Lazy approach to categories $\mathcal{O}$, V.

1) Something old.

2) Something new.

1) Something old.

A concrete goal of this lecture is to establish a derived equivalence between certain blocks of affine & quantum categories $\mathcal{O}$.

1.1) Affine category $\mathcal{O}$

Recall that G is a simple alg'c group. Let $(\cdot,\cdot)\in S^2(\mathfrak{g}^*)^G$ be normalized so that $(\alpha^\vee,\alpha^\vee)=2$ $\forall$ short coroot α.

Form the affine Lie algebra $\hat{\mathfrak{g}}$ using the form $(\cdot,\cdot)$. Let $c\in\hat{\mathfrak{g}}$ denote the unit element in the central ideal. Let $\mathfrak{h}^a=\mathfrak{h}\oplus\mathbb{C}c$ be the affine Cartan.

Pick $\tilde{\lambda}\in\mathfrak{h}^{a,*}$. We write κ for $\langle\tilde{\lambda},c\rangle+h^\vee$, where $h^\vee$ is the dual Coxeter number, so that $\kappa=0$ corresponds to the critical level. We can define the category $\mathcal{O}^a_{\tilde{\nu}}$ similarly to Lec 1: as a full subcategory in the category of Λ^a-graded $\hat{\mathfrak{g}}$-modules, where Λ^a is the affine root lattice.

For $\tilde{\lambda} \in \Lambda^a$ we have the Verma module $\Delta^a(\tilde{\lambda})$ and its simple quotient $L^a(\tilde{\lambda})$.

Rem: Here we already see a bit of (important) difference from both the usual & quantum categories $\mathcal{O}$: the labelling set for simples is larger. We also note that $\Lambda^a \not\hookrightarrow \mathfrak{h}^{a,*}$ instead $\Lambda^a \hookrightarrow \mathfrak{h}^{ea,*} \twoheadrightarrow \mathfrak{h}^{a,*}$ for the extended affine Cartan $\mathfrak{h}^{ea} = \mathfrak{h}^a \oplus \mathbb{C}d$. We can view $\tilde{\nu}$ as an element of $\mathfrak{h}^{ea,*}$ by specifying, in fact, any pairing w. d.

Now set $R := \mathbb{C}[[\mathfrak{h}^{a,*}]]$. We can consider the deformed version $\mathcal{O}^a_{\nu,R}$ similarly to the above. The categories $\mathcal{O}^a_{\tilde{\nu}}$ & $\mathcal{O}^a_{\tilde{\nu},R}$ are highest weight (over $\mathbb{C}$ & over R, respectively) with poset Λ^a that is interval finite (see Sec 2 of part IV for the discussion of such highest weight categories).

1.2) Subgeneric behavior.

Choose $\rho^a \in \mathfrak{h}^{rea,*}$ w. $\langle \rho^a, \alpha^\vee \rangle = 1$ for all simple coroots $\alpha^\vee$ (so that $\langle \rho^a, c \rangle = h^\vee$)

Def:

- A real affine root β is integral for $\tilde{\nu}$ if $\langle \tilde{\nu}, \beta^\vee \rangle \in \mathbb{Z}$ ($\Leftrightarrow \langle \tilde{\nu} + \rho^a, \beta^\vee \rangle \in \mathbb{Z}$).
- We say that δ is integral for $\tilde{\nu}$ if $\kappa (= \langle \tilde{\nu} + \rho^a, c \rangle) = 0$.

The following is known but nontrivial (cf. the discussion in Sec 3.2 below).

Fact: 1) $\mathcal{O}^a_{\tilde{\nu}}$ is s/simple $\Leftarrow$ there are no integral roots.

2) If the only integral roots for $\tilde{\nu}$ are a pair of opposite <u>real</u> roots, then $\mathcal{O}^a_{\tilde{\nu}}$ splits as the direct sum of blocks of category $\mathcal{O}$ for $\mathfrak{sl}_2$.

We will explain below (in the "New" part) what happens when the only integral root is δ.

1.3) Whittaker coinvariants

We have the triangular decomposition $\hat{\mathfrak{g}} = \mathfrak{n}_-^a \oplus \mathfrak{h}^a \oplus \mathfrak{n}^a$ so can consider the Whittaker coinvariant functor. Let $\tilde{\psi}: \mathfrak{n}_-^a \to \mathbb{C}$ be a character that is nonzero on the root space $\hat{\mathfrak{g}}_{-\alpha}$ $\forall$ simple roots α. Set $Wh := \mathbb{C}_{\tilde{\psi}} \otimes_{\mathcal{U}(\mathfrak{n}_-^a)} ?: \hat{\mathfrak{g}}\text{-mod} \to \mathbb{C}$. This gives rise to functors $\mathcal{O}^a_{\tilde{\nu}} \to Vect$, $\mathcal{O}^a_{\tilde{\nu},R} \to R\text{-mod}$.

