Running example $U_q(\widehat{sl}_2)$

- Drinfeld–Jimbo presentation

Generators $E_0, E_1, K_0, K_1, F_0, F_1$

Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

Relations: $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$

$K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i$

$E_i^3 E_j - (q^{-2} + 1 + q^2) E_i^2 E_j E_i + (q^{-2} + 1 + q^2) E_i E_j E_i^2 - E_j E_i^3 = 0$

$i \neq j$

$F_i^3 F_j - (q^{-2} + 1 + q^2) F_i^2 F_j F_i + (q^{-2} + 1 + q^2) F_i F_j F_i^2 - F_j F_i^3 = 0$

- Coproduct

$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta K_i = K_i \otimes K_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$

- $K = K_0 K_1$ – central

- One can add d or q^{2d}

$[d, E_1] = [d, F_1] = [d, K_1] = 0$, $[d, E_0] = E_0$, $[d, F_0] = -F_0$

- In not q-deformed setting

Two presentations

- Kac-Moody presentation

$$\begin{matrix} f_1 & h_0 & e_0 \\ f_1 & h_1 & e_1 \end{matrix}$$

- Loop presentation X_n^-, X_n^0, X_n^+, K $\quad n \in \mathbb{Z}$

$$[X_n^{\varepsilon}, X_{n'}^{\varepsilon'}] = [X^{\varepsilon}, X^{\varepsilon'}]_{n+n'} + n(X^{\varepsilon}, X^{\varepsilon'}) K \delta_{n+n',0}$$

$X^+ = e$

$X^0 = h$

$X^- = f$

$n \in \mathbb{Z}$

Advantage: PBW basis

Braid group action

Def Lusztig's Braid group

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$$

$$T_i(K_j) = K_j K_i^{-a_{ij}}$$

as reflections

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}$$

Here $E_i^{(r)} = \dfrac{E_i^r}{[r]_q!}$

Rk $T_i(E_j) = ad_{\Delta^{op}, E_i^{(-a_{ij})}} E_j = \dfrac{1}{[-a_{ij}]_q!} ad_{q, E_i}^{-a_{ij}} E_j$

$ad_{q, X} Y = XY - q^{(wt(X), wt(Y))} YX$

$$T_1(E_0) = E_1^{(2)} E_0 - q E_1 E_0 E_1 + q^2 E_0 E_1^{(2)}$$

Th T_i - automorphisms of quantum group as an algebra satisfy braid group relations

add $\tau: E_0 \mapsto E_1 \quad K_0 \mapsto K_1 \quad F_0 \mapsto F_1$

$E_1 \mapsto E_0 \quad K_1 \mapsto K_0 \quad F_1 \mapsto F_0$

$$Br^{ae} = \langle T_0, T_1, \tau \mid \tau T_0 \tau^{-1} = T_1,\ \tau T_1 \tau^{-1} = T_0,\ \tau^2 = e \rangle$$

no braid relation

- Inverse map

$$T_i^{-1}(E_i) = -K_i^{-1} F_i, \quad T_i^{-1}(F_i) = -E_i K_i, \quad T_i^{-1}(K_j) = K_j K_i^{-a_{ij}}$$

