Lazy approach to categories $\mathcal{O}$, III.

1) Description of $\mathcal{O}^{\Delta}_{\nu,R,\Xi}$

1.0) Recap

Let $\nu\in\mathfrak{h}^*$, $R=\mathbb{C}[\mathfrak{h}^*]^{\wedge_0}$, $\mathbb{K}=\mathrm{Frac}(R)$, $\iota:\mathfrak{h}\hookrightarrow R$ be the natural inclusion. In Sec 2 of Part 2 we have produced a functor $Wh:\mathcal{O}_{\nu,R}\to Z(\mathfrak{g})\otimes R\text{-mod}$ and proved that:

- It's faithful on $\mathcal{O}^{\Delta}_{\nu}$
- & fully faithful on $\mathcal{O}^{\Delta}_{\nu,R}$

Our goal now is to give a description of the full subcategory $Wh(\mathcal{O}^{\Delta}_{\nu,R,\Xi})\subset Z(\mathfrak{g})\otimes R\text{-mod}$. An additional ingredient is the analysis of subgeneric behavior done in Sec 1 of Part 2.

1.1) Target category. Recall (Sec 2.1 of Part 2) that $Wh(\Delta_{\nu,R}(\lambda))\simeq R$ where $Z(\mathfrak{g})$ acts via:

$$Z(\mathfrak{g})\simeq \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}\hookrightarrow S(\mathfrak{h})\longrightarrow R$$
$$\mathfrak{h}\ni x\mapsto \iota(x)+\langle\lambda+\nu,x\rangle$$

In particular, let $\mathfrak{m}_\Xi \subset Z(\mathfrak{g})$ denote the maximal ideal of $\lambda+\nu$ for $\lambda\in\Xi$ (the same for all such λ). We see that

$$\mathfrak{m}_\Xi \, Wh(\Delta_{\nu,\mathfrak{p}}(\lambda)) \subset Wh(\Delta_{\nu,\mathfrak{p}}(\lambda))\mathfrak{m}.$$

Since every object $M\in \mathcal{O}_{\nu,\mathfrak{p},\Xi}$ has a finite filtration by quotients of $\Delta_{\nu\mathfrak{p}}(\lambda)$, $\lambda\in\Xi$, we have $\mathfrak{m}_\Xi^k Wh(M)\subset Wh(M)\mathfrak{m}$, where k is the length of the filtration.

It follows that $Z(\mathfrak{g}) \curvearrowright Wh(M)$ canonically extends to the completion $Z(\mathfrak{g})^{\wedge\Xi}$ at $\mathfrak{m}_\Xi$.

Now we examine the structure of $\Xi = W\cdot(\lambda+\nu)\cap \nu+\Lambda$ where Λ is the root lattice. Note that for $\lambda\in\Lambda$ we have $w\cdot(\lambda+\nu)\in \nu+\Lambda \Leftrightarrow w\nu-\nu\in\Lambda \Leftrightarrow w\in im[Stab_{W\ltimes\Lambda}(\nu)\hookrightarrow W]$

Since $W\ltimes\Lambda$ is a reflection group, Stab & its image are reflection subgroups to be denoted by $W_{[\nu]}$. Every Ξ is a $W_{[\nu]}$-orbit hence contains a unique element $\bar\lambda=\bar\lambda_\Xi$ s.t. $\bar\lambda+\nu$ is anti-dominant for $W_{[\nu]}$ (for the positive root system consisting of positive roots of W). Let $W^\circ = Span_{W_{[\nu]}}(\bar\lambda+\nu)$.

Note that $Z(\mathfrak{g})^{\wedge\Xi}$ is identified with R^{W°. More precisely, we have the following elementary but important

Exercise 1: 1) the action of $Z(\mathfrak{g})^{\wedge\Xi}$ on $Wh(\Delta_{\nu,R}(\lambda^-))\simeq R$ is via an embedding $Z(\mathfrak{g})^{\wedge\Xi}\hookrightarrow R$ whose image is R^{W°. Denote it by η.

2) The action of $Z(\mathfrak{g})^{\wedge\Xi}$ on $Wh(\Delta_{\nu,R}(w\lambda^-))$ for $w\in W_{[\nu]}$ is via $w\circ\eta$, where we view w as an automorphism of R.

We need to shrink the target category (technical!)

