Lecture 2 (Pavel)

1) Reps of $U_q(\hat{\mathfrak{g}})$

Def'n in Sec 1.5 was corrected. Thx to Frank for catching a mistake!

1.1) Algebra $U_q(\hat{\mathfrak{sl}}_2)$

$q \in \mathbb{C}^\times$, $q \neq \sqrt[\cdot]{1}$

Take $\mathfrak{g} = \mathfrak{sl}_2$: Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

generators: $e_i, f_i, K_i^{\pm 1}$, $i = 0, 1$

$K_i e_i K_i^{-1} = q^2 e_i$, $K_i f_i K_i^{-1} = q^{-2} f_i$

$K_i e_j K_i^{-1} = q^{-2} e_j$, $K_i f_j K_i^{-1} = q^2 f_j$ $j \neq i$.

$K_i K_j = K_j K_i$

$[e_i, f_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}}$, $[e_i, f_j] = 0$ $i \neq j$

+ q-Serre relations

Set $K = K_0 K_1$ - central

We care about fin. dim. type I reps (informally, $K_i = q^{h_i}$ w. h_i acting w. integral e-values).

$\boxed{1}$

Exercise: In any fin. dim. rep'n $K=1$ (cf. Lec 1)

- true for any $\mathfrak{g}$.

1.2) Evaluation & twists by loop rotations.

Evaluation homomorphism $U_q(\hat{\mathfrak{sl}}_2) \xrightarrow{\varphi} U_q(\mathfrak{sl}_2)$ of algebras:

$\varphi(e_1) = \varphi(f_0) = e, \quad \varphi(f_1) = \varphi(e_0) = f, \quad \varphi(K_1) = \varphi(K_0^{-1}) = K$

- not a Hopf algebra homom.

$\forall \mathfrak{g}\ \exists\ \mathbb{Z}$-grading on $U_q(\hat{\mathfrak{g}})$ (by energy) $\rightsquigarrow$

loop rotation action $\mathbb{C}_m^\times \curvearrowright U_q(\hat{\mathfrak{g}})$, $z \mapsto \tau_z$

$\rightsquigarrow$ for $\mathfrak{sl}_2$ ($\&\ \mathfrak{sl}_n$) $\varphi_z := \varphi \circ \tau_z$

$\rightsquigarrow \varphi_z^*: \operatorname{Rep} U_q(\mathfrak{sl}_2) \longrightarrow \operatorname{Rep} U_q(\hat{\mathfrak{sl}}_2)$

$Y(z) = \varphi_z^* Y$ for $Y \in \operatorname{Rep} U_q(\mathfrak{sl}_2)$.

Rem: For general $\mathfrak{g}$, if W is a $U_q(\hat{\mathfrak{g}})$-rep'n $\rightsquigarrow W(z) := \tau_z^* W$.

Properties: $\forall\ W \in \operatorname{Rep}(U_q(\hat{\mathfrak{g}}))$: $W(z)(u) = W(zu)$

$(X \otimes Y)(z) = X(z) \otimes Y(z), \quad Y(z)^* = Y^*(z)$.

1.3) Failure of braiding/semisimplicity

We'll see: if $V, W \in \mathrm{Rep}\, U_q(\hat{\mathfrak{sl}}_2)$, then $(V \otimes W)(z) \not\simeq V(z) \otimes W(z)$ b/c φ is not Hopf alg. homomom. Similarly, $V(z)^* \not\simeq V^*(z)$.

Rem: Irreps of $U_q(\mathfrak{sl}_2)$ are V_a of $\dim = a+1$, $a \in \mathbb{Z}_{\geq 0}$ $\rightsquigarrow$ $V_a(z)$

For $a=1$: $V_a(z)$ in matrices:

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ K_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$e_0 \mapsto \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix},\ f_0 \mapsto \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix},\ K_0 \mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

Exer: Any 2-dim. nontriv. $U_q(\hat{\mathfrak{sl}}_2)$ rep is $V_1(z)$ for unique z.

Corollary: $V_1(z)^* \simeq V_1(w)$ for unique w.

