Lazy approach to categories $\mathcal{O}$, $\mathcal{I}$.

0) Intro

1) Highest weight structure

0.1) BGG category $\mathcal{O}$...

Notation: Base field $\mathbb{C}$

G connected reductive group, $\mathfrak{g} = Lie(G)$

$H \subset B \subset G$, Cartan & Borel. $\Lambda := Hom(H, \mathbb{C}^\times)$

Def: Pick $\nu \in \mathfrak{h}^*$ & view ν as element of $\mathfrak{b}^*$ via $\mathfrak{h}^* \hookrightarrow \mathfrak{b}^*$.

$\mathcal{O}_\nu$ is the full subcategory in $\mathcal{U}(\mathfrak{g})\text{-mod}_{fg}$ consisting of all M s.t. the action of $\mathfrak{b}$ on M given by

$x \cdot m = \overbrace{xm}^{\text{initial action}} - \langle \nu, x \rangle m$ integrates to a B-action.

Standard consequences:

- Weight decomposition: $M \in \mathcal{O}_\nu \Rightarrow M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ w.

$M_\lambda = \{ m \in M \mid xm = \langle \lambda + \nu, x \rangle m \ \forall x \in \mathfrak{h} \}$; $\dim M_\lambda < \infty$.

• $\{\lambda \mid M_\lambda \neq 0\}$ is bounded from above with respect to the usual order: $\lambda_1 \le \lambda_2$ if $\lambda_2 - \lambda_1 \in \mathrm{Span}_{\mathbb{Z}_{\geq 0}}$(positive roots).

• Can form Verma module $\Delta_\nu(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda+\nu}$ & its simple quotient $L_\nu(\lambda)$ so that $\Lambda \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O}_\nu)$, $\lambda \leftrightarrow L_\nu(\lambda)$.

• For $\mu \in \Lambda$ have $\mathcal{O}_\nu \xrightarrow{\sim} \mathcal{O}_{\nu+\mu}$ w. $L_\nu(\lambda) \leftrightarrow L_{\mu+\nu}(\lambda - \mu)$.

0.2) ... and its siblings.

$\mathcal{O}_\nu$ is a "finite type" category (is "controlled" by the Hecke category associated to a subgroup of W, the Weyl group of G). It also has "affine" and, potentially, "double affine" analogs to be briefly mentioned now and, hopefully, elaborated later.

Affine world: is inhabited by:

• Categories $\mathcal{O}$ over affine Lie algebras that exhibit 3 possible behaviors: "negative", "positive" and "critical" level.

• Modular/quantum-at-a-root-of-1 categories $\mathcal{O}$.

Most of these (except the critical affine category) are directly controlled by the affine Hecke category.

There are also various geometric relatives of the above categories.

Double affine world: we haven't seen many categories living here but one of the families should be:

quantum-at-a-root-of-1-rational-level-affine categories $\mathcal{O}$ and their modular counterparts.

Likely, there are many more but all of them (incl. quantum affine ones) are very complicated.

0.3) Goals & tools.

Category $\mathcal{O}$ (& its siblings) split into direct sums of blocks. Our goal is to establish (derived) equivalences between blocks of different categories $\mathcal{O}$. The most basic & crucial tool here is highest weight structures to be discussed in the main part of the lecture.

1) Highest weight structures

1.1) General - and abstract - definition

Let $\mathbb{F}$ be a field & $\mathcal{C}$ be an $\mathbb{F}$-linear abelian category

Definition: The structure of a highest weight category w. finite poset on $\mathcal{C}$ is a finite poset, $\mathcal{T}$, and a collection of standard objects $\Delta(\tau)\in\mathcal{C}$, $\tau\in\mathcal{T}$, satisfying the following:

(HW1) $\dim_{\mathbb{F}} \operatorname{Hom}_{\mathcal{C}}(\Delta(\tau), M)<\infty \ \forall \ \tau\in\mathcal{T}, M\in\mathcal{C}$

(HW2) $\operatorname{Hom}_{\mathcal{C}}(\Delta(\tau),\Delta(\tau'))\neq 0 \Rightarrow \tau\leqslant\tau'$

(HW3) $\mathbb{F}\xrightarrow{\sim}\operatorname{End}_{\mathcal{C}}(\Delta(\tau)) \ \forall \ \tau\in\mathcal{T}$

(HW4) $\forall \ M\in\mathcal{C}, M\neq 0 \ \exists \ \tau\in\mathcal{T}$ s.t. $\operatorname{Hom}_{\mathcal{C}}(\Delta(\tau), M)\neq 0.$

(HW5) $\forall \ \tau\in\mathcal{T} \ \exists$ projective $P_\tau\in\mathcal{C}$ & $P_\tau\twoheadrightarrow\Delta(\tau)$ s.t. $\ker[P_\tau\twoheadrightarrow\Delta(\tau)]$ admits a finite filtration by $\Delta(\tau')$'s w. $\tau'>\tau$.

