MATH 720, PROBLEM SET 2

Problem 1, 4pts

Prove that the universal cover mentioned in Example of Section 3 in Lecture 6 is indeed $\mathbb{C}^n \setminus \{0\}$ with its natural action of Sp_n, and the covering map $\mathbb{C}^n \setminus \{0\} \to \mathbb{O}$ is the moment map (viewed as a map to its image).

Problem 2, 5pts

Recall the resolution $G \times^P \mathfrak{g}_{\geqslant 2} \to \overline{\mathbb{O}}$ from Section 3 in Lecture 8. Prove that the extension of the Kirillov-Kostant form to this resolution (that appeared in the proof the main theorem in that section) is symplectic if and only if $e \in \mathbb{O}$ is *even* meaning that $\mathfrak{g}_i = \{0\}$ for odd i.

Problem 3

This problem introduces another class of conical symplectic singularities – the symplectic quotient singularities. Let V be a finite dimensional symplectic vector space and $\Gamma \subset \mathrm{Sp}(V)$ is a finite subgroup. Consider the variety $X = V/\Gamma$.

1, 1pt) Show that $\mathbb{C}[V]^\Gamma$ is a graded Poisson subalgebra of $\mathbb{C}[V]$. Conclude that $\mathbb{C}[V]^\Gamma$ is a positively graded Poisson algebra.

The rest of the problem is devoted to checking condition 1 in the definition of a singular symplectic variety.

2, 1pt) Show that Γ contains no complex reflections.

3, 2pts) Deduce that $(V/\Gamma)^{reg}$ consists exactly of the points in V/Γ corresponding to the free Γ-orbits. Denote this locus in V by V^0 so that the quotient morphism $\pi_\Gamma : V \to V/\Gamma$ restricts to a finite etale morphism $V^0 \to (V/\Gamma)^{reg}$.

4, 2pts) Show that the Poisson structure on $(V/\Gamma)^{reg}$ is given by the unique symplectic form ω^{reg} such that $\pi_\Gamma^* \omega^{reg}$ is the restriction to V^0 of the initial form on V.

Condition 2) in the definition of singular symplectic varieties is verified in Beauville's paper.

1. Aside

One can ask when the variety V/Γ from the previous problem has a symplectic resolution. Here are some special cases when this happens.

1) $\dim V = 2$ so that Γ can be viewed as a finite subgroup of SL_2. Such subgroups up to conjugation are parameterized by Dynkin diagrams via the McKay correspondence. As any surface, V/Γ has the unique minimal resolution of singularities. This resolution is actually a symplectic resolution. The varieties V/Γ and their minimal resolutions play an important role in what we going to do in this course (and, more generally, in the Geometric representation theory). Later in the course we will cover them (including their Lie theoretic construction) in more detail.

2) $V = (\mathbb{C}^2)^n$ with symplectic form that is the direct sum of n copies of the symplectic form on $\mathbb{C}^2$, and $\Gamma = S_n$ permuting the n copies. The variety V/Γ can be interpreted as

parameterizing the unordered n-tuples of points in $\mathbb{C}^2$. Its symplectic resolution is given by the *Hilbert scheme* $\mathrm{Hilb}_n(\mathbb{C}^2)$ parameterizing the codimension n ideals in $\mathbb{C}[x, y]$. This variety is smooth and symplectic. It admits a morphism to V/Γ by sending an ideal I to its support counted with multiplicities. This morphism is a resolution of singularities, moreover, it is a morphism of Poisson varieties.

3) Next, there is a class of examples generalizing both 1) and 2): we take $V = (\mathbb{C}^2)^n$ (with the same form as in 2)), and $\Gamma_n = S_n \ltimes \Gamma_1^n$ for a finite subgroup $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$. Here S_n permutes the n copies of $\mathbb{C}^2$, and the n copies of Γ_1 act on the corresponding copies of $\mathbb{C}^2$. Then V/Γ_n has a symplectic resolution (one can take, for example, the Hilbert scheme of n points on the minimal resolution of $\mathbb{C}^2/\Gamma_1$).

In fact, all resolutions above can be obtained as Nakajima quiver varieties. And there are just two known exceptional examples, both in dimension 4. Conjecturally, there is nothing else.