Rem: $z = \mathrm{tr}_{V_1(z)}(e_0 e_1)$

$$w = \mathrm{tr}_{V_1(z)^*}(S(e_0)^* S(e_1)^*) = \mathrm{tr}(S(e_1)S(e_0)) = \mathrm{tr}(-e_1K_1^{-1} \cdot (-e_0K_0))$$

$$= q^2\,\mathrm{tr}(e_1e_0) = q^2 z$$

$\Rightarrow V(z)^{**} = V(q^4 z) \Rightarrow \mathrm{Rep}\, U_q(\hat{\mathfrak{g}})$ is <u>not</u> braided

In any rigid tensor category $\mathcal{C}$ if $X \in \mathcal{C}$, then we have
$X^* \otimes X \xrightarrow{ev_X} \mathbb{1}$, $X \otimes X^* \xleftarrow[coev]{} \mathbb{1}$

Claim: If X is simple & either of these maps splits $\Rightarrow X^{**} \simeq X$.

Proof: Suppose ev_X splits $\Rightarrow X^* \otimes X \simeq Y \otimes \mathbb{1}$ & $\mathbb{1} \hookrightarrow X^* \otimes X$

$$\mathbb{1} \overset{i}{\hookrightarrow} X^* \otimes X \rightsquigarrow {}^*X \xhookrightarrow{i \otimes 1} X^* \otimes X \otimes {}^*X \to X^*, \qquad \alpha_i : {}^*X \to X^*$$

Exer: This defines isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{1}, X^* \otimes X) & \xrightarrow{\sim} & \mathrm{Hom}({}^*X, X^*) \\ \Psi & & \Psi \\ i & \longmapsto & \alpha_i \end{array}$$

Since ${}^*X, X^*$ are isomorphic, by Schur lemma, ${}^*X \simeq X^*$. $\square$

Exer: $\mathbb{1} \xhookrightarrow[coev]{} V_1(z) \otimes V_1(q^2 z) \to V_2(qz) \to 0$ -nonsplit (*)
(if $Y \in \mathrm{Rep}\, U_q(\hat{\mathfrak{sl}}_2)$, $Y|_{U_q(\mathfrak{sl}_2)}$ is irred $\Rightarrow Y \simeq V_a(z)$ for some z)

Dualize (*): $0 \to V_2(qz) \to V(q^2 z) \otimes V(z) \to \mathbb{C} \to 0$
$\Rightarrow V(q^2 z) \otimes V(z) \not\simeq V(z) \otimes V(q^2 z)$

But: if $w \neq q^2 z \Rightarrow V(z)\otimes V(w)$ is irreducible (exercise) isomorphic to $V(w)\otimes V(z)$. In fact, it's defined by an R-matrix.

Rem: For general $\mathfrak{g}$ & $\forall$ irred. X, Y: $X(z)\otimes Y$ is irred & isom to $Y\otimes X(z)$ for all z but fin. many.

1.4) Double dual.

For general $\mathfrak{g}$: if Y is fin. dim. rep'n of $U_q(\hat{\mathfrak{g}}) \Rightarrow$

$Y^{**} = Y(q^{2h^\vee})$ $h^\vee$ is dual Coxeter number (for $\mathfrak{sl}_2$: $Y^{**} \simeq Y(q^4)$)

Q: Why $h^\vee$?

A: For q-triangular Hopf alg. (H,R) w. $R = \sum_i a_i \otimes b_i$, R-matrix invertible, $R\Delta(x) = \Delta^{op}(x)R$ & $(\Delta\otimes 1)(R) = R_{13}R_{23}$, $(1\otimes\Delta)(R) = R_{13}R_{12}$

Thm (Drinfeld): for $u = \sum_i S(b_i)a_i \Rightarrow uxu^{-1} = S^2(x) \rightsquigarrow$

$u : X \xrightarrow{\sim} X^{**}$

For $U_q(\mathfrak{g})$, $u = v q^{2\rho}$ (v is ribbon element, central)

For affine Lie algebra $\hat{\rho} = \rho + h^\vee d \Rightarrow q^{2\hat{\rho}} = q^{2\rho} \underbrace{q^{2h^\vee d}}_{\text{shifts } z!}$

1.5) Classif'n of fin. dim. irreps for $U_q(\hat{\mathfrak{sl}}_2)$.

