q-Opers, QQ-systems, Bethe Ansatz

① (G,q)-opers : G - simple, simply-connected complex Lie group

· Type A (can also do $GL(r+1)$)

Consider $M_q : \mathbb{P}^1 \to \mathbb{P}^1$, $\quad q \in \mathbb{C}^\times$

$z \mapsto z \cdot q$

Def: A meromorphic $(SL(r+1), q)$ - oper on $\mathbb{P}^1$ is $(E, A, \mathcal{L}^\bullet)$,

E - vector bundle of rank $r+1$ over $\mathbb{P}^1$, $\mathcal{L}^\bullet$

$$\mathcal{L}^{r+1} \subset \mathcal{L}^{r} \subset \mathcal{L}^{r-1} \subset \dots \subset \mathcal{L}^{1} = E$$

line

s.t. meromorphic q-connection $A \in \mathrm{Hom}_{\mathcal{O}_U}(E, E^q)$, where U - open Zariski dense subset, E^q - pullback under M_q satisfies

i) $A\,\mathcal{L}^i \subset \mathcal{L}^{i-1}$

ii) $\bar{A}_i : \mathcal{L}^i / \mathcal{L}^{i+1} \xrightarrow{\sim} \left(\mathcal{L}^{i-1}/\mathcal{L}^{i}\right)^q$ is an isomorphism

restriction on $V = U \cap M_q^{-1}(U)$. $\quad \det A = 1 \Rightarrow SL$ condition.

$$A = \begin{pmatrix} * & & & 0 \\ * & * & & \\ & \ddots & \ddots & \\ * & & * & * \end{pmatrix}$$

Changing trivialization $g(z) \in SL(r+1)(z)$

$$A(z) \mapsto g(qz)\, A(z)\, g(z)^{-1}$$

$\mathcal{L}^{r+1} = \mathrm{Span}\,(s(z))$

$$\mathcal{W}_i(s)(z) = \left. s(z) \wedge A^{-1}(z) s(qz) \wedge A^{-1}(z) A^{-1}(qz) s(q^2 z) \wedge \dots \prod_{k=0}^{i} A^{-1}(q^k z) \cdot s(q^i z) \right|_{\Lambda^i \mathcal{L}^{r+1}}$$

$i = 2, \dots r+1$

q-Oper conditions : $\mathcal{W}_i \neq 0$.

Def: An $(SL(r+1), q)$-oper has regular singularities at roots of $\{\Lambda_i(z)\}_{i=1,\dots r}$

s.t. $\quad W_i(s)(z) = \Lambda_i(z)$

roots are q-distinct.

$$\Lambda_i(z) = \prod_{k=1}^{\ell_i} (z - a_{i,k})$$

non-degeneracy : $\dfrac{a_{i,k}}{a_{j,n}} \neq q^{\mathbb{Z}}$

· An $(SL(r+1), q)$-oper is called Z-twisted if $\exists\, g(z) \in SL(r+1)(z)$

s.t. $\quad g(qz)\, A(z)\, g(z)^{-1} = Z^{-1}, \qquad Z \in H \subset H(z) \subset G(z)$

$$Z = \prod_{i=1}^{r} \zeta_i^{\check{\alpha}_i} \qquad \frac{\zeta_i}{\zeta_j} \neq q^{\mathbb{Z}}$$

<u>Def</u>: Miura $(SL(r+1), q)$-oper is $(E, A, \mathcal{L}^\bullet, \hat{\mathcal{L}}^\bullet)$, where $(E, A, \mathcal{L}^\bullet)$ is an $(SL(r+1), q)$-oper and $\hat{\mathcal{L}}^\bullet$ is preserved by $A(z)$

$$A(z) = \begin{pmatrix} & & 0 \\ & \diagdown\diagdown & \\ 0 & & \end{pmatrix}$$

$$(\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$$

$$B_+ \backslash G / B_- \simeq W_G$$

$$\mathcal{F}_{G,z} \simeq G$$

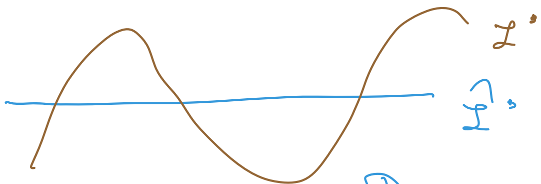
$$\mathcal{F}_{B_+,z} \simeq aB_+, \quad \mathcal{F}_{B_-,z} \simeq bB_-$$