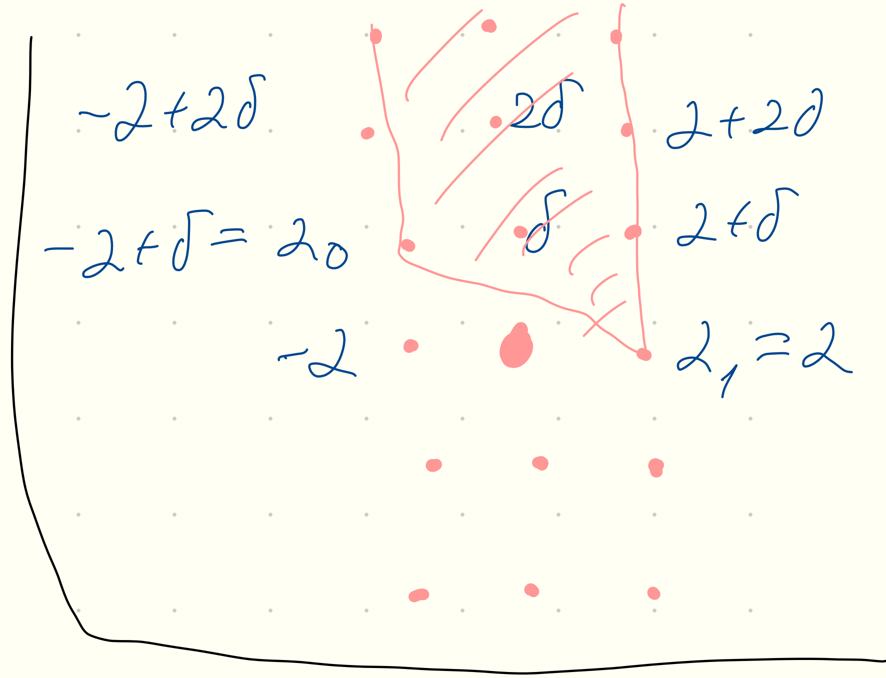
$$T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)} \qquad T_i^{-1}(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)}$$

- Weyl group $\langle s_0, s_1, \tau \mid \tau s_0 \tau^{-1} = s_1,\ \tau s_1 \tau^{-1} = s_0,\ s_1^2 = s_0^2 = \tau^2 = e \rangle$

translations $s_0 s_1$ - by root $\qquad \tau s_0,\ \tau s_1$ - by weight

Def $E_{\alpha+n\delta} = (\tau T_1)^{-n} E_1$

$E_{-\alpha+(n+1)\delta} = (\tau T_1)^{n} E_0$ $\quad n \geq 0$



Question How to define E_δ (q-analog $[e_1, e_0]$)

natural choices E_δ

$$ad_{q,E_1} E_0 = E_1E_0 - q^{-2}E_0E_1$$

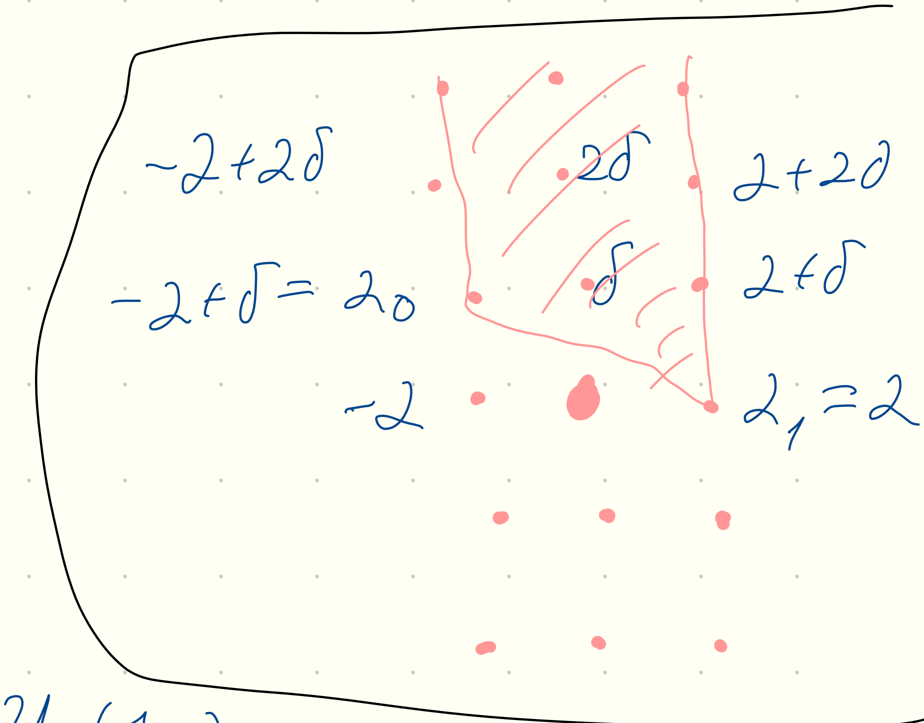
$$ad_{q,E_0} E_1 = E_0E_1 - q^{-2}E_1E_0$$