Exercise 2: Use 2) and $\mathcal{O}_{\nu,R,\Xi}$ being highest weight to show $\exists$ an ideal $I\subset R^{W^\circ}\otimes R$

a) $Wh(\mathcal{O}_{\nu R,\Xi})\subset (R^{W^\circ}\otimes R)/I\text{-mod}$

b) $R^{W^\circ}\otimes R/\sqrt{I}=R^{W^\circ}\otimes_{R^W}R$ ($\Rightarrow R^{W_\circ}\otimes R/I$ is fin. gen. over R)

& I is generically radical ($\Leftrightarrow [R^{W_\circ}\otimes R/I]\otimes_R \mathbb{K}\simeq \mathbb{K}^{\oplus|W_\nu/W^\circ|}$)

One can make a much more precise (& elegant) statement - especially if one is Soergel:

Fact: We can take $R^{W^\circ}\otimes R/I=R^{W^\circ}\otimes_{R^W}R$.

Conclusion: we have seen that the target category for Wh as well as images of standards are recovered from a reflection group, $W_{[\nu]}$, and its parabolic subgroup W° (and a reflection representation of $W_{[\nu]}$).

In Sec 1.3 we'll see that a similar claim is true for $Wh(\mathcal{O}^{\Delta}_{\nu,R,\Sigma})$.

1.2) Abstract nonsense

Suppose

- R is a regular complete Noetherian local ring, $\mathbb{F} := R/\mathfrak{m}$.
- $\mathcal{C}_R$ is a highest weight category over R
- $\underline{\mathcal{C}}_R$ is an R-linear abelian category equivalent to $\underline{A}_R\text{-mod}_{fg}$, where $\underline{A}_R$ is an associative R-algebra that is a finitely generated R-module.
- $\pi_R: \mathcal{C}_R \to \underline{\mathcal{C}}_R$ is a right exact R-linear functor

Note that π_R is given by $B_R \otimes_{A_R} \bullet$, where B_R is an $\underline{A}_R$-A_R-bimodule (w $\mathcal{C}_R \simeq A_R\text{-mod}_{fg}$). So for an R-algebra

S, we can consider $A_S := S\otimes_R A_R$, $\underline{A}_S := S\otimes_R \underline{A}_R$, $\mathcal{C}_S = A_S\text{-mod}_{fg}$, $\underline{\mathcal{C}}_S$, $\pi_S := B_S\otimes_{A_S}\bullet$, etc.

π_R is supposed to satisfy the following conditions:

(a) $\mathcal{C}_{\mathbb{K}}, \underline{\mathcal{C}}_{\mathbb{K}}$ are split semisimple $\mathbb{K}$-linear categories & $\pi_{\mathbb{K}}: \mathcal{C}_{\mathbb{K}} \xrightarrow{\sim} \underline{\mathcal{C}}_{\mathbb{K}}$

(b) $\pi_R(\Delta_R(\tau))$ is flat over R & $L_i\pi_R(\Delta_R(\tau)) = 0$ $\forall i>0, \forall \tau$.

(c) $\pi_{\mathbb{F}}$ is faithful on $\mathcal{C}_{\mathbb{F}}^{\Delta}$.

We call π_R an RS (Rouquier-Soergel) functor. For example, take $\mathcal{C}_R = \mathcal{O}_{\nu,R,\Xi}$ & let $\underline{\mathcal{C}}_R = R^{W^\circ}\otimes R/I\text{-mod}$, $\pi_R = Wh$.

Here are consequences of the axioms (a)-(c). First, π_R is fully faithful on $\mathcal{C}_R^{\Delta}$, cf. Premium Exercise 3 from Sec 2.2 of Lec 2. The Yoneda description of Ext^1 then shows that $\pi_R: \mathcal{C}_R^{\Delta} \hookrightarrow \underline{\mathcal{C}}_R$ is injective on Ext^1's.

Moreover, we can recover Ext^1 between objects of $\mathcal{C}_R^{\Delta}$. Since $\underline{\mathcal{C}}_{\mathbb{K}}$ is semisimple there's a divisor $D\subset \mathrm{Spec}(R)$, with the following property:

if $\underline{M}_R, \underline{N}_R \in \underline{\mathcal{C}}_R$ are flat over R, then $\mathrm{Ext}^1_{\underline{\mathcal{C}}_R}(\underline{M}_R, \underline{N}_R)$

is supported on D. Let $\mathfrak{p}_1, \dots \mathfrak{p}_k \subset R$ be the prime ideals corres-ponding to the components of D. Let

$$L(R) := \bigoplus_{i=1}^{k} R_{\mathfrak{p}_i} \text{ – a localization of } R.$$

We have maps $\pi_R : \mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R) \hookrightarrow \mathrm{Ext}^1_{\underline{\mathcal{C}}_R}(\pi_R M_R, \pi_R N_R)$ $\forall M_R, N_R \in \mathcal{C}_R^{\Delta}$, & similarly $\pi_{L(R)}$.

