_{(a,b,c)} \neq (0,0,0)$ for $(a,b,c)\neq(0,0,0)$. Then the genus of $\bar{X}$ is $\frac{1}{2}(d-1)(d-2)$.

For example:

$d=1$: this is just $\mathbb{P}^1$ given by a linear equation in $\mathbb{P}^2$.

$d=2$: this is the smooth quadric in $\mathbb{P}^2$. It's classically known to be isomorphic to $\mathbb{P}^1$ (via the projection from a point in the quadric). But more is true: every genus 0 curve is isomorphic to $\mathbb{P}^1$.

$d=3$: up to a linear change of variables every polynomial F whose zeroes in $\mathbb{P}^2$ is a smooth curve is given by

$$-y^2z+x^3+pxz^2+qz^3=0 \quad (p,q\in\mathbb{C},\ 4p^3+27q^2\neq 0)$$

The latter inequality means that the polynomial x^3+px+q has no repeated roots.

The curves of this form are known as elliptic curves. In fact, all genus 1 curves are of this form.

Now we get back to the general situation.

Fact: Let g denote the genus of $\bar{X}$. As a group $Cl^0(\bar{X})$ is isomorphic to the quotient of $\mathbb{C}^{\oplus g}$ (the group w.r.t. addition) by a rank $2g$ lattice (i.e. the subgroup generated by some $\mathbb{R}$-vector space basis).

A proof is roughly as follows: one shows that $Cl^0(\bar{X})$ is a compact connected complex Lie group and then shows that the dimension is g. Any such group is isomorphic to a quotient of $\mathbb{C}^g$ by a lattice as explained in the fact.

Exercise: Prove the claim of Fact for $\bar{X}=\mathbb{P}^1$.

Example: We get back to the example of $\overline{X}$ given by $-y^2z+x^3+pxz^2+qz^3=0$. There's an elementary construction of a product on $\overline{X}$ w. $[0:1:0]$ being 0: for $\alpha,\beta,\gamma\in\overline{X}$ we declare $\alpha+\beta+\gamma=0$ if α,β,γ are the intersection points (counted w. multiplicity) of a projective line in $\mathbb{P}^2$ w. $\overline{X}$. Then we have a homomorphism $\alpha\mapsto\alpha-0: \overline{X}\longrightarrow Cl^0(\overline{X})$ and one can show that it is a group isomorphism.