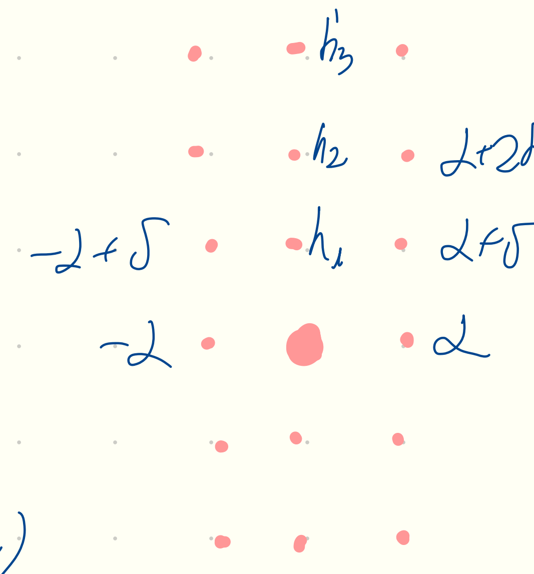
Running example $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$



- Drinfeld-Jimbo presentation

  generators: $E_0, E_1, K_0^{\pm 1}, K_1^{\pm 1}, F_0, F_1$

  relations

- Root generators $E_\beta, F_{-\beta}$ $\beta \in \Delta_+^{(a)}$

  convex order
  $\alpha+\beta$ between α, β

  Braid group action

  PBW basis $E_{-\alpha+\delta}^{a_1} E_{-\alpha+2\delta}^{a_2} \cdots h_1^{b_1} h_2^{b_2} \cdots E_{\alpha+\delta}^{c_2} E_{\alpha}^{c_1}$

- New Drinfeld realization

  $x^-(z)$ $\psi^+(z)$ $\psi^-(z)$ $x^+(z)$

  $\mathcal{U}_q(\hat{n}_-) \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(\hat{n}_+)$

Coproduct and Braid group

For any β define

$$\overline{R}_\beta = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_\beta^n \otimes F_{-\beta}^n$$

$$R_i = R_{\alpha_i}$$

α_i - simple root

$$\overline{R}_\beta^{-1} = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_\beta^n \otimes F_{-\beta}^n$$

Thm $\quad \overline{R}_i \, \Delta(x) \, \overline{R}_i^{-1} = (T_i^{-1} \otimes T_i^{-1}) \Delta(T_i(x)) =: \Delta^{T_i}(x)$

Using this we study $\Delta E_\beta \quad \forall \beta$

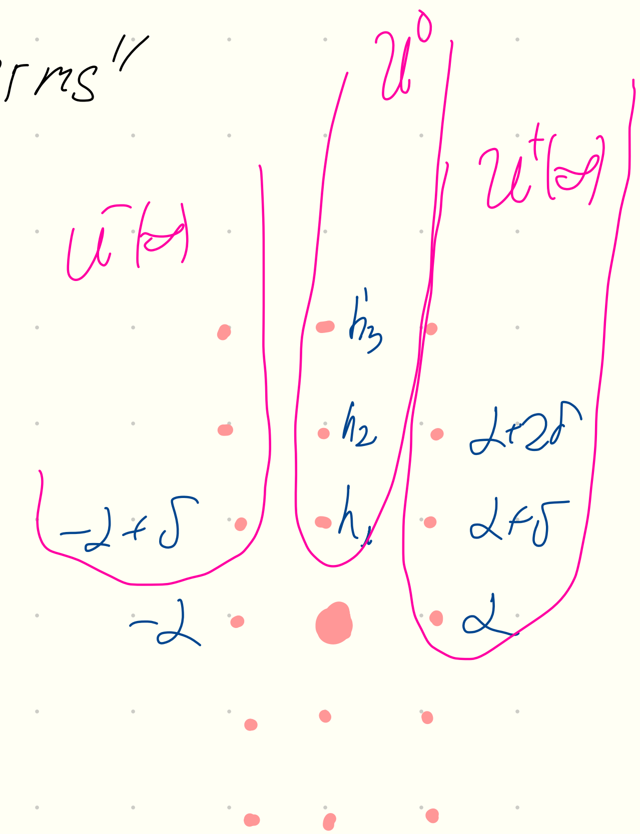
Triangularity property

Lemma (a) $\Delta E_{\alpha+n\delta} = E_{\alpha+n\delta}\otimes K_{\alpha+n\delta} + 1\otimes E_{\alpha+n\delta} +$ "lower terms"