generic relative position, $\quad a^{-1}b = 1$

Generic Miura condition

$$A(z) = \prod_i y_i(z)^{\check{\alpha}_i} \exp \frac{\Lambda_i(z)}{y_i(z)} e_i$$

$$\Lambda_i(z) \in \mathbb{C}[z]$$
$$y_i(z) \in \mathbb{C}(z)$$



$\hat{\mathcal{L}}^\bullet: \quad e_1, e_2, \dots, e_{r+1}$

$$D_k(s)(z) = e_1 \wedge \dots \wedge e_{r+1-k} \wedge s(z) \wedge Z\, s(qz) \wedge \dots \wedge Z^{k-1} s(q^k z)$$

$$D_k(s)(z) \neq 0$$

$D_k(s)(z) = \det M_k(z)$

Miura q-oper condition:

$$D_k(s)(z) = \alpha_k \Lambda_k(z) \cdot V_k(z)$$

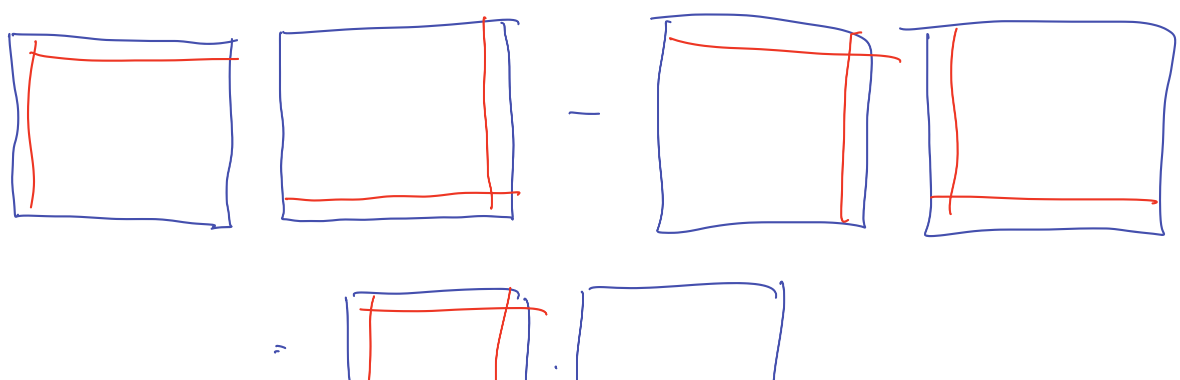
<u>Theorem</u>: Polynomials $V_k(z)$ satisfy the QQ-system:

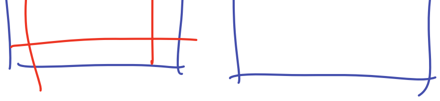
$$V_i(z) = Q_i^+(z) \qquad i = 1 \dots r$$

$$\zeta_{i+1} Q_i^+(qz)\, Q_i^-(z) - \zeta_i Q_i^+(z)\, Q_i^-(qz) = \alpha_i \Lambda_i(z)\, Q_{i-1}^+(z)\, Q_{i+1}^+(qz),$$

where $Q_i^\pm$ can be obtained as minors of $M_i(z)$

Lewis-Carroll identity





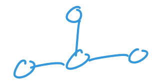
Theorem: $\left\{\begin{array}{c}\text{Space of} \\ \text{Non-degenerate solutions} \\ \text{of } QQ\text{-system}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Space of} \\ \text{Miura } SL(r+1,q)\text{-opers} \\ \text{on } \mathbb{P}^1 \text{ } Z\text{-twisted} \\ \text{w/ sing. of } \Lambda_i(z)\end{array}\right\}$

For general case, (G,q)-opers:

$$\left\{\begin{array}{c} QQ\text{-system} \\ \text{for } {}^L G\end{array}\right\} \longleftrightarrow \left\{ \text{Miura } (G,q)\text{-opers} \right\}$$

$$\xi_i \, Q_i^+(qz)\, Q_i^-(z) - \tilde{\xi}_i \, Q_i^+(z)\, Q_i^-(qz) = \Lambda_i(z) \prod_{j<i} Q_j^+(z)^{-a_{ji}} \cdot \prod_{j>i} Q_j^+(qz)^{-a_{ji}}$$

$$\xi_i = \zeta_i \prod_{j<i} \zeta_j, \qquad \tilde{\xi}_i = \zeta_i^{-1} \prod_{j>i} \zeta_j^{-1}$$



$$Z = \prod_{i=1}^{r} \zeta_i^{\alpha_i^\vee}$$

Generalized minors [Fomin-Zelevinski]