Semi-conjecture: $Wh: \mathcal{O}^a_{\tilde{\nu}} \to Vect$ is faithful on standardly filtered objects if $\kappa \neq 0$ (non-critical level).

As a consequence, $Wh: \mathcal{O}^a_{\tilde{\nu},R} \to R\text{-mod}$ is fully faithful on standardly filtered objects (for $\kappa \neq 0$). We'll discuss what happens at the critical level in Sec 2 (spoiler: big center happens).

Rem: Let's discuss the status of the "semi-conjecture". I don't know an algebraic proof (and would like to have one hoping it will also work in the quantum affine setting). A rough idea of proof in the negative level setting ($\kappa \notin \mathbb{Q}_{\geq 0}$) is: use the Kashiwara-Tanisaki localization to relate blocks of $\mathcal{O}^a_{\tilde{\nu}}$ to Hecke category (perverse sheaves on finite or thin affine flag variety) and use an analogous statement for that category, which is known. I'll skip the details here.

1.4) Blocks and highest weight order.

The affine Weyl group $W^a := W \ltimes \Lambda^\vee$ (the coroot lattice) acts on $\mathfrak{h}^{rea}$ and hence on $\mathfrak{h}^{rea,*}$: for a real root β, the reflection $s_{\beta^\vee}$ acts by $x \mapsto x - \langle \beta^\vee, x \rangle \beta$ ($x \in \mathfrak{h}^{rea,*}$).

Exercise: Write down how $wt_{\lambda^\vee}$ ($w \in W$, $\lambda^\vee \in \Lambda^\vee$) acts.

Assume for the rest of the section that $\kappa \neq 0$. Embed Λ^a into $\mathfrak{h}^{rea,*}$ via $\tilde{\lambda} \mapsto \tilde{\lambda} + \rho^a + \tilde{\nu}$. The image is stable under $W_{[\tilde{\nu}]}$ giving an action of $W_{[\tilde{\nu}]}$ on Λ^a. The following is a theorem of Kac.

Thm: The blocks in $\mathcal{O}^a_{\tilde{\nu}}$ correspond to $W_{[\tilde{\nu}]}$-orbits in Λ^a: for such an orbit Ξ we set $\mathcal{O}^a_{\tilde{\nu},\Xi}$ = Serre span of $\Delta^a_{\tilde{\nu}}(\tilde{\lambda})$, then $\mathcal{O}^a_{\tilde{\nu}} = \bigoplus_{\Xi} \mathcal{O}^a_{\tilde{\nu},\Xi}$.

We note that the highest wt. order on Ξ is generated by $s_{\beta^\vee}\cdot\tilde{\lambda} > \tilde{\lambda}$ if the difference is of the form $m(\alpha+n\delta)$ (w. $m>0$, and $n>0$ or $\alpha>0$). An interesting case is when $\kappa\in\mathbb{Q}$ & $\exists$ an integral real root so that $W_{[\tilde{\nu}]}$ is an affine Weyl group. One can deduce the following from Exercise above:

1) If $\kappa\in\mathbb{Q}_{<0}$, Ξ contains a unique integrally anti-dominant element $\tilde{\lambda}_-$. Identifying $W_{[\tilde{\nu}]}/W^\circ$ (where $W^\circ = \mathrm{Stab}_{W_{[\tilde{\nu}]}}\tilde{\lambda}_-$) w. Ξ via $wW^\circ \mapsto w\cdot\tilde{\lambda}_-$ we get the usual Bruhat order on $W_{[\tilde{\nu}]}/W^\circ$.

2) If $\kappa\in\mathbb{Q}_{>0}$, Ξ contains a unique integrally dominant element $\tilde{\lambda}_+$. Identifying $W_{[\tilde{\nu}]}/W^\circ$ (where $W^\circ = \mathrm{Stab}_{W_{[\tilde{\nu}]}}\tilde{\lambda}_+$) w. Ξ via $wW^\circ \mapsto w\cdot\tilde{\lambda}_+$ we get the opposite Bruhat order on $W_{[\tilde{\nu}]}/W^\circ$.

Rem: note that both orders are different from the order for quantum category $\mathcal{O}$ (cf. Rem. in Sec 4 of Part IV): that order was invariant under $x \mapsto t_\lambda x$, while neither the positive nor the negative level orders are. So there's no highest weight equivalence between these two categories!

2) Something new.

2.0) Discussion.

It is easy to recover the "quantum" order on $\Xi = W_{[\lambda]}/W^\circ$ in the affine setup. Assume $\kappa \in \mathbb{Q}_{<0}$. Set $\Delta^a_{\infty/2} = \{\alpha + n\delta \mid \alpha > 0 \text{ or } \alpha = 0 \ \& \ n > 0\}$. This is a system of positive roots in the affine root system Δ^a. We can consider the order $>^{\infty/2}$ on Ξ given by $s_{\beta^\vee} \cdot \tilde{\lambda} > \tilde{\lambda}$ if $\langle \tilde{\lambda}, \beta^\vee \rangle \beta$ is a positive multiple of a root in $\Delta^a_{\infty/2}$. This is what we have in the quantum case.