"lower terms" $\in \mathbb{C}(E_{\alpha}, E_{\alpha+\delta}, \dots, E_{\alpha+(n-1)\delta})\otimes \mathcal{U}^+\mathcal{U}^0$

(b) $\Delta h_r = h_r\otimes K_{r\delta} + 1\otimes h_r +$ "lower terms"

"lower terms" $\in \mathcal{U}^+(\succ)\otimes \mathcal{U}^-(\succ)\mathcal{U}^0$



- **Theorem** $\langle F_{\alpha-\delta}^{a_1} F_{\alpha-2\delta}^{a_2} \cdots h_{-1}^{b_1} h_{-2}^{b_2} \cdots F_{-\alpha-\delta}^{c_2} F_{-\alpha}^{c_1},\ E_{-\alpha+\delta}^{a'_1} E_{-\alpha+2\delta}^{a'_2} \cdots h_1^{b'_1} h_2^{b'_2} \cdots E_{\alpha+\delta}^{c'_2} E_{\alpha}^{c'_1} \rangle =$

$$= \prod \delta_{a_n,a'_n} \frac{[a_n]!}{q^{\binom{a_n}{2}}} \langle F_{\alpha-n\delta}, E_{-\alpha+n\delta}\rangle^{a_n}\ \delta_{b_n,b'_n}\, b_n! \langle h_{-n}, h_n\rangle^{b_n}\ \delta_{c_n,c'_n} \frac{[c_n]!}{q^{\binom{c_n}{2}}} \langle F_{-\alpha-n\delta}, E_{\alpha+n\delta}\rangle^{c_n}$$

KhT D

- **Thm** $R = \bar{R}_H \sum \# E_{-J} \otimes F_J$

$R_{-\alpha+r\delta}$

$R_{\alpha+r\delta}$

$$= \bar{R}_H \overrightarrow{\prod_{r\geq 0}} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_{-\alpha+r\delta}^n \otimes F_{\alpha-r\delta}^n \right) \left(\prod_{r>0} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{r(q-q^{-1})}{[2r]}\right)^n h_r^n \otimes h_{-r}^n \right) \overleftarrow{\prod_{r\geq 0}} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_{\alpha+r\delta}^n \otimes F_{-\alpha-r\delta}^n \right)$$

Here $\bar{R}_H = e^{\hbar(\frac{1}{2} H_1 \otimes H_1 + k\otimes d + d \otimes k)}$ with $K_1 = e^{\hbar H_1}$, $K = e^{\hbar k}$, $q = e^{\hbar}$

honest exponent

$$R_{r\delta} = \exp\left(\frac{r(q-q^{-1})}{[2r]} h_r \otimes h_{-r} \right)$$

- We have $\bar{R}_H \, E_\alpha \otimes F_\alpha = (K_\alpha^{-1} E_\alpha \otimes F_\alpha K_\alpha) \bar{R}_H$. Hence

$R = R^- R^0 R^+$ where

$$R^- = \prod_{r>0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} (K_{\alpha - r\delta} E_{-\alpha+r\delta})^n \otimes (F_{\alpha - r\delta} K_{-\alpha+r\delta})^n \right)$$

$$R^0 = R_H \left(\prod_{r>0} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{r(q-q^{-1})}{[2r]} \right)^n h_r^n \otimes h_{-r}^n \right) \qquad R^+ = \prod_{r\geq 0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{\alpha + r\delta} \otimes F^n_{-\alpha - r\delta} \right)$$