Exercises:

1) $A:=\operatorname{End}_{\mathcal{C}}(\bigoplus_\tau P_\tau)$ is finite & $\operatorname{Hom}_{\mathcal{C}}(\bigoplus_\tau P_\tau, \bullet):\mathcal{C}\xrightarrow{\sim} A^{opp}\text{-mod}_{fd}$.
2) Each $\Delta(\tau)$ has unique simple quotient, $L(\tau)$, and $\tau\mapsto L(\tau)$ is a bijection $\mathcal{T}\xrightarrow{\sim}\operatorname{Irr}(\mathcal{C})$.

1.2) Example: infinitesimal blocks of $\mathcal{O}$

$\mathcal{O}_\nu$ itself is not highest wt. in the sense of Sec 1.1 but is the infinite direct sum of such.

Recall the Harish-Chandra isomorphism:

$HC: Z(U(\mathfrak{g})) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$, $w\cdot\lambda = w(\lambda+\rho)-\rho$

so that $z \in Z(U(\mathfrak{g}))$ acts on $\Delta_\nu(\lambda)$ by $HC_z(\lambda+\nu)$

Consider the equivalence relation $\sim_\nu$ on Λ: $\lambda_1 \sim_\nu \lambda_2$ if $\lambda_1 + \nu = w\cdot(\lambda_2+\nu)$.

We get the decomposition $\mathcal{O}_\nu = \bigoplus_{\Xi} \mathcal{O}_{\nu,\Xi}$, where Ξ runs over the equivalence classes for $\sim_\nu$.

Exercise: Each $\mathcal{O}_{\nu,\Xi}$ is a highest weight category with standard objects $\Delta_\nu(\lambda)$, $\lambda \in \Xi$, and order restricted from $\leqslant$.

1.3) Deformation.

Let R be a Noetherian ring and $\mathcal{C}_R$ be an R-linear abelian category. Note that for $M \in \mathcal{C}_R$ we get a right exact functor $M \otimes_R ?: R\text{-mod}_{fg} \to \mathcal{C}_R$, we say that M is R-flat if this functor is exact.

One can generalize the definition of a highest weight category to $\mathcal{C}_R$: we require that $\Delta_R(\tau)$ are flat over R & modify (HW1) & (HW5) as follows:

(HW1'): $\mathrm{Hom}_{\mathcal{C}_R}(\Delta_R(\tau), M)$ is fin. generated over R.

(HW5'): $\ker[P_\tau \twoheadrightarrow \Delta_R(\tau)]$ is filtered by objects of the form $Q^{\tau'} \otimes_R \Delta_R(\tau')$ for $\tau' > \tau$ & $Q^{\tau'}$ fin. gen'd projective R-module.

Exercise: $\mathrm{End}_{\mathcal{C}_R}(\bigoplus_\tau P_\tau)$ is fin. gen'd projective R-module.

Main example: $R := \mathbb{C}[\mathfrak{h}^*]^{\wedge_0}$ completion at 0. Let c be the composition $\mathfrak{h} \hookrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \hookrightarrow R$.

$\mathcal{O}_{\nu,R}$ is the full subcategory in $\mathcal{U}(\mathfrak{g}) \otimes R\text{-mod}_{fg}$ consisting of all M s.t. the action of $\mathfrak{b}$ on M given by

$x \cdot m = \overbrace{xm}^{\text{initial action}} - (\langle \nu, x \rangle + c(x))\, m$ integrates to a B-action.

We have the same properties as for $\mathcal{O}_\nu$, e.g. weight decomposition $M = \bigoplus_\lambda M_\lambda$ w. fin. gen'd R-modules M_λ & weights bounded from above. We can form Verma modules

$\Delta_{\nu,R}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} R_{\lambda+\nu}$, where $R_{\lambda+\nu} \simeq R$ w. $\mathfrak{h} \curvearrowright R$ by
$x \mapsto \iota(x) + \langle \lambda+\nu, x\rangle$.

Exercise: $\mathcal{O}_\nu$ is identified w. the full subcategory in $\mathcal{O}_{\nu,R}$ consisting of all objects M where R acts via $R \twoheadrightarrow \mathbb{C}$.