Lemma $(\tau T_1)(E_0E_1 - q^{-2}E_1E_0) = E_0E_1 - q^{-2}E_1E_0$

Def $E_{n\delta} = E_{-\alpha+\delta}E_{\alpha+(n-1)\delta} - q^{-2}E_{\alpha+(n-1)\delta}E_{-\alpha+\delta}$

Lemma $[E_{\delta_1}, E_{\alpha+n\delta}] = [2]_q E_{\alpha+(n+1)\delta}$

$[E_{\delta_1}, E_{-\alpha+n\delta}] = -[2]_q E_{-\alpha+(n+1)\delta}$



Pf $n=0$ – computation

use τT_1 $\square$

- Let $U_q(\hat{n}_+)$ – subalgebra generated by E_0, E_1

Corollary $E_{\alpha+n\delta}, E_{(n+1)\delta}, E_{-\alpha+(n+1)\delta} \in U_q(\hat{n}_+)$ $\quad n \geq 0$

Relations

Lemma $E_{\alpha+(n+1)\delta} E_{\alpha+m\delta} - q^2 E_{\alpha+n\delta} E_{\alpha+(m+1)\delta} + E_{\alpha+(m+1)\delta} E_{\alpha+n\delta} -$

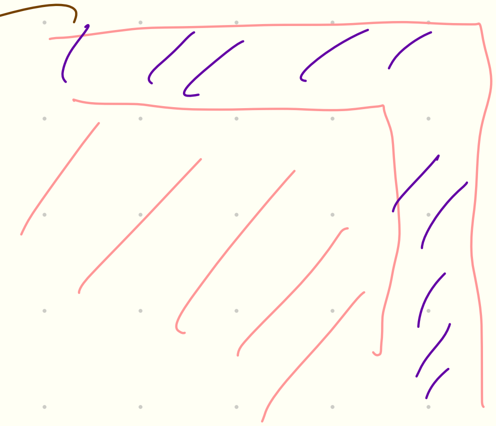
$- q^2 E_{\alpha+m\delta} E_{\alpha+(n+1)\delta} = 0$

<u>Def</u> Half-current $e^+(z) = \sum_{n\geq 0} E_{\alpha+n\delta} z^{-n}$

Relation $e^+(z)e^+(w)(z-q^2w) + e^+(w)e^+(z)(w-q^2z)$

$= (1-q^2)(z e^+(w)^2 + w e^+(z)^2)$

boundary term



<u>Def</u> Half currents

$e^-(z) = \sum_{n\geq 0} E_{-\alpha+n\delta} z^{-n}$ $\qquad e_\delta = 1 + (q-q^{-1})\sum_{n>0} E_{n\delta} z^{-n}$

- $(z-q^2w)\, e_\delta(z)\, e^+(w) = (z-q^{-2}w)\, e^+(w)\, e_\delta(z)$
- $(z-q^{-2}w)\, e_\delta(z)\, e^-(w) = (z-q^2w)\, e^-(w)\, e_\delta(z)$
- $e^-(z)e^-(w)(z-q^{-2}w) + e^-(w)e^-(z)(w-q^{-2}z) =$

$= (1-q^{-2})(z e^-(w)^2 - w e^-(z)^2)$

- $[E_{n\delta}, E_{m\delta}] = 0$
- $E_{-\alpha+(p-r)\delta} E_{\alpha+r\delta} - q^{-2} E_{\alpha+r\delta} E_{-\alpha+(p-r)\delta} = E_{p\delta}$