Rem For f.d (evaluation) reps $V_1 \otimes V_2$, $k=0$

$(\rho_{V_1} \otimes \rho_{V_2}) R = R_- R_0 R_+$ — Gauss decomposition

Example $R: \ \mathbb{C}^2(u_1) \otimes_\Delta \mathbb{C}^2(u_2) \to \mathbb{C}^2(u_1) \otimes_{\Delta^{op}} \mathbb{C}^2(u_2)$

$$R = f\left(u_1/u_2\right)\begin{pmatrix} 1 & & & \\ & 1 & 0 & \\ & \frac{u_1(q-q^{-1})}{u_2-u_1} & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & \frac{q(u_1-u_2)}{u_1-q^2u_2} & 0 & \\ & 0 & \frac{q^2u_1-u_2}{q(u_1-u_2)} & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & \frac{u_2(q-q^{-1})}{u_2-u_1} & \\ & 0 & 1 & \\ & & & 0 \end{pmatrix}$$

$$= f\left(u_1/u_2\right)\begin{pmatrix} 1 & & & \\ & \frac{q(u_1-u_2)}{u_1-q^2u_2} & \frac{u_2(1-q^2)}{u_1-q^2u_2} & \\ & \frac{u_1(1-q^2)}{u_1-q^2u_2} & \frac{q(u_1-u_2)}{u_1-q^2u_2} & \\ & & & 1 \end{pmatrix}$$

here

$$f(u) = \frac{(u;q^4)_\infty (q^4u;q^4)_\infty}{(q^2u;q^4)_\infty^2}$$

$$(x;p)_\infty = \prod_{k=0}^{\infty}(1-p^k x)$$

(Some details)

$$\mathbb{C}^2 = \langle \xi_+, \xi_- \rangle$$

$$\rho_u(E_{\alpha+r\delta}) = \begin{pmatrix} 0 & u^r \\ 0 & 0 \end{pmatrix} \qquad \rho_u(F_{-\alpha-r\delta}) = \begin{pmatrix} 0 & 0 \\ u^{-r} & 0 \end{pmatrix}$$

$$(\rho_{u_1}\otimes\rho_{u_2})R_+ = (\rho_{u_1}\otimes\rho_{u_2})\left(\prod_{r\geq 0}\left(\sum_{n\geq 0} q^{\binom{n}{2}}\frac{(q-q^{-1})^n}{[n]_q!}E^n_{\alpha+r\delta}\otimes F^n_{-\alpha-r\delta}\right)\right) =$$

$$= (\rho_{u_1}\otimes\rho_{u_2})\left(\prod_{r\geq 0}\left(1+(q-q^{-1})E_{\alpha+r\delta}\otimes F_{-\alpha-r\delta}\right)\right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (q-q^{-1})\sum_{r\geq 0}\left(\frac{u_1}{u_2}\right)^r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{(q-q^{-1})u_2}{u_2-u_1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- For generic $\mathfrak{g}$

$$R = R_H \prod_{\alpha \in \Phi_I^+} \prod_{r>0} \left(\sum_{n \geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{-\alpha+r\delta} \otimes F^n_{\alpha-r\delta} \right) \quad \text{— } q\text{-exp}$$

$$\exp\left(\sum_{r>0} \frac{r(q-q^{-1})}{[r]_q} \widetilde{B}_{i,j}(q^r)\, h_{i,r} \otimes h_{j,-r} \right)$$

$$\prod_{\alpha \in \Phi_I^+} \prod_{r \geq 0} \left(\sum_{n \geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{\alpha+r\delta} \otimes F^n_{-\alpha-r\delta} \right) \quad \text{— } q\text{-exp}$$

here

$\widetilde{B}_{ij}(q)$ inverse to $B(q) = ([a_{ij}]_q)$ — Cartan matrix

<u>Example</u> $\mathfrak{sl}_2$

$$B(q) = \left(\frac{q^2 - q^{-2}}{q - q^{-1}} \right) \qquad \widetilde{B} = \frac{q - q^{-1}}{q^2 - q^{-2}} \qquad \widetilde{B}(q^r) = \frac{q^r - q^{-r}}{q^{2r} - q^{-2r}}$$