We also have natural maps induced by localization functor L:

$$L : \mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}_{L(R)}}(M_{L(R)}, N_{L(R)}),$$

& similar maps for $\underline{\mathcal{C}}_R$.

Here's the required description of $\mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R)$.

Thm (I.L.23) The following diagram is Cartesian.

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R) \arrow[r, "L"] \arrow[d, hook, "\pi_R"] & \mathrm{Ext}^1_{\mathcal{C}_{L(R)}}(M_{L(R)}, N_{L(R)}) \arrow[d, hook, "\pi_{L(R)}"] \\
\mathrm{Ext}^1_{\mathcal{C}_R}(\underline{M}_R, \underline{N}_R) \arrow[r, "L"] & \mathrm{Ext}^1_{\mathcal{C}_{L(R)}}(\underline{M}_{L(R)}, \underline{N}_{L(R)})
\end{tikzcd}$$

where $\underline{M}_R := \pi_R(M_R)$, etc, & $M_R, N_R \in \mathcal{C}_R^{\Delta}$.

Note that the bottom arrow depends only on $\underline{\mathcal{C}}_R$, while the right arrow only depends on the inclusions

$\mathcal{C}^{\Delta}_{R_{\beta_i}} \hookrightarrow \underline{\mathcal{C}}_{R_{\beta_i}}$. Informally, once we have an RS functor, $\mathcal{C}_R$ is recovered from the target category & its subgeneric behavior.

1.3) Back to $\mathcal{O}$.

As our first application of Sec 1.2 we give a proof of the following result due to Soergel.

Thm: A regular block of $\mathcal{O}_{\nu,\mathbb{Z}}$ (one w. $W^\circ = \{1\}$) is determined (up to an equivalence of highest weight categories) by $W_{[\nu]}$.

There's an immediate generalization to singular blocks, which is proved similarly & is left as premium exercise.

Sketch of proof.

For $w \in W_{[\nu]}$, we write R_w for the R-bimodule R, where R acts from the right by $r \mapsto r$ and from the left by $r \mapsto w(r)$, so that $Wh(\Delta_R(w\cdot\lambda)) = R_w$.

Important (commutative algebra) exercise 1

$\mathrm{Ext}^1_{R\otimes R}(R_u, R_w) \neq 0 \Rightarrow u^{-1}w = 1$ or s_α. Moreover, in the latter case this R-bimodule is

$$R_w / R_w \alpha \simeq R_u / R_u \alpha.$$

Thx to this exercise we can take $\mathcal{D} = \bigcup \mathrm{Spec}(R/(\alpha))$, where the union is taken over the positive roots of $W_{[\lambda]}$. Consider the corresponding localization $\mathcal{O}^{\Delta}_{\lambda, R_{(\alpha)}, \Xi}$. It splits into $|W|/2$ blocks and so does $\underline{\mathcal{C}}_{R_{(\alpha)}}$, the blocks correspond to s_α-orbits in Ξ. The functor $\pi_{R_{(\alpha)}}$ goes between blocks. Let $\mathbb{F}_\alpha$ be the residue field of $R_{(\alpha)}$.

Important exercise 2: Let $\lambda \in \Xi$ satisfy $\langle \lambda + \rho, \alpha^\vee \rangle < 0$. Then $\mathrm{Ext}^1_{\mathcal{O}_{\lambda, R_{(\alpha)}}}(\Delta_{R_{(\alpha)}}(\lambda), \Delta_{R_{(\alpha)}}(s_\alpha \cdot \lambda)) \neq 0$ hence $\mathbb{W}h$ induces isomorphism with $\mathrm{Ext}^1_{\underline{\mathcal{C}}_{R_{(\alpha)}}}(R_{w,(\alpha)}, R_{ws_\alpha,(\alpha)}) = \mathbb{F}_\alpha$ for $\lambda = w \cdot \lambda^-$.

This implies the following characterization of the image of the block: it consists of all objects M s.t. $\exists$ SES

$$0 \longrightarrow R_{ws_\alpha,(\alpha)}^{\oplus ?} \longrightarrow M \longrightarrow R_{w,(\alpha)}^{\oplus ?} \longrightarrow 0$$

(w. $w \in W_{[\nu]}$ shortest in its S_α-coset). Informally: we get all extensions in the right direction and none in the wrong direction. Thm in Sec 1.2 now shows that Ext^1 between two objects in $Wh(\mathcal{O}^\Delta_{\nu,R,\Xi})$ can be fully recovered inside their Ext^1 in $\underline{\mathcal{C}}_R$ without actually knowing $\mathcal{O}^\Delta_{\nu,R,\Xi}$. To finish the proof is left as an exercise $\square$