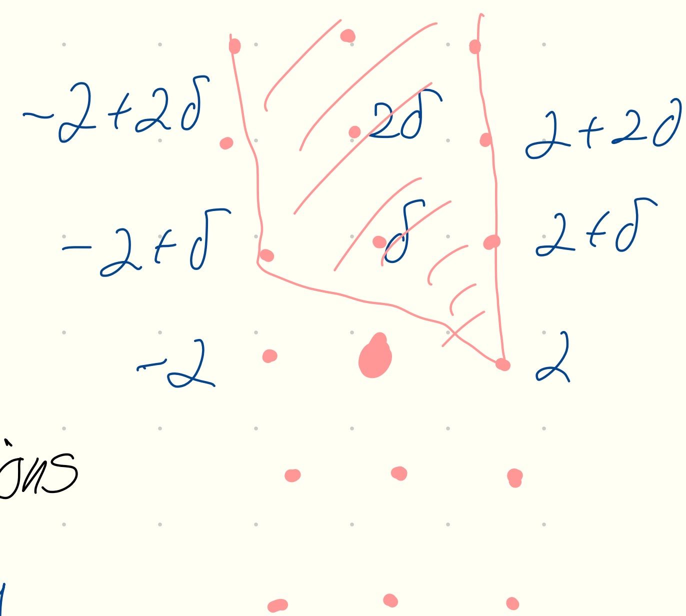
Th (PBW) Elements

$$\{ E_{-\alpha+\delta}^{a_1} E_{-\alpha+2\delta}^{a_2} \cdot \ldots \cdot E_{\delta}^{b_1} E_{2\delta}^{b_2} \cdot \ldots \cdot \ldots E_{\alpha+2\delta}^{c_2} E_{\alpha+\delta}^{c_1} E_{\alpha}^{c_0} \}$$

form a Basis in $U_q(\hat{n}_+)$

Rk Convex order

$$-\alpha-\delta < -\alpha+2\delta < \ldots < \underset{\vdots}{\overset{\delta}{2\delta}} < \ldots < \alpha+\delta < \alpha$$



Pf Generating set follows from relations

Linear indep. follows from $q \to 1$

- $U_q(\hat{n}_-)$

automorphism

$$\varphi(E_i)=F_i \qquad \varphi(F_i)=E_i \qquad \varphi(K_i)=K_i \qquad \varphi(q)=q^{-1}$$

<u>Def</u>

$$\sigma\varphi(E_{\alpha+n\delta}) = (\sigma T_1)^n F_0 = F_{\alpha-(n+1)\delta}$$

$$\sigma\varphi(E_{-\alpha+(n+1)\delta}) = (\sigma T_1)^{-n} F_1 = F_{-\alpha-n\delta}$$

$$\sigma\varphi(E_{n\delta}) = F_{-n\delta}$$

$\Rightarrow$ PBW property

- Full currents

<u>Def</u> $X_n^+ = (\sigma T_1)^{-n} E_1 \qquad X_n^- = (\sigma T_1)^n F_1 \qquad n\in\mathbb{Z}$

<u>Remark</u> $n\geq 0$ $\quad X_n^+ = E_{\alpha+n\delta} \qquad X_{-n}^- = F_{-\alpha-n\delta}$

<u>But</u> $n>0$ $\quad X_{-n}^+ = -(F_{\alpha-n\delta}K^n)K_1^{-1} \notin U_q(\hat{n}_-) \qquad X_n^- = -K_1K^{-n}E_{-\alpha+n\delta} \notin U_q(\hat{n}_+)$

<u>Def</u> $x^+(z) = \sum_{n\in\mathbb{Z}} x_n^+ z^{-n} = e^+(z) - f^+(Kz)K_1^{-1}$

$x^-(z) = \sum_{n\in\mathbb{Z}} x_n^- z^{-n} = -K_1 e^-(Kz) - f^-(z)$

$K_1^{-1}\psi^+(z) = 1 + (q-q^{-1})\sum_{n>0} E_{n\delta} z^{-n} = \exp\left(\sum_{n>0}(q-q^{-1}) h_n z^{-n}\right)$

$K_1\psi^-(z) = 1 + (q^{-1}-q)\sum_{n>0} F_{-n\delta} z^{n} = \exp\left(\sum_{n>0}(q^{-1}-q)\, h_{-n} z^{n}\right)$

- <u>Th</u> $U_q(\hat{sl}_2)$ has presentation with generators $x_n^+,\ x_n^-,\ n\in\mathbb{Z},\ h_r,\ h_{-r}\ r\in\mathbb{Z}_{>0},\ K^{\pm 1}, K_1^{\pm 1}$ and relations

- K - central
- $K_1 x_n^+ = q^2 x_n^+ K_1$ $\qquad\qquad$ $K_1 x_n^- = q^{-2} x_n^- K_1$

