Lecture 1 (Pavel)

1) Affine Lie algebras & their finite dim'l reps
2) Intro to quantum groups

1) $\mathfrak{g}$ fin. dim. simple Lie alg.

$\hat{\mathfrak{g}} := \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$, $[a(t), b(t)] = [a,b](t) + \mathrm{Res}_{t=0}(a(t), b(t))\frac{dt}{t}K$

Q: what are fin. dim reps of $\hat{\mathfrak{g}}$?

Lem: $K=0$ on every fin. dim. rep'n

Proof: $\langle e_i, h_i, f_i \rangle \subset \hat{\mathfrak{g}}$ root $\mathfrak{sl}_2$-triple, $i=0,\dots,\mathrm{rk}\,\mathfrak{g}$

$K = \sum k_i h_i$; $[e_i, f_i] = h_i \Rightarrow tr_V h_i = 0 \Rightarrow tr_V K = 0$

But K is sum of commuting s/simple el'ts $\Rightarrow K=0$. $\square$

So we reduce to studying fin. dim. reps of $L\mathfrak{g} = \mathfrak{g}[t^{\pm 1}]$.

$z \in \mathbb{C}^\times \rightsquigarrow ev_z : L\mathfrak{g} \to \mathfrak{g}$, $a(t) \mapsto a(z)$, it's surjective

$V \in \mathrm{Rep}\,\mathfrak{g} \rightsquigarrow V(z) = ev_z^*(V)$.

For $\lambda \in \Lambda^+$ let V_λ denote $\mathfrak{g}$-irrep. w. highest wt λ

$\rightsquigarrow$ irreps $V_\lambda(z)$ of $L\mathfrak{g}$.

Prop 1: $V_{\lambda_1}(z_1) \otimes \ldots \otimes V_{\lambda_n}(z_n)$ w. $\lambda_i \neq 0$ is irrep

iff z_i are pairwise distinct.

Proof: $\Rightarrow$: need to prove: $X, Y \neq \mathbb{C} \Rightarrow X \otimes Y$ is reducible

Observe: $\mathbb{C}, \mathfrak{g}$ are direct summands in $X \otimes X^*, Y \otimes Y^*$ so

$\dim \mathrm{Hom}_{\mathfrak{g}}(X \otimes Y, X \otimes Y) = \dim \mathrm{Hom}_{\mathfrak{g}}(X \otimes X^*, Y \otimes Y^*) \geq 2$

$\Leftarrow$: exercise (hint: $L\mathfrak{g} \twoheadrightarrow \mathfrak{g}^{\oplus k}$ via $(ev_{z_1}, \ldots ev_{z_n})$) $\square$

Question: which of tensor products in Prop 1 are isomorphic?

Prop 2: These $\otimes$-products are pairwise non-iso.

Proof: $h \in \mathfrak{h} \subset \mathfrak{g}$, $h_+(z) := -\sum_{n=0}^{\infty} (h \otimes t^{-n-1}) z^n$

Apply $h_+(z)$ to $v := v_{\lambda_1} \otimes \dots \otimes v_{\lambda_n} \in V_{\lambda_1}(z_1) \otimes \dots \otimes V_{\lambda_n}(z_n)$,

unique up to scaling vector of wt $\lambda_1 + \dots + \lambda_n$ for $\mathfrak{g} \subset L\mathfrak{g}$

$h_+(z) v = \sum_{k,n} -\lambda_k(h) \left(\frac{z}{z_k}\right)^n = \sum_k \frac{\lambda_k(h)}{z - z_k}$

has poles z_k & residues $-\lambda_k(h)$.

$n_{ik} := \lambda_k(h_i) \in \mathbb{Z}_{\geq 0}$ so $h_{i+}(z) v = \left(\sum_k \frac{n_{ik}}{z - z_k}\right) v = \frac{P_i'(z)}{P_i(z)} v$, where

$P_i(z) := \prod_k (z - z)^{n_{ik}}$

So the action of $\mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}]$ is encoded by $P_1, \dots P_r$, which yields the proof. $\square$

Prop 3: Every fin. dim $L\mathfrak{g}$-irrep is isomorphic to some $V_{\lambda_1}(z_1) \otimes \dots \otimes V_{\lambda_n}(z_n)$.

