MATH 720, PROBLEM SET 4, DUE NOV 17

Unlike the previous homeworks, this one concentrates on proving some statements in Lecture 17 left as exercises. Note that you can use previous parts to prove subsequent ones even if you didn't solve the former.

Problem 1, 8pts

This problem provides missing proofs for 3) of Lemma in Section 1.2 showing that $X_{\mathbb{C}\lambda} \to \overline{\mathbb{O}} \times \mathbb{C}$ lifts to an isomorphism $X_{\mathbb{C}\lambda} \xrightarrow{\sim} X \times \mathbb{C}$. Let $e \in \mathbb{O}$ and let τ denote the finite morphism $X_{\mathbb{C}\lambda} \to \overline{\mathbb{O}} \times \mathbb{C}$.

1, 3pts) Show that $\tau^{-1}(\{e\} \times \mathbb{C})$ is the disjoint union of affine lines each mapping isomorphically to $\mathbb{C}\lambda$ (hint: use that the locus where τ is etale is open and $G \times \mathbb{C}^\times$-stable; moreover, use that the scheme-theoretic fiber of 0 under $X_{\mathbb{C}\lambda} \to \mathbb{C}\lambda$ is the reduced scheme X to show that the locus of etality intersects the zero fiber).

2, 3pts) Pick one of these affine lines, say ℓ. Show that the action morphism $G \times \ell \to X_{\mathbb{C}\lambda}, (g, x) \mapsto gx$, descends to a G-equivariant open embedding $\widetilde{\mathbb{O}} \times \ell \hookrightarrow X_{\mathbb{C}\lambda}$, hence giving a G-equivariant morphism $X \times \ell \to X_{\mathbb{C}\lambda}$.

3, 2pts) Show that the morphism $X \times \ell \to X_{\mathbb{C}\lambda}$ is $\mathbb{C}^\times$-equivariant. Deduce that it is an isomorphism (hint: graded Nakayama).

Problem 2, 7pts

This problem establishes the claim in the end of Section 1.2. Let A be a finitely generated positively graded commutative algebra. Let A' be a $\mathbb{Z}_{\geqslant 0}$-filtered commutative algebra with an isomorphism $A \cong \operatorname{gr} A'$.

1, 1pt) Show that if A is reduced (= has no nonzero nilpotent elements), then so is A'.

2, 2pts) Show that if A is CM, then so is A'. *Hint: the main ingredient is to show that A admits a maximal regular sequence of homogeneous elements and that any lift of this sequence to A' is still regular.*

3, 2pts) Show that $\operatorname{codim}_{X_\lambda} X_\lambda \setminus \widetilde{\mathbb{O}}_\lambda \geqslant 2$.

4, 1pt) Use the fact (Thm 18.15 in Eisenbud) that if a finite type Cohen-Macaulay scheme is regular outside of codimension 2, then it is normal. Then deduce that X_λ is normal.

5, 1pt) Complete the proof of the isomorphism $\mathbb{C}[X_\lambda] \xrightarrow{\sim} \mathbb{C}[\widetilde{\mathbb{O}}_\lambda]$.