Remark: One can informally view R as the algebra of functions on a tiny neighborhood around ν. Then $\mathcal{O}_{\nu,R}$ is a family of categories over this neighborhood whose fiber at a point ν' in the neighborhood is $\mathcal{O}_{\nu'}$ (note that strictly speaking $\mathrm{Spec}(R)$ only has one $\mathbb{C}$-point).

We can extend the infinitesimal block decomposition for $\mathcal{O}_\nu = \bigoplus_{\Xi} \mathcal{O}_{\nu,\Xi}$ to $\mathcal{O}_{\nu,R}$. Let $\mathfrak{m} \subset R$ denote the maximal ideal. Set $\mathcal{O}_{\nu,R,\Xi} := \{M \in \mathcal{O}_{\nu,R} \mid M/\mathfrak{m}^k M$ is filtered by objects in $\mathcal{O}_{\nu,\Xi}, \forall k\}$

Exercise 1) $\mathcal{O}_{\nu,R} = \bigoplus_{\Xi} \mathcal{O}_{\nu,R,\Xi}$

2) $\mathcal{O}_{\nu,R,\Xi}$ is highest weight w. standards $\Delta_{\nu,R}(\lambda)$, $\lambda \in \Xi$

1.4) Category of standardly filtered objects

Def: An object in $\mathcal{C}_R$ is called ==standardly filtered== if it admits a finite filtration by $Q^{\tau'}\otimes_R \Delta_R(\tau')$, $\tau'\in\mathcal{T}$, where $Q^{\tau'}$ is finitely generated projective R-module. The full subcategory of stand. filtered objects will be denoted by $\mathcal{C}_R^{\Delta}$.

E.g. $\Delta_R(\tau')$ & P_τ from (HW5') are in $\mathcal{C}_R^{\Delta}$.

The following claims require introducing "costandard" objects – the readers familiar with the notion could try to prove them.

Facts: 1) Every projective in $\mathcal{C}_R$ is in $\mathcal{C}_R^{\Delta}$

2) If $M,N\in\mathcal{C}_R^{\Delta}$ & $\varphi: M\twoheadrightarrow N$, then $\ker\varphi\in\mathcal{C}_R^{\Delta}$.

The proof of the following corollary of these fact is left as an exercise.

Corollary: For $M \in \mathcal{C}_R^\Delta$ TFAE:

a) M is projective

b) $\mathrm{Ext}^1_{\mathcal{C}_R}(M, N) = 0 \ \forall N \in \mathcal{C}_R^\Delta$

c) $\mathrm{Ext}^1_{\mathcal{C}_R}(M, \Delta_R(\tau)) = 0 \ \forall \tau \in \mathcal{T}$.

The importance of this corollary is as follows. We note that $\mathcal{C}_R^\Delta$ is an "exact category" (an additive category with a good notion of short exact sequences). Fact 1 shows that the additive category of projectives $\mathcal{C}_R$-proj is contained in $\mathcal{C}_R^\Delta$ & Corollary allows to recover $\mathcal{C}_R$-proj inside $\mathcal{C}_R^\Delta$. And once we know $\mathcal{C}_R$-proj we can recover the abelian category $\mathcal{C}_R$.

1.5) What's next?

Here's the "lazy approach" to understand the categories $\mathcal{O}_{\nu,\Sigma}$ (the most interesting case is $\nu=0$). We will construct a "nice" right exact functor $\mathcal{O}_{\nu,R,\Sigma} \xrightarrow{\mathbb{V}} \mathcal{C}_R$, where $\underline{\mathcal{C}}_R$ is an "easy" category that very roughly depends on "combinatorics" of $\mathcal{O}_{\nu,R,\Xi}$. We will see that $\mathbb{V}$ is acyclic on the standard objects & is fully faithful on $\mathcal{O}^{\Delta}_{\nu,R,\Xi}$. So we need to describe $\mathbb{V}(\mathcal{O}^{\Delta}_{\nu,R,\Xi})$. It turns out that since $\mathbb{V}$ is "nice", it suffices to only describe the localizations of the categories & the functor at ht 1 prime ideals (which morally amounts to understanding the cases when ν is generic on a root hyperplane).

The resulting description of $\mathcal{O}^{\Delta}_{\nu,R,\Xi}$ (and hence indirectly $\mathcal{O}_{\nu,\Xi}$) is very implicit, yet it allows to prove equivalences between different such categories.

Finally, what makes this approach "lazy" is that one needs to know relatively little to get equivalences: essentially one needs to have a nice functor to a "combinatorial category" & to understand its localizations to neighborhoods of generic points on hyperplanes.