Pf: Let V be fin. dim. rep

$\varphi : \mathbb{C}[t^{\pm 1}] \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathrm{End}(V))$

$\varphi(g)(a) := \pi_V(a \otimes g)$

Claim: $I = \ker \varphi$ is an ideal.

Pf of Claim: $a,b\in\mathfrak{g}$, $q\in I$, $p\in\mathbb{C}[t^{\pm 1}]$

$\pi_V([a,b]\otimes pq)=[\pi_V(ap),\pi_V(bq)]=0$ of Claim

Since $\mathrm{Span}_{\mathbb{C}}([\mathfrak{g},\mathfrak{g}])=\mathfrak{g}\Rightarrow \pi_V(cpq)=0\ \forall c\in\mathfrak{g}\Rightarrow pq\in I$. $\square$

So $I=(q)$, w. $q=\prod_{i=1}^{d}(t-t_i)^{n_i}$ so

$\mathfrak{g}[t,t^{-1}]\longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ factors through $\mathfrak{g}\otimes(\mathbb{C}[t^{\pm 1}]/(q))=:\mathfrak{a}$

$\Rightarrow \mathfrak{a}=\mathfrak{a}_{ss}\ltimes \mathrm{Rad}(\mathfrak{a})$, where $\mathfrak{a}_{ss}\simeq\bigoplus_{i=1}^{d}\mathfrak{g}$

Since $\mathrm{Rad}(\mathfrak{a})$ is nilpotent, it acts by 0 on $\forall$ irrep

$\Rightarrow V$ has to be $\otimes$ of irreps of simple summands of $\mathfrak{a}_{ss}\Rightarrow$

$V\simeq V_{\lambda_1}(z_1)\otimes\ldots\otimes V_{\lambda_k}(z_k)$ for some $\lambda_1,\ldots,\lambda_k$. $\square$

Rem: • Direct analog of the classification of irreps holds for $\mathfrak{g}\otimes_{\mathbb{C}}A$ for any fin. gen'd commut. $\mathbb{C}$-algebra A.

• $\otimes$ simples is s/simple.

• Indecomposable reps are of $L\mathfrak{g}$ are still interesting.

2) Intro to quantum groups

Presentation of Kac-Moody Lie algebras:

$a_{ij} \in \mathbb{Z}$ s.t. $a_{ii} = 2$, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ &

$a_{ij} \leq 0$ for $i \neq j$.

Assume: $\exists\, d_i$ s.t. $d_i a_{ij} = d_j a_{ji}$ (symmetrizable KM) & fix them.

Generators: h_i, e_i, f_i

$[h_i, h_j] = 0$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$

$[e_i, f_j] = \delta_{ij} h_i$

Serre rel'ns: $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0$

Define $\tilde{\mathfrak{g}}(A)$ by same w/o Serre rel'ns.

$\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$

free in generators e_i for $\tilde{\mathfrak{n}}_+$ & f_i for $\tilde{\mathfrak{n}}_-$

$\exists!\, I \subset \tilde{\mathfrak{g}}(A)$ largest graded ideal w. $I \cap \mathfrak{h} = \{0\}$

$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is identified w. $\tilde{\mathfrak{g}}(A)/I$ (Gabber-Kac thm).

q-deformation: take $q \in \mathbb{C}^\times$ not a root of 1 (or work over $\mathbb{C}(q)$)

Formally set $q_i = q^{d_i}$ & $K_i = q_i^{"h_i"}$

$U_q(\mathfrak{g}(A)) = \langle K_i^{\pm 1}, e_i, f_i \rangle / \text{rel'ns:}$

Rel'n: $[K_i, K_j] = 0$, $K_i e_j K_i^{-1} = q_i^{a_{ij}} e_j$

$$K_i f_j K_i^{-1} = q_i^{-a_{ij}} f_j$$

$$[e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$(ad_{q_i} e_i)^{1-a_{ij}} e_j = 0$ – q-Serre relation w. $(ad_q x)(y) = xy - qyx$.

Same recipe as before allows to bypass Serre rel'ns:

$U_q(\tilde{\mathfrak{g}}(A)) = U_q(\tilde{n}_+) \otimes U_q(\mathfrak{h}) \otimes U_q(\tilde{n}_-)$

Mod out similar ideal to I to get $U_q(\mathfrak{g}(A))$.