- $$[h_r, h_s] = \frac{[2r]}{r} \frac{K^r - K^{-r}}{q - q^{-1}} \delta_{r+s,0}$$

- $$[h_r, X^+(w)] = \frac{[2r]}{r} w^r X^+(w) \qquad [h_{-r}, X^+(w)] = \frac{[2r]}{r} K^{-r} w^{-r} X^+(w)$$

  $$[h_r, X^-(w)] = -K^r \frac{[2r]}{r} w^r X^-(w) \qquad [h_{-r}, X^-(w)] = -\frac{[2r]}{r} w^{-r} X^-(w)$$

- $$[X^+(z), X^-(w)] = \frac{1}{q - q^{-1}} \left(\psi^+(z)\, \delta\!\left(\frac{Kw}{z}\right) - \psi^-(w)\, \delta\!\left(\frac{w}{Kz}\right) \right)$$

- $$X^+(z) X^+(w)(z - q^2 w) + X^+(w) X^-(z)(w - q^2 z) = 0$$

  $$X^-(z) X^-(w)(z - q^{-2} w) + X^-(w) X^-(z)(w - q^{-2} z) = 0$$

Here $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$

Pf One can construct homomorphisms in both directions $\square$

Rk Should work for q root of 1 (perhaps for $q^4 \neq 1$)

- In general affine KM algebra $\Longleftrightarrow X_n^{(k)}$
  $\bar{I}$ - set of vertices of X_n

<u>Def</u> $\mathcal{U}^D(X_n^{(k)})$ (for simplicity $k=1$, $X=ADE$)
is $\mathbb{C}(q)$ algebra
generators $x_{i,n}^+$, $x_{i,n}^-$, $h_{i,r}$, $h_{i,-r}$, $K_i^{\pm 1}$, $K^{\pm 1}$
$i \in \bar{I}$, $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$

relations

- $K_i K_j = K_j K_i$, K - central
- $K_i X_{j,n}^+ = q^{a_{ij}} X_{j,n}^+ K_i$ $\qquad$ $K_i X_{j,n}^- = q^{-a_{ij}} X_{j,n}^- K_i$
- $[h_r, X^+(w)] = \frac{[r a_{ij}]}{r} w^r X^+(w)$ $\qquad$ $[h_{-r}, X^+(w)] = \frac{[r a_{ij}]}{r} K^{-r} w^{-r} X^+(w)$
  $[h_r, X^-(w)] = -K^r \frac{[r a_{ij}]}{r} w^r X^-(w)$ $\qquad$ $[h_{-r}, X^-(w)] = -\frac{[r a_{ij}]}{r} w^{-r} X^-(w)$

- $$[h_{i,r}, h_{j,s}] = \frac{[r a_{ij}]}{r} \frac{K^r - K^{-r}}{q - q^{-1}} \delta_{r+s,0}$$

- $$[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{q-q^{-1}} \left(\psi_i^+(z)\, \delta\!\left(\frac{Kw}{z}\right) - \psi_i^-(w)\, \delta\!\left(\frac{w}{Kz}\right) \right)$$

- $$X_i^+(z) X_j^+(w)(z - q^{a_{ij}} w) + X_j^+(w) X_i^+(z)(w - q^{a_{ij}} z) = 0$$
$$X_j^-(z) X_i^-(w)(z - q^{-a_{ij}} w) + X_i^-(w) X_j^-(z)(w - q^{-a_{ij}} z) = 0$$

- $$\mathrm{Sym}\left[\sum_{p=0}^{1-a_{ij}} (-1)^p \begin{bmatrix} 1-a_{ij} \\ p \end{bmatrix}_q X^+_{i n_1} \cdots X^+_{i n_p} X^+_{j m} X^+_{i n_{p+1}} \cdots X^+_{i n_{1-a_{ij}}} \right] = 0$$

  Symmetrization on $n_1, \dots, n_{1-a_{ij}}$

- <u>Th</u> (Drinfeld, Beck, Damiani) $\mathcal{U}_q^{DJ} \simeq \mathcal{U}_q^{D}$

- <u>Corollary</u> Let $J \subset I$, There is $\mathcal{U}_q(\hat{\mathfrak{g}}_J) \hookrightarrow \mathcal{U}_q(\hat{\mathfrak{g}}_I)$

  In particular $i \in I$ $\quad \mathcal{U}_q(\hat{\mathfrak{sl}}_2)_i \hookrightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$