MATH 720, PROBLEM SET 1

1. Existence and uniqueness of moment maps

This problem investigates the questions of existence of moment maps for Lie group actions on symplectic manifolds. Let G be a connected Lie group, M be a connected manifold with a symplectic form ω. Let G act on M preserving ω. *We start with uniqueness, which is easier.*

1, 2pts) Let $\mu : M \to \mathfrak{g}^*$ be a moment map for the G-action on M. A map $\mu' : M \to \mathfrak{g}^*$ is a moment map iff $\mu' - \mu$ is a constant map taking values in $(\mathfrak{g}^*)^G$.

Now we turn to the existence. First, we need conditions for vector fields $\xi_M, \xi \in \mathfrak{g}$, to be Hamiltonian, i.e. to lie in the image of $C^\infty(M)$ under the skew-gradient map $v : C^\infty(M) \to \mathrm{Vect}(M)$.

2, 1pt) Suppose that $H^1(M, \mathbb{R}) = 0$. Then ξ_M is Hamiltonian for all $\xi \in \mathfrak{g}$.

3, 1pt) Show that $[\xi, \eta]_M = v(\omega(\xi_M, \eta_M))$ for all $\xi, \eta \in \mathfrak{g}$. Deduce that ξ_M is Hamiltonian for all $\xi \in \mathfrak{g}$ provided $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

2) and 3) give rise to $\xi \mapsto H_\xi : \mathfrak{g} \to C^\infty(M)$ but it doesn't need to be G-equivariant (equivalently, a Lie algebra homomorphism).

4, 1pt) Let $M = V$ be a symplectic vector space and $G = (V, +)$ act on M by translations. Show that all vector fields ξ_M are Hamiltonian, but the action is not Hamiltonian.

In fact, assuming G is simply connected, we can always find a central extension of G by $(\mathbb{R}, +)$ whose action on M is Hamiltonian (the copy of $(\mathbb{R}, +)$ acts trivially). Any such central extension of a semisimple group is trivial, and so every action of a semisimple group by symplectomorphisms is Hamiltonian.

2. From formal quantizations to filtered ones

Suppose that A is a $\mathbb{Z}_{\geqslant 0}$-graded Poisson algebra so that we can talk about its filtered and formal quantizations. Let $\mathcal{A}_\hbar$ be a formal quantization. By a *grading* on $\mathcal{A}_\hbar$ we mean a collection of algebra gradings on the quotients $\mathcal{A}_\hbar/\hbar^n\mathcal{A}_\hbar$ such that

- $\deg \hbar = 1$,
- the projections $\mathcal{A}_\hbar/\hbar^{n+1}\mathcal{A}_\hbar \to \mathcal{A}_\hbar/\hbar^n\mathcal{A}_\hbar$ are graded,
- $\iota : \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \xrightarrow{\sim} A$ is graded.

For $k \geqslant 0$, define $\mathcal{A}_\hbar^k := \varprojlim_n (\mathcal{A}_\hbar/\hbar^n\mathcal{A}_\hbar)^k$, where the superscript denotes the kth graded component. Set $\mathcal{A}_\hbar^{fin} := \bigoplus_k \mathcal{A}_\hbar^k$.

1, 1pt) Show that $\mathcal{A}_\hbar^{fin}$ is a $\mathbb{C}[\hbar]$-subalgebra in $\mathcal{A}_\hbar$. Equip it with an algebra grading.

2, 2pts) Show that $\mathcal{A}_\hbar^{fin}/(\hbar - 1)\mathcal{A}_\hbar^{fin}$ is a filtered quantization of A.

3, 2pts) Show that the assignments $\mathcal{A} \mapsto \hat{R}_\hbar(\mathcal{A})$ and $\mathcal{A}_\hbar \mapsto \mathcal{A}_\hbar^{fin}/(\hbar - 1)\mathcal{A}_\hbar^{fin}$ give mutually inverse bijections between the set of isomorphism classes of filtered quantizations and the set of isomorphism classes of formal quantizations with a grading (in the latter case you need to explain what one means by an isomorphism).

3. Classical and Quantum formal Darboux theorems

The classical Darboux theorem states that every point in a symplectic manifold has a neighborhood with a coordinate system, where the symplectic form is constant (Darboux coordinates). Here we investigate an algebraic analog of this theorem and its extension to quantizations.

1, 1pt) Let A be a Poisson algebra and $\mathfrak{m}$ be its maximal ideal with $A/\mathfrak{m} = \mathbb{C}$. Show that the Poisson bracket on A induces a skew-symmetric form on $\mathfrak{m}/\mathfrak{m}^2$.

2, 2pts) Suppose that $A = \mathbb{C}[[x_1, \ldots, x_{2n}]]$ (so that there's the unique maximal ideal $\mathfrak{m}$). Further, suppose the form on $\mathfrak{m}/\mathfrak{m}^2$ is nondegenerate. Prove that there are elements $x'_i \in \mathfrak{m}$ such that

- the elements $x'_i + \mathfrak{m}^2, i = 1, \ldots, 2n$, form a basis in $\mathfrak{m}/\mathfrak{m}^2$,
- and $\{x'_i, x'_j\} \in \mathbb{C}$ for all i, j.

In other word, after a change of coordinates, the Poisson bivector on A becomes constant. A hint for a solution: lift x_i's order by order.

3, 2pts) Now let A be as in part 2), and $\mathcal{A}_\hbar$ be its formal (=deformation) quantization (in the sense of the original definition in Lecture 3). Show that there are elements $\hat{x}'_i \in \mathcal{A}_\hbar$ such that

- $\hat{x}'_i + \hbar\mathcal{A}_\hbar = x'_i$,
- and $\frac{1}{\hbar}[\hat{x}'_i, \hat{x}'_j] \in \mathbb{C}$ for all i, j.

In other words, A has only one quantization up to isomorphism and it is the formal version of the Weyl algebra.