Prop. $U_q(\mathfrak{g}(A))$ is Hopf algebra: $\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i$

$$\Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i$$

$$\Delta(K_i) = K_i \otimes K_i$$

Antipode $S(e_i) = -e_i K_i^{-1}$, $S(f_i) = -K_i f_i$, $S(K_i) = K_i^{-1}$.

Important observation:

$U_q(\mathfrak{g}(A))$ is almost Drinfeld double of $U_q(\mathfrak{b}_+)$.

Recall: quantum double. Let H be (fin. dim.) Hopf algebra

Then $D(H) = H \otimes H^{*,cop}$ (opposite coproduct) as coalgebra

multiplication: $H, H^{*,cop}$ are subalgebras

commutation law: $b \in H^{*,cop}, a \in H \rightsquigarrow ba$

$\Delta_3 a = a_1 \otimes a_2 \otimes a_3, \quad \Delta_3 b = b_1 \otimes b_2 \otimes b_3$

$ba := (S^{-1}(a_1), b_1)(a_3, b_3)\, a_2 b_2$

Point: $\mathrm{Rep}(D(H))$ is braided

Def: If $\mathcal{C}$ is monoidal category, then its Drinfeld center $Z(\mathcal{C})$ is category w. objects (X, φ_X) w $X \in \mathcal{C}$,

$\varphi_X : X \otimes \bullet \xrightarrow{\sim} \bullet \otimes X$

s.t.

$$\begin{tikzcd} X \otimes M \otimes N \arrow[d, "\varphi_{X,M} \otimes 1"'] \arrow[dr, "\varphi_{X, M\otimes N}"] & \\ M \otimes X \otimes N \arrow[r, "1_M \otimes \varphi_{X N}"'] & M \otimes N \otimes X \end{tikzcd}$$

$\leftarrow$ hexagon rel'n.
+ the other way around.

Then $Z(\mathcal{C})$ is a monoidal category, in fact, braided

Thm (Drinfeld): $\mathcal{Z}(\mathrm{Rep}\,H) \xrightarrow{\sim} \mathrm{Rep}\,\mathcal{D}(H)$, where braiding on $\mathrm{Rep}\,\mathcal{D}(H)$ is defined using universal R-matrix
$\sum_i a_i \otimes a^i$, where a_i is basis of H & a^i is dual basis

$$c_{X,Y} (= \varphi_{X,Y}) = \underset{\substack{\uparrow \\ \text{permutation}}}{P} \circ R|_{X\otimes Y} : X \otimes Y \to Y \otimes X.$$

Important **properties**: • $R\Delta(x) = \Delta^{op}(x) R \quad \forall x \in \mathcal{D}(H)$.
• Hexagon rel'ns $\Rightarrow (\Delta \otimes 1)(R) = R_{13} R_{23}$, $(1 \otimes \Delta)(R) = R_{13} R_{12}$

The Drinfeld double construction can be carried to some inf. dim. cases but now $R \in \mathcal{D}(H) \hat{\otimes} \mathcal{D}(H)$

Example ($U_q(\mathfrak{sl}_2)$ as almost Drinfeld double)
$H := U_q(\mathfrak{b}_+) = \langle K^{\pm 1}, e \rangle$
$KeK^{-1} = q^2 e$, $\Delta(K)$, $\Delta(e)$ as before
Take restricted dual $H^* = U_q(\mathfrak{b}_-) = \langle \tilde{K}, f \rangle$
$\tilde{K} f \tilde{K}^{-1} = q^{-2} f$, $\Delta(\tilde{K}) = \tilde{K} \otimes \tilde{K}$, $\Delta(f) = f \otimes 1 + \tilde{K}^{-1} \otimes f$.

$D(H) = H \otimes H^{*,\mathrm{cop}} = \langle e, f, K, \tilde{K} \rangle$

but $C := \tilde{K} K^{-1}$ is central $\rightsquigarrow$ $\bar{D}(H) = D(H)/(C-1) \simeq U_q(\mathfrak{sl}_2)$

Drinfeld commut'n rel'n: $[e,f] = \frac{K - K^{-1}}{q - q^{-1}}$

Get R-matrix for free.

$$R = q^{h \otimes h/2} \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(q-q^{-1})^k}{[k]_q!} e^k \otimes f^k.$$

Rem: R gives braiding on cat. $\mathcal{O}$ of $U_q(\mathfrak{sl}_2)$-reps.

The Drinfeld double construction extends to all KM algebras (starting w. $U_q(\mathfrak{b}_+)$)