Reference: Chari-Pressley

Prop 1: All irreps of $U_q(\hat{\mathfrak{sl}}_2)$ are of the form

$$V_{a_1}(z_1) \otimes V_{a_2}(z_2) \otimes \ldots \otimes V_{a_n}(z_n)$$

Key question: when is it irreducible?

Can rule out: $a_i = a_{i+1} = 1$, $z_i / z_{i+1} = q^{\pm 2}$. Can also do the same when $a_i = a_j = 1$ w $|i-j| > 1$.

To state the answer we need a combinatorial constr'n:

Attach to $V_a(z)$ a q^2-string $(q^{-a+1} z, q^{-a+3} z, \ldots, q^{a-1} z)$

Def: A collection of strings $S_1, \ldots S_n$ is in special position if $\exists\, i,j \mid S_i \cup S_j \supsetneq S_i, S_j$ & $S_i \cup S_j$ is a q^2-string. Otherwise we say $S_1, \dots S_n$ is in general position

Thm: $V_{a_1}(z_1) \otimes \dots \otimes V_{a_n}(z_n)$ is irred $\Leftrightarrow$ strings of factors are in general position. The product doesn't depend on the order.

This generalizes the case $V(z) \otimes V(w)$ as strings are z & w.

Prop 2: Any finite multi-subset of $\mathbb{C}^\times$ can be uniquely written as union of strings in general position (up to permutation).

Conclusion: $U_q(\hat{\mathfrak{sl}}_2)$-irreps $\xleftrightarrow{\sim}$ multisubsets of $\mathbb{C}^\times$ $\xleftrightarrow{\sim}$ polynomials w. nonzero const term (up to scaling). This is Drinfeld polynomial, usually normalized to have const. term 1

1.2) R-matrices w. spectral parameter.

$U_q(\hat{\mathfrak{sl}}_2)/(K-1)$ has universal R-matrix, $R = \sum_i a_i \otimes a^i$, where $a_i \in U^+$ & $a^i \in U^-$

Can we make sense of $R|_{X \otimes Y}$? Not in general.

What about $X(z) \otimes Y$ for formal variable z?

$R(z) = \sum_i \tau_z(a_i) \otimes a^i$

has only nonneg. powers of z.

$\rightsquigarrow R(z)|_{X \otimes Y} \in \mathrm{End}(X \otimes Y)[[z]]$

Thm (Drinfeld) (for all $\mathfrak{g}$) This gives a converging series in some neigh-d of 0, $\{z \mid |z| < r\}$, where $r = r_{XY}$.

Get operator $R_{XY}(z): X(z) \otimes Y \to X(z) \otimes Y$.

Prop: This operator extends to a meromorphically to $\mathbb{C}$

If X, Y irred. $\Rightarrow X(z) \otimes Y$ irred. for generic z, cf. Thm in Sec 1.5

Fact: $R_{XY}(z) = \overline{R}_{XY}(z) f_{XY}(z)$

rational matrix function | scalar function.

Further facts: This $\bar{R}_{XY}(z)$ can be normalized to satisfy:

$$\bar{R}(z)\bar{R}(z^{-1}) = 1\otimes 1, \quad \bar{R}_{XZ}(z)\,\bar{R}_{YZ}(z) = \bar{R}_{X\otimes Y,Z}(z)$$

$$\bar{R}_{XZ}(z)\,\bar{R}_{XY}(z) = \bar{R}_{X,Y\otimes Z}(z)$$

$$\Rightarrow \bar{R}^{12}_{XX}(z_1/z_2)\,\bar{R}^{13}_{XX}(z_1/z_3)\,\bar{R}^{23}_{XX}(z_2/z_3)$$

= opposite order

– QYBE.

Rem: Kind of commutative like vertex algebra