MATH 720, PROBLEM SET 3

Problem 1, 5pts

Prove iv) of Proposition in Section 2.1 of Lecture 10 for $\mathfrak{g} = \mathfrak{sl}_n$ by considering the restriction of π_G to the $(n-1)$-dimensional affine subspace in $\mathfrak{g}$ consisting of all matrices (a_{ij}) with $a_{i,i+1} = 1$ for $i = 1, \ldots, n-1$, and $a_{ij} = 0$ unless $j = 1, i > 1$ or $j = i+1$. For example, for $n = 4$ this locus looks like

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \\ a_4 & 0 & 0 & 0 \end{pmatrix}$$

Hints: what is the characteristic polynomial of this matrix? And you are supposed to use Commutative algebra results mentioned in the proof.

Problem 2, 6pts

This problems concerns the properties of being $\mathbb{Q}$-factorial and terminal for symplectic quotient singularities. Let V be a finite dimensional symplectic vector space, and $\Gamma \subset \mathrm{Sp}(V)$ be a finite subgroup. Set $X := V/\Gamma$.

1, 2pts) By a *symplectic reflection* in Γ we mean an element γ with $\mathrm{rk}(\gamma - \mathrm{id}_V) = 2$. Show that V/Γ is terminal if and only if there are no symplectic reflections in Γ.

2, 1pt) *In the next four parts we'll see that V/Γ is always $\mathbb{Q}$-factorial.* Let V^0 denote the locus in V consisting of all points with trivial stabilizer in Γ. Show that $\mathrm{Pic}(V^0) = \{0\}$.

3, 1pt) Show that $\mathfrak{X}(\Gamma) \xrightarrow{\sim} \mathrm{Pic}^\Gamma(V^0)$ via $U \mapsto U \otimes \mathcal{O}_{V^0}$.

4, 1pt) Let $\pi : V^0 \to (V/\Gamma)^{reg}$ be the restriction of the quotient morphism. Show that π^* and $(\pi_*(\bullet))^\Gamma$ define mutually inverse bijections between $\mathrm{Pic}^\Gamma(V^0)$ and $\mathrm{Pic}((V/\Gamma)^{reg})$.

5, 1pt) Conclude that V/Γ is $\mathbb{Q}$-factorial.

Problem 3, 4pts

Consider the group $G := \mathrm{Sp}_4$ and the nilpotent orbit $\mathbb{O}$ corresponding to the partition $(2,2)$. The fundamental group is $\mathbb{Z}/2\mathbb{Z}$. Let $\tilde{\mathbb{O}}$ be the two-fold equivariant cover of $\mathbb{O}$. Now let P_1, P_2 be two parabolic subgroup of G: P_1 is the stabilizer of a line in $\mathbb{C}^4$, while P_2 is the stabilizer of a lagrangian subspace. Show that the open G-orbit in $T^*(G/P_1)$ is isomorphic to $\tilde{\mathbb{O}}$, while the open G-orbit in $T^*(G/P_2)$ is isomorphic to $\mathbb{O}$. *Hint: look at the moment maps for the G-actions on $T^*(G/P_i)$.*