q-Opers, QQ-systems, Bethe Ansatz

① (G,q)-opers : G - simple, simply-connected complex Lie group

· $G = SL(r+1)$

Consider $M_q : \mathbb{P}^1 \to \mathbb{P}^1$, $\quad q \in \mathbb{C}^\times$
$\qquad z \mapsto z\cdot q$

Def: A meromorphic $(SL(r+1), q)$ - oper on $\mathbb{P}^1$ is $(E, A, \mathcal{L}^\bullet)$,
E - vector bundle of rank $r+1$ over $\mathbb{P}^1$, $\mathcal{L}^\bullet$

$$\mathcal{L}^{r+1} \subset \mathcal{L}^{r} \subset \mathcal{L}^{r-1} \subset \dots \subset \mathcal{L}^{1} = E$$

line

s.t. meromorphic q-connection $A \in \mathrm{Hom}_{\mathcal{O}_U}(E, E^q)$, where U - open Zariski dense subset, E^q - pullback under M_q satisfies

i) $A\,\mathcal{L}^i \subset \mathcal{L}^{i-1}$

· ii) $\bar{A}_i : \mathcal{L}^i/\mathcal{L}^{i+1} \xrightarrow{\sim} \left(\mathcal{L}^{i-1}/\mathcal{L}^{i}\right)^q$ is an isomorphism

restriction on $V = U \cap M_q^{-1}(U)$. $\quad \det A = 1 \Rightarrow SL$ condition.

$$A = \begin{pmatrix} * & * & & 0 \\ & * & * & \\ & \ddots & \ddots & * \\ * & & & * \end{pmatrix}$$

Changing trivialization $g(z) \in SL(r+1)(z)$

$$A(z) \mapsto g(qz)\,A(z)\,g(z)^{-1}$$

$\mathcal{L}^{r+1} = \mathrm{Span}\,(s(z))$

$$\mathcal{W}_i(s)(z) = \left. s(z) \wedge A^{-1}(z)s(qz) \wedge A^{-1}(z)A^{-1}(qz)s(q^2z) \wedge \dots \prod_{k=0}^{i} A^{-1}(q^k z)\cdot s(q^i z) \right|_{\Lambda^i \mathcal{L}^{r+1}}$$

$$i = 2, \dots r+1$$

q-Oper conditions : $\mathcal{W}_i \neq 0$.

Def: · An $(SL(r+1), q)$-oper has regular singularities at roots of $\{\Lambda_i(z)\}_{i=1,\dots r}$
s.t. $\qquad \mathcal{W}_i(s)(z) = \Lambda_i(z)$

roots are q-distinct.

$$\Lambda_i(z) = \prod_{k=1}^{\ell_i} (z - a_{i,k})$$

non-degeneracy : $\dfrac{a_{i,k}}{a_{j,n}} \neq q^{\mathbb{Z}}$

· An $(SL(r+1), q)$-oper is called Z-twisted if $\exists\, g(z) \in SL(r+1)(z)$

s.t. $g(qz)\, A(z)\, g(z)^{-1} = Z^{-1}$, $\quad Z \in H \subset H(z) \subset G(z)$

$$Z = \prod_{i=1}^{r} \zeta_i^{\check{\alpha}_i} \qquad \frac{\zeta_i}{\zeta_j} \neq q^{\mathbb{Z}}$$

Def: Miura $(SL(r+1), q)$-oper is $(E, A, \mathcal{L}^\bullet, \hat{\mathcal{L}}^\bullet)$, where $(E, A, \mathcal{L}^\bullet)$ is an $(SL(r+1), q)$-oper and $\hat{\mathcal{L}}^\bullet$ is preserved by $A(z)$

$$A(z) = \begin{pmatrix} \ddots & \ddots & 0 \\ & \ddots & \ddots \\ 0 & & \ddots \end{pmatrix}$$

$$(\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$$

$$B_+ \backslash G / B_- \simeq W_G$$

$\mathcal{F}_{G,z} \simeq G$

$\mathcal{F}_{B_+,z} \simeq aB_+$, $\quad \mathcal{F}_{B_-,z} \simeq bB_-$

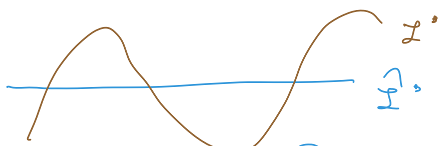
generic relative position, $a^{-1}b = 1$

Generic Miura condition

$$A(z) = \prod_i y_i(z)^{\check{\alpha}_i} \exp \frac{\Lambda_i(z)}{y_i(z)} e_i$$

$\Lambda_i(z) \in \mathbb{C}[z]$

$y_i(z) \in \mathbb{C}(z)$



$\hat{\mathcal{L}}^\bullet$: $e_1, e_2, \dots, e_{r+1}$

$$\mathcal{D}_k(s)(z) = e_1 \wedge \dots \wedge e_{r+1-k} \wedge s(z) \wedge Z s(qz) \wedge \dots \wedge Z^{k-1} s(q^k z)$$

$$\mathcal{D}_k(s)(z) \neq 0$$

$\mathcal{D}_k(s)(z) = \det M_k(z)$

Miura q-oper condition:

$$\mathcal{D}_k(s)(z) = \alpha_k \Lambda_k(z) \cdot V_k(z)$$

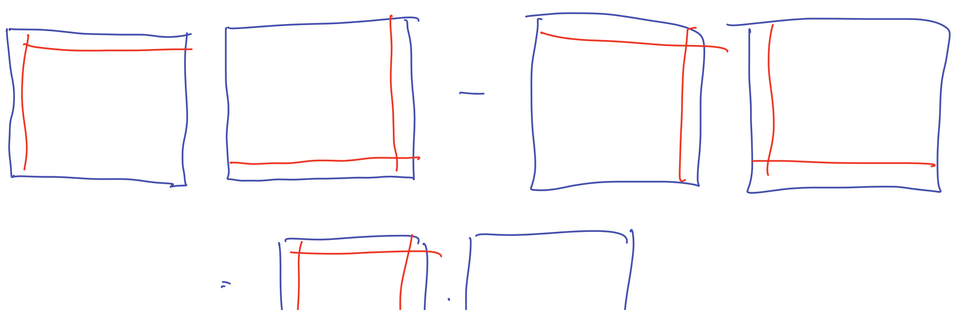
Theorem: Polynomials $V_k(z)$ satisfy the QQ-system: $\quad i = 1 \dots r$

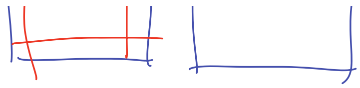
$V_i(z) = Q_i^+(z)$

$$\zeta_{i+1} Q_i^+(qz) Q_i^-(z) - \zeta_i Q_i^+(z) Q_i^-(qz) = \alpha_i \Lambda_i(z) Q_{i-1}^+(z) Q_{i+1}^+(qz),$$

where $Q_i^{\pm}$ can be obtained as minors of $M_i(z)$

Lewis-Carroll identity





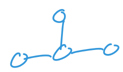
Theorem: $\left\{ \begin{array}{c} \text{Space of} \\ \text{Non-degenerate solutions} \\ \text{of } QQ\text{-system} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Space of} \\ \text{Miura } SL(r+1,q)\text{-opers} \\ \text{on } \mathbb{P}^1 \ Z\text{-twisted} \\ \text{w/ sing. at } \Lambda_i(z) \end{array} \right\}$

For general case, (G,q)-opers:

$$\left\{ \begin{array}{c} QQ\text{-system} \\ \text{for } {}^L G \end{array} \right\} \longleftrightarrow \left\{ \text{Miura } (G,q)\text{-opers} \right\}$$

$$\xi_i Q_i^+(qz) Q_i^-(z) - \tilde{\xi}_i Q_i^+(z) Q_i^-(qz) = \Lambda_i(z) \prod_{j<i} Q_j^+(z)^{-a_{ji}} \cdot \prod_{j>i} Q_j^+(qz)^{-a_{ji}}$$

$$\xi_i = \zeta_i \prod_{j<i} \zeta_j, \qquad \tilde{\xi}_i = \zeta_i^{-1} \prod_{j>i} \zeta_j^{-1}$$



$$Z = \prod_{i=1}^{r} \zeta_i^{\check{\alpha}_i}$$

Generalized minors
[Fomin–Zelevinski]

$$G_0 = N_- H N_+$$
$$g = n_- \cdot h \cdot n_+$$

V_i^+ - irrep of G w/ highest weight ω_i wrt. B_+

$$h\, v^+_{\omega_i} = [h]^{\omega_i} v^+_{\omega_i}$$

Def: Principal minor: $\Delta : G_0 \to \mathbb{C}^\times$

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}$$

Generalized minors: $u, v \in W_G$ – Weyl group of G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\underbrace{\tilde{u}^{-1} g \tilde{v}}_{\in G_0})$$

Theorem ([FZ])

$$\Delta_{u\omega_i, v\omega_i} \cdot \Delta_{uv} - \Delta \cdot \Delta = \prod \Delta^{-a_{ji}}$$

Theorem: Let $A(z) = v(qz)\, Z\, v(z)^{-1}$, $v(z) \in B_+(z)$

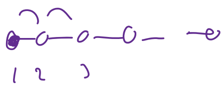
$w \in W_G$

$$\Delta_{w\cdot\omega_i, \omega_i}\left(v^{-1}(z)\right) = Q_i^w(z).$$

$$w = 1 \qquad Q_i^w(z) = Q_i^+(z)$$

$w = s_i \quad Q_i^{w}(z) = Q_{i'}(z)$

$Q_r^{w}(z)$ satisfy generalized QQ-system



$$S(z) = \begin{pmatrix} \vdots \\ Q^-_{1,2,3}(z) \\ Q^-_{1,2}(z) \\ Q^-_{1}(z) \\ Q^+_{1}(z) \end{pmatrix}$$