K

$F \quad E$

K^{-1}

ψ^+

$x^- \quad x^+ \quad K$

ψ^-

New coproduct

$$\Delta^D X^+(z) = 1 \otimes X^+(z) + X^+(K_{(2)} z) \otimes \psi^-(K_{(2)} z) \qquad \Delta^D K = K \otimes K$$

$$\Delta^D \psi^+(z) = \psi^+(z K_{(2)}^{-1}) \otimes \psi^+(z)$$

$$\Delta^D X^-(z) = X^-(z) \otimes 1 + \psi^+(K_{(1)} z) \otimes X(K_{(1)} z)$$

$$\Delta^D \psi^-(z) = \psi^-(z) \otimes \psi(K_{(1)}^{-1} z)$$

<u>Remark</u> Δ^D is topological

$\Delta^D(x)$ is well defined on $V_1 \otimes V_2$ for $\forall x \in \widehat{sl}_2$

V_1, V_2 – h.w. reps

Thm (kh-T) coproducts

$$(\varsigma T_1)^{-n} \otimes (\varsigma T_1)^{-n} \Delta((\varsigma T_1)^n x)$$ tend to Δ^D

- Corollary For $R^D = \prod_{r\geq 0} \bar{R}_{\alpha+r\delta} \bar{R}_H \prod_{r>0} \bar{R}_{-\alpha+r\delta} \prod_{r>0} \bar{R}_{r\delta}$

R matrix for Drinfeld coproduct

we have $R^D \Delta^D = \Delta^{D,op} R^D$

- $\mathcal{U}_q(\mathfrak{g})$ — $\mathcal{U}(\mathfrak{g})$

$\mathcal{U}_q(\mathfrak{g})$ — $\mathbb{C}[G^*]$ — dual Poisson-Lie group

$\{(L^+, L^-) \mid pr_+ L^+ \, pr_- L^- = 1\} \subset B_+ \times B_-$

$pr_\pm : B_\pm \to H$

Def $U(R)$ – Hopf algebra with generators

$$e^{+}_{ij}[n],\ ,\ e^{-}_{ji}[-n],\qquad 1\le i\le j\le 2$$

$$e^{+}_{ji}[n+1]\qquad e^{-}_{ij}[-n-1]\qquad 1\le i< j\le 2$$

$$e^{\pm}_{ij}=\sum e^{\pm}_{ij}[n]\,z^{-n}$$

$$L^{+}(z)=\begin{pmatrix} e^{+}_{11}(z) & e^{+}_{12}(z)\\ e^{+}_{21}(z) & e^{+}_{22}(z)\end{pmatrix},\qquad L^{-}(z)=\begin{pmatrix} e^{-}_{11}(z) & e^{-}_{12}(z)\\ e^{-}_{21}(z) & e^{-}_{22}(z)\end{pmatrix}$$

- with relations $e^{+}_{ii}[0]\,e^{-}_{ii}[0]=1$

$$R(z/w)\,L^{\pm}_{1}(z)\,L^{\pm}_{2}(w)=L^{\pm}_{2}(w)\,L^{\pm}_{1}(z)\,R(z/w)$$ – RLL relation

$$R\left(K^{-1}\tfrac{z}{w}\right)L^{-}_{1}(z)\,L^{+}_{2}(w)=L^{+}_{2}(w)\,L^{-}_{1}(z)\,R\left(K\tfrac{z}{w}\right)$$