Morally, a choice of a positive root system in Δ^a should give rise to a highest weight t-structure on (suitably completed) $D^6(\mathcal{O}^a_{\tilde{\lambda}, \Xi})$. For example:

0) The usual positive root system Δ_+^a gives the usual t-structure (& standards = Vermas).

1) The affine Hecke category $D^b_{cons}(I_{\tilde{\lambda}}\backslash \hat{G}/_{\tilde{\lambda}} I)$ acts on $D^b(\mathcal{O}^a_{\tilde{\lambda},\Xi})$, this yields the action (denoted by $*$) of the braid group $Br_{W[\tilde{\lambda}]}$ on $D^b(\mathcal{O}^a_{\tilde{\lambda},\Xi})$ by equivalences. The twist of the usual t-structure by T_x corresponds to $x(\Delta_+^a)$.

2) $-\Delta_+^a$ gives rise to the Ringel dual t-structure on the (ind-completion) of $D^b(\mathcal{O}^a_{\tilde{\lambda},\Xi})$. This ind-completion is equivalent to a suitable version of the derived category for the positive level category and this equivalence is t-exact (for the default positive level t-structure).

3) $\Delta^a_{\infty/2}$ gives rise to a t-structure on $D^b(\mathcal{O}^a_{\tilde{\lambda},\Xi})$ called "Frenkel-Gaitsgory"/"new"/"stabilized"/"semi-infinite" t-structure. The standards are the Wakimoto modules.

2.1) New t-structure.

Assume for simplicity that $\tilde{\lambda}$ is integral $\Rightarrow$ the relevant braid group is Br^a. Then $\Lambda^\vee \hookrightarrow Br^a$ lifting $\lambda \mapsto t_\lambda$. Let

$J_\lambda \in Br^a$ denote the image of λ.

Facts (Frenkel-Gaitsgory / Bezrukavnikov-Lin):

1) $\exists$ (autom. unique) "new" t-structure on $D^b(\mathcal{O}^a_{\tilde{\nu},\Xi})$ s.t.

$$D^b(\mathcal{O}^a_{\tilde{\nu},\Xi})^{\leq 0,\, new} = \{\mathcal{F} \in D^b(\mathcal{O}^a_{\tilde{\nu},\Xi}) \mid J_\lambda * \mathcal{F} \in D^b(\mathcal{O}^a_{\tilde{\nu},\Xi})^{\leq 0},\ \forall \lambda\}$$

for usual t-structure

2) $D^b(\mathcal{O}^a_{\tilde{\nu},\Xi}) = D^b(\text{heart of new t-structure})$.

Note that, with the usual normalizations, J_λ is costandard for λ dominant ($\Rightarrow J_\lambda * ?$ is right t-exact) & standard for λ anti-dominant. One consequence is that $J_\lambda * \Delta(t_{-\lambda}x)$ is indep. of λ for all λ sufficiently dominant (how dominant, depends on x). These "stabilized" objects are denoted by $\Delta^{st}(x)$. They lie in $\mathcal{O}^a_{\tilde{\nu},\Xi}$.

More facts: 1-Arkhipov: these are the Wakimoto modules (as defined via the free field realization)

2- I.L.: the heart of the new t-structure is highest weight w. standards $\Delta^{st}(x)$. And the heart deforms over R.

Using this (and techniques from Lec 3) one can now establish a highest wt equiv. between quantum cat. $\mathcal{O}$ & affine cat. $\mathcal{O}$ w. new t-structure (more precisely between the blocks w. matching combinatorics).

2.2) And what about the critical level?

Our goal here is to understand the structure of $\mathcal{O}^{a}_{\tilde{\nu}}$ in the case when there are no integral real roots (what we want is a generic parameter at the critical level or a parameter near such).

We have the Wakimotization functor

$\mathrm{Wak}: \mathcal{O}^{a}_{\tilde{\nu}'}(\hat{\mathfrak{h}}) \longrightarrow \mathcal{O}^{a}_{\tilde{\nu}}(\hat{\mathfrak{g}})$,

where $\tilde{\nu}'$ is obtained from $\tilde{\nu}$ by a shift independent of $\tilde{\nu}$ (the only important thing about this shift is that the critical level corresponds to the critical level)

This functor can shown to be an equivalence when there are no integral real roots, cf. Remark in Sec 3.2 of Part IV.

When $\kappa=0$, $\mathcal{O}^{a}_{\tilde{\nu}'}(\hat{\mathfrak{h}})$ is essentially the category of graded modules (w. grading bounded from above) over the polynomial algebra in infinitely many variables. Note that such a category cannot admit a functor to Vect faithful on standard objects.