$G_0 = N_- H N_+$, $g = n_- \cdot h \cdot n_+$

V_i^+ - irrep of G w/ highest weight ω_i wrt. B_+

$$h\, v^+_{\omega_i} = [h]^{\omega_i} v^+_{\omega_i}$$

Def: Principal minor: $\Delta: G_0 \to \mathbb{C}^\times$

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}$$

Generalized minors: $u, v \in W_G$ - Weyl group of G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\underbrace{\tilde{u}^{-1} g\, \tilde{v}}_{\text{lift}})$$

Theorem ([FZ])

$$\Delta_{u\omega_i, v\omega_i} \cdot \Delta_{uv} - \Delta \cdot \Delta = \prod \Delta^{-a_{ji}}$$

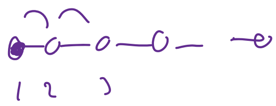
Theorem: Let $A(z) = v(qz)\, Z\, v(z)^{-1}$, $v(z) \in B_+(z)$

$w \in W_G$ $\qquad \Delta_{w\cdot\omega_i, \omega_i}\left(v^{-1}(z)\right) = Q_i^w(z)$.

$w = 1 \qquad Q_i^1(z) = Q_i^+(z)$

$w = s_i \quad Q_i^{w}(z) = Q_i(z)$

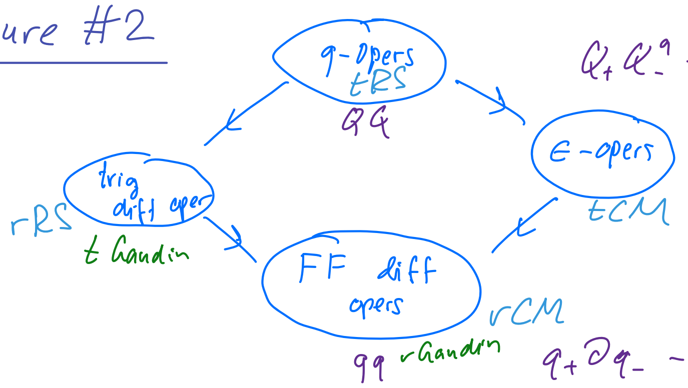
$Q_r^{w}(z)$ satisfy generalized QQ-system



$$S(z) = \begin{pmatrix} \vdots \\ Q^-_{1,2,3}(z) \\ Q^-_{1,2}(z) \\ Q^-_1(z) \\ Q^+_1(z) \end{pmatrix}$$

Lecture #2

[K] 2312. ...



$Q_+ Q_-^q - Q_+^q Q_- = 1$ $\quad M_q: \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto qz$ $\quad q = e^{\epsilon}$

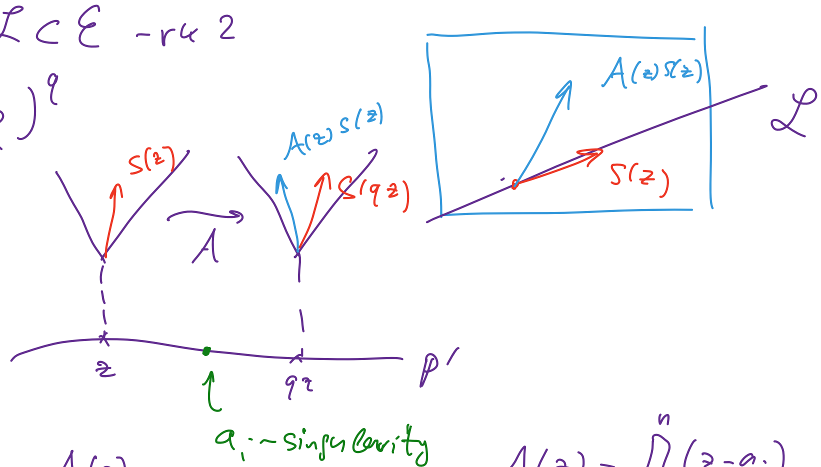
$M_\epsilon: z \mapsto z + \epsilon$

$q_+ \partial q_- - q_- \partial q_+ = 1$

Ld $G = SL(2)$ $\quad \mathcal{L} \subset \mathcal{E}$ - rk 2

$\bar{A}: \quad \mathcal{L} \xrightarrow{\sim} (\mathcal{E}/\mathcal{L})^q$

$\mathcal{L} = \operatorname{Span} s(z)$



$$\underbrace{s(qz) \wedge A(z)\, s(z)}_{Z} = \alpha \Lambda(z) \qquad \Lambda(z) = \prod_{i=1}^{n} (z - a_i)$$

$$S(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}$$

Z-twisted: $\exists\, g(z) \in G(z)$ s.t.

$$g(qz)\, A(z)\, g(z)^{-1} = \underbrace{Z}_{\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}} \subset H \subset H(z)$$

$$\begin{vmatrix} Q_+(zq) & \zeta\, Q_+(z) \\ Q_-(zq) & \zeta^{-1} Q_-(z) \end{vmatrix} = \alpha \Lambda(z)$$

$$\zeta^{-1} Q_+(qz)\, Q_-(z) - \zeta\, Q_+(z)\, Q_-(qz) = \alpha \Lambda(z) \quad \leftarrow QQ\text{-system}$$

- Bethe Ansatz: $\quad Q_+(z) = \prod_{i=1}^{k} (z - w_i)$

  $U_q(\widehat{sl}_2)$

Evaluate QQ-system at roots of $Q_+(z)$:

$$\zeta^{-1} Q_+(q w_i)\, Q_-(w_i) = \alpha \Lambda(w_i)$$

Shift by q^{-1}

$$\zeta^{-1} Q_+(z)\, Q_-(zq^{-1}) - \zeta\, Q_+(zq^{-1}) Q_-(z) = \alpha \Lambda(zq^{-1})$$

$$-\zeta\, Q_+(q^{-1}w_i)\, Q_-(w_i) = \alpha\, \Lambda(w_i q^{-1})$$

$$-\zeta^{-2}\prod_{j=1}^{k}\frac{q w_i - w_j}{q^{-1}w_i - w_j} = \prod_{j=1}^{n}\frac{w_i - a_j}{q^{-1}w_i - a_j}$$

$\leftarrow$ XXZ Bethe equations for sl_2 spin chain on n sites w/ k excitations

$\uparrow \to \mathbb{C}^2(a_i)$

$\mathbb{C}^2(a_1)\otimes \dots \otimes \mathbb{C}^2(a_n)$

- Many-Body system

trigonometric Ruijsenaars-Schneider (tRS) $\leftarrow$ $(M, T, u, v) \sim (g^{-1}Mg, g^{-1}Tg, gu, gv)$

$(GL(2), q)$-oper

$$qMT - TM = u\otimes v^T \quad (*)$$

$M = \mathrm{diag}(m_1 \dots m_n)$

T – Lax matrix for tRS
Macdonald

$\det(z-T) = \sum (-1)^i z^i H_{n-i}$

$$\begin{vmatrix} Q_+(zq) & \zeta_1 Q_+(z) \\ Q_-(zq) & \zeta_2 Q_-(z) \end{vmatrix} = \alpha\,\Lambda(z)$$

$Q_+(z) = z - p_1$

$Q_-(z) = z - p_2$

$$\det\underbrace{\begin{pmatrix} qz - p_1 & \zeta_1 z - \zeta_1 p_1 \\ qz - p_2 & \zeta_2 z - \zeta_2 p_2 \end{pmatrix}}_{= M(z)} = \det\Big(z\overbrace{\begin{pmatrix} q & \zeta_1 \\ q & \zeta_2 \end{pmatrix}}^{V} + M(0)\Big) = \underset{\det V}{\underset{\downarrow}{\alpha}}\,\Lambda(z)$$

$$\det\big(z + V^{-1}M(0)\big) = \Lambda(z) = (z-a_1)\cdot(z-a_2)$$

Ex: $-V^{-1}M(0) = T$ from $(*)$

$$T_{ij} = \frac{\prod_k (\zeta_k - q\zeta_