$L_1=L\otimes 1$

$L_2=1\otimes L$

- Coproduct

$$\Delta L^-(z) = (1\otimes L^-(K_{(1)}^{-1} z))(L^-(z)\otimes 1)$$

$$\Delta L^+(z) = (1\otimes L^+(z))(L^+(K_{(2)}^{-1} z)\otimes 1)$$

More explicitly

$$\Delta \ell_{ij}^-(a) = \sum_k \ell_{kj}^-(a)\otimes \ell_{ik}^-(a K_{(1)}^{-1})$$

$$\Delta \ell_{ij}^+(a) = \sum_k \ell_{kj}^+(a K_{(2)}^{-1})\otimes \ell_{ik}^+(a)$$

$$R = \begin{pmatrix} 1 & & & \\ & \frac{q(z-w)}{z-q^2w} & \frac{w(1-q^2)}{z-q^2w} & \\ & \frac{z(1-q^2)}{z-q^2w} & \frac{q(z-w)}{z-q^2w} & \\ & & & 1 \end{pmatrix}$$

[Faddev, Reshetikhin, Takhtajan]
[Reshetikhin, Semenov-Tian-Shansky]
[Ding, Frenkel]

$U_q(\widehat{gl}_2)$

- Algebra generated by

$$K_1^{\pm}(z)=\sum_{\pm r\geq 0} K_{1,r}\, z^{-r}, \quad K_2^{\pm}(z)=\sum_{\pm r\geq 0} K_{2,r}\, z^{-r}, \quad X^{\pm}(z)=\sum_{n\in\mathbb{Z}} X_n^{\pm}\, z^{-n}$$

- Relations

K

- $K_{i,0}^{+} K_{i,0}^{-} = K_{i,0}^{-} K_{i,0}^{+} = 1 \qquad i,j=1,2$
- $K_i^{\pm}(z) K_j^{\pm}(w) = K_j^{\pm}(w) K_i^{\pm}(z)$
- $K_i^{+}(z) K_i^{-}(w) = K_i^{-}(w) K_i^{+}(z)$
- $$\frac{zK^{\pm 1}-w}{zK^{\pm 1}q^{-1}-wq} K_1^{\mp}(z) K_2^{\pm}(w) = K_2^{\pm}(w) K_1^{\mp}(z) \frac{zK^{\mp 1}-w}{zK^{\mp 1}q^{-1}-wq}$$
- $X^{+}(z)X^{+}(w)(z-q^{2}w) + X^{+}(w)X^{+}(z)(w-q^{2}z)=0$
- $X^{-}(z)X^{-}(w)(z-q^{-2}w) + X^{-}(w)X^{-}(z)(w-q^{-2}z)=0$

- $$[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}} \left(K_2^+(z) K_1^+(z)^{-1} \delta\left(\frac{Kw}{z}\right) - K_2^-(w) K_1^-(w)^{-1} \delta\left(\frac{w}{Kz}\right) \right)$$

- $$K_1^-(z) X^+(w) = \frac{zq^{-1} - wKq}{z - wK} X^+(w) K_1^-(z)$$

$$K_2^-(z) X^+(w) = \frac{zq - wKq^{-1}}{z - wK} X^+(w) K_2^-(z)$$

$$K_1^+(z) X^+(w) = \frac{zq^{-1} - wq}{z - w} X^+(w) K_1^+(z)$$

$$K_2^+(z) X^+(w) = \frac{zq - wq^{-1}}{z - w} X^+(w) K_2^+(z)$$

$$K_1^-(z) X^-(w) = \frac{z - w}{zq^{-1} - wq} X^-(w) K_1^-(z)$$

$$K_2^-(z) X^-(w) = \frac{z - w}{zq - wq^{-1}} X^-(w) K_2^-(z)$$

$$K_1^+(z) X^-(w) = \frac{z - wK}{zq^{-1} - wKq} X^-(w) K_1^+(z)$$

$$K_2^+(z) X^-(w) = \frac{z - wK}{zq - wKq^{-1}} X^+(w) K_2^+(z)$$

Gauss decomposition

$$L^+(z) = \begin{pmatrix} e^+_{11}(z) & e^+_{12}(z) \\ e^+_{21}(z) & e^+_{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (q^{-1}-q)\,\mathcal{E}^+(z) & 1 \end{pmatrix} \begin{pmatrix} K_1^+(z) & 0 \\ 0 & K_2^+(z) \end{pmatrix} \begin{pmatrix} 1 & (q^{-1}-q)\,\mathcal{F}^+(z) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} K_1^+(z) & (q^{-1}-q) K_1^+(z)\,\mathcal{F}^+(z) \\ (q^{-1}-q)\,\mathcal{E}^+(z) K_1^+(z) & K_2^+(z) + (q^{-1}-q)^2\,\mathcal{E}^+(z) K_1^+(z)\,\mathcal{F}^+(z) \end{pmatrix}$$

$$L^-(z) = \begin{pmatrix} e^-_{11}(z) & e^-_{12}(z) \\ e^-_{21}(z) & e^-_{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (q-q^{-1})\,\mathcal{E}^-(z) & 1 \end{pmatrix} \begin{pmatrix} K_1^-(z) & 0 \\ 0 & K_2^-(z) \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})\,\mathcal{F}^-(z) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} K_1^-(z) & (q-q^{-1}) K_1^-(z)\,\mathcal{F}^-(z) \\ (q-q^{-1})\,\mathcal{E}^-(z) K_1^-(z) & K_2^-(z) + (q^{-1}-q)^2\,\mathcal{E}^-(z) K_1^-(z)\,\mathcal{F}^-(z) \end{pmatrix}$$

$\mathcal{E}^\pm$, $\mathcal{F}^\pm$, $K_i^\pm$ — half currents

Claim $X^+(w) = \mathcal{E}^+(w) + \mathcal{E}^-(\kappa w)$
$X^-(w) = \mathcal{F}^+(\kappa w) + \mathcal{F}^-(w)$

Compare quadratic relations for half-currents

- For $\mathfrak{gl}_N$, $N>2$ in RLL presentation relations are quadratic while new Drinfeld has Serre rel.
- In RLL generators we have PBW property

- From universal R matrix

Recall $\rho_z : U_q(\hat{sl}_2) \to End(\mathbb{C}^2(z))$

$R = R^- R^0 R^+$ where

$$R^- = \prod_{r>0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} (K_{\alpha - r\delta} E_{-\alpha+r\delta})^n \otimes (F_{\alpha - r\delta} K_{-\alpha+r\delta})^n \right)$$

$$R^0 = R_H \left(\prod_{r>0} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{r(q-q^{-1})}{[2r]} \right)^n h_r^n \otimes h_{-r}^n \right) \qquad R^+ = \prod_{r\geq 0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{\alpha+r\delta} \otimes F^n_{-\alpha-r\delta} \right)$$

- Let $q^{d\otimes k} L^-(z) = (\rho_z \otimes id) R \qquad L^+(z) q^{-k\otimes d} = (id \otimes \rho_z) R^{-1}$

- Factorization $R \rightsquigarrow$ Gauss factorization L^+, L^-

Yang-Baxter $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$. apply $\rho_{u_1} \otimes \rho_{u_2} \otimes id$

$R L^- L^- = L^- L^- R$

RLL relations

(Some difference between $\hat{sl}_2$ and $\widetilde{sl}_2$)

- Coproduct $(\Delta\otimes id)R = R_{13}R_{23}$ hence

$$(\Delta\otimes id)R^{-1} = R_{23}^{-1}R_{13}^{-1}$$

Using $L^+(a)q^{-k\otimes d} = (id\otimes \rho_a)R^{-1}$

$$\Delta L^+(a)q^{-(k_1+k_2)\otimes d} = L_2^+(a)q^{-k_2\otimes d}L_1^+(a)q^{-k_1\otimes d} = L_2^+(a)L_1^+(K_{(2)}a)q^{-(k_1+k_2)d}$$

Hence $\Delta L^+(a) = L_2^+(a)L_1^+(K_{(2)}a)$

In modes $\Delta \ell_{ij}^+(a) = e_{kj}^+(aK_2^{-1})\otimes \ell_{ik}^+(a)$

- Finite coproduct, Drinfeld-Jimbo one

Explicit formula for $\forall$ root vector, triangularity.