Lazy approach to categories $\mathcal{O}$, II

0) Recap
1) (Sub)generic behavior
2) Whittaker coinvariants.

0) Recap: $\nu \in \mathfrak{h}^*$, $R := \mathbb{C}[\mathfrak{h}^*]^{\wedge_0}$ completion at 0. Let ι be the composition $\mathfrak{h} \hookrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \hookrightarrow R$.

$\mathcal{O}_{\nu,R}$ is the full subcategory in $\mathcal{U}(\mathfrak{g}) \otimes R\text{-mod}_{fg}$ consisting of all M s.t. the action of $\mathfrak{b}$ on M given by

$$x \cdot m = \overbrace{xm}^{\text{initial action}} - (\langle \nu, x \rangle + \iota(x))\, m \text{ integrates to a } B\text{-action.}$$

Remark: Let S be an R-algebra. Similarly to $\mathcal{O}_{\nu,R}$ we can consider the category $\mathcal{O}_{\nu,S}$, the full subcategory in $\mathcal{U}(\mathfrak{g}) \otimes S\text{-mod}$ w. the same integrability condition (where we replace ι with the composition $\mathfrak{h} \overset{\iota}{\hookrightarrow} R \longrightarrow S$).

Recall the equivalence $\sim_\nu$ on Λ (root lattice): $\lambda_1 \sim_\nu \lambda_2$ if $\lambda_1 + \nu \in W \cdot (\lambda_2 + \rho)$. Then $\mathcal{O}_{\nu R} = \bigoplus_{\Xi} \mathcal{O}_{\nu, R, \Xi}$, where $\mathcal{O}_{\nu, R, \Xi} = \text{Serre span}(\Delta_{\nu, R}(\lambda) \mid \lambda \in \Xi)$

Later we'll see that $\mathcal{O}_{\nu R, \Xi}$ may decompose further.

Recall also that $\mathcal{O}_{\nu R, \Xi}$ is highest weight with poset Ξ and standards $\Delta_{\nu, R}(\lambda)$, $\lambda \in \Xi$.

Goal: Describe the category $\mathcal{O}^{\Delta}_{\nu, R, \Xi}$ of standardly filtered objects.

1) (Sub)generic behavior

Exercise 1: 1) If $\mathcal{O}_\nu$ is not semisimple, then $\exists$ root α with $\langle \nu, \alpha^\vee \rangle \in \mathbb{Z}$.

2) Let $\mathbb{K} = \operatorname{Frac}(R)$. Then $\mathcal{O}_{\nu, \mathbb{K}}$ is semisimple.

Next consider a very generic element ν on the hyperplane $\langle \nu, \alpha^\vee \rangle = n$ (for $n \in \mathbb{Z}$): we require that each equivalence class Ξ for $\sim_\nu$ has at most 2 elements, the corresponding locus is the complement of countably many hyperplanes.

If $|\Xi|=1$, then $\mathcal{O}_{\nu,\Xi}\simeq \mathrm{Vect}$.

Let $|\Xi|=2$, then $\Xi=\{\lambda_-<\lambda_+\}$

Fact (Ch. 4 in Humphreys) $\dim \mathrm{Hom}(\Delta_\nu(\lambda_-),\Delta_\nu(\lambda_+))=1$

Observation: BGG reciprocity holds $\Rightarrow$ indec. projective $P_\nu(\lambda_-)$ fits into SES $0\to\Delta_\nu(\lambda_+)\to P_\nu(\lambda_-)\to\Delta_\nu(\lambda_-)\to 0$

Premium exercise 2: Use Fact & Observation to establish an equivalence of highest weight categories between $\mathcal{O}_{\nu,\Xi}$ and the principal block of the category $\mathcal{O}$ for $\mathfrak{sl}_2$.

Remark: A similar but more technical statement is true in a deformed setup. Very informally: near a point ν generic w. $\langle\nu,\alpha^\vee\rangle=n$ as above, category $\mathcal{O}$ looks like the category $\mathcal{O}$ for $\mathfrak{sl}_2$ near 0.

2) Whittaker coinvariants.

2.1) Construction of the functor.

Let $\mathfrak{n}^-$ denote the opposite max. nilpotent subalgebra &

$\psi: \mathfrak{n}^- \to \mathbb{C}$, $\psi(x) = (\sum_{i=1}^{\operatorname{rk}\mathfrak{g}} e_i, x)$, a nondegenerate character.

For $M \in \mathcal{U}(\mathfrak{g})$-mod, consider its Whittaker coinvariants

$Wh(M) = M/\{x - \psi(x) \mid x \in \mathfrak{n}^-\}M$.

Note that the center $Z(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ acts on $Wh(M)$, so we get a right exact functor $Wh: \mathcal{U}(\mathfrak{g})\text{-mod} \longrightarrow Z(\mathfrak{g})\text{-mod}$.

For $M \in \mathcal{O}_{\nu,R}$, have commuting R-actions leading to

$Wh: \mathcal{O}_{\nu,R} \longrightarrow Z(\mathfrak{g}) \otimes R\text{-mod}$

Exercise/example:

1) $Wh(\Delta_\nu(\lambda)) \simeq \mathbb{C}$ as vector space (hint: $\Delta_\nu(\lambda) \underset{\mathfrak{n}^-}{\simeq} \mathcal{U}(\mathfrak{n}^-)$) w. action of $Z(\mathfrak{g}) = \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$ given by evaluation at $\lambda+\nu$.

2) $Wh(\Delta_{\nu,R}(\lambda)) \simeq R$ as right R-module w. $Z(\mathfrak{g}) = \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$ acting via $\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)} \hookrightarrow S(\mathfrak{h}) \xrightarrow{(*)} R = S(\mathfrak{h})^{\wedge_0}$ w.

$(*)$: $x \in \mathfrak{h} \mapsto \iota(x) + \langle \lambda+\nu, x \rangle \in R$.

3) Wh is acyclic on $\Delta_\nu(\lambda)$ & $\Delta_{\nu,R}(\lambda)$.

2.2) Faithfullness.

Our goal in this section is to prove the following

Thm: 1) $Wh: \mathcal{O}_\gamma \to Vect$ is faithful on (=injective on Homs between) standardly filtered objects
2) $Wh: \mathcal{O}^\Delta_{\gamma,R} \to Z(\mathfrak{g})\otimes R\text{-mod}$ is fully faithful on (=bijective on Homs between) standardly filtered objects.

There are two ways to prove 1): geometric & rep. theoretic We'll use the former. The latter requires a connection between category $\mathcal{O}$ & Whittaker modules.

Proof of 1): Consider the algebre
$$\mathcal{U}_\hbar(\mathfrak{g}) = T(\mathfrak{g})[\hbar]/(x\otimes y - y\otimes x - \hbar[x,y]),$$
equivalently the Rees algebra of $\mathcal{U}(\mathfrak{g})$ under the PBW filtration. It's a graded flat $\mathbb{C}[\hbar]$-algebra w. $\mathcal{U}_\hbar(\mathfrak{g})/(\hbar) \xrightarrow{\sim} S(\mathfrak{g})$.

Consider the category $\mathcal{O}_{\nu\hbar}$ of graded finitely generated $\mathcal{U}_\hbar(\mathfrak{g})$-modules $\mathcal{M}$ that are equipped w. rational B-action s.t.

- $\mathcal{U}_\hbar(\mathfrak{g})\otimes\mathcal{M}\longrightarrow\mathcal{M}$ is B-equivariant.
- For $x\in\mathfrak{b}$ we write $x_{\mathcal{M}}\in End(\mathcal{M})$ for the element given by the differential of the B-action. Then we have $\hbar x_{\mathcal{M}} m = xm - \hbar\langle\nu,x\rangle\ \forall x\in\mathfrak{b}, m\in\mathcal{M}$.

In particular, $\mathcal{M}/(\hbar-1)\mathcal{M}\in\mathcal{O}_\nu$, while $\mathcal{M}/\hbar\mathcal{M}\in Coh^{B\times\mathbb{G}_m}[(\mathfrak{g}/\mathfrak{b})^*]$.

We still have a functor $Wh:\mathcal{O}_{\nu\hbar}\longrightarrow\mathbb{C}[\hbar]$-mod as above. Moreover, we observe that $Wh(\mathcal{M})$ is naturally graded. Namely, let $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ be the principal grading. Define the modified grading on $\mathcal{U}_\hbar(\mathfrak{g})$ by putting $x\in\mathfrak{g}(i)$ in degree $i+1$ (while $\hbar$ is still in deg 1). Then $\{x-\psi(x)\,|\,x\in\mathfrak{n}^-\}$ is homogeneous. We can modify the grading on any T-equivariant graded $\mathcal{U}_\hbar(\mathfrak{g})$-module, N, to make it graded for the modified grading on $\mathcal{U}_\hbar(\mathfrak{g})$ (namely, $T\times\mathbb{G}_m$-acts on N and we consider the $\mathbb{G}_m$-action given by $(\rho^\vee, id): \mathbb{G}_m \longrightarrow T\times \mathbb{G}_m)$. This upgrades Wh to a functor

$$\mathcal{O}_{\nu\hbar} \longrightarrow \mathbb{C}[\hbar]\text{-grmod}.$$

Consider the full subcategory in $\mathcal{O}_{\nu\hbar}$ consisting of all objects where $\hbar$ acts by 0, it's identified with $Coh^{B\times\mathbb{G}_m}((\mathfrak{g}/\mathfrak{b}')^*)$. The restriction of Wh to this subcategory is $Wh: N\longmapsto N_\psi$ (fiber), where we view ψ as a point of $(\mathfrak{g}/\mathfrak{b}')^*$ via identification $\mathfrak{g}/\mathfrak{b}'\simeq \mathfrak{n}^-$. Here's the crucial property of $\psi\in(\mathfrak{g}/\mathfrak{b}')^*$:

Exercise 1: 1) Show that $B\psi$ is dense in $(\mathfrak{g}/\mathfrak{b}')^*$.
2) Deduce that the functor $M\mapsto M_\psi$ is fully faithful on the full subcategory of $Coh^{B\times\mathbb{G}_m}((\mathfrak{g}/\mathfrak{b}')^*)$ consisting of torsion-free modules.

Now for $\lambda\in\Lambda$, $m\in\mathbb{Z}$ we can consider the Verma module $\Delta_{\nu\hbar}(\lambda,m)\in\mathcal{O}_{\nu\hbar}$ with highest weight vector of weight λ in degree m. The following exercise finishes the proof.

Exercise 2: 1) Use 2) of Exercise 1 to show that Wh is faithful on the full subcategory of $\mathcal{O}_{\nu,\hbar}$ whose objects are $\Delta_{\nu,\hbar}(\lambda, m)$.

2) Deduce that Wh is faithful on the full subcategory of $\mathcal{O}_\nu$ w. objects $\Delta_\nu(\lambda)$ (hint: use the Rees construction) and hence of $\mathcal{O}_\nu^\Delta$.

Sketch of proof of 2): Let $\mathbb{K} = \operatorname{Frac}(R)$. According to Sec 0 we can consider the $\mathbb{K}$-linear category $\mathcal{O}_{\nu,\mathbb{K}}$. It's semisimple by Exercise 1 in Sec 1.

Next it is easy to show that

$\mathrm{Wh}: \mathcal{O}_{\nu,\mathbb{K}} \longrightarrow Z(\mathfrak{g}) \otimes \mathbb{K}\text{-mod}$ is fully faithful.

The next (very formal) exercise finishes the proof.

Premium exercise 3: Deduce that

$\mathrm{Wh}: \mathcal{O}_{\nu,R}^\Delta \longrightarrow Z(\mathfrak{g}) \otimes R\text{-mod}$ is fully faithful from

- $\mathrm{Wh}: \mathcal{O}_\nu^\Delta \longrightarrow \mathrm{Vect}$ is faithful
- $\mathrm{Wh}: \mathcal{O}_{\nu,\mathbb{K}}^\Delta \longrightarrow Z(\mathfrak{g}) \otimes \mathbb{K}\text{-mod}$ is fully faithful.

Hint: Prove that $Wh: \mathcal{O}_{\nu, S} \longrightarrow Z(\mathfrak{g}) \otimes S\text{-mod}$ is faithful for S being any localization of any quotient of R.

Rem: The category $Coh^{B \times \mathbb{G}_m}((\mathfrak{g}/\mathfrak{b})^*)$ that appeared in the proof of 1) is an example of a category from the affine world.

Premium exercise 4: Show that $Wh: \mathcal{O}_\nu \longrightarrow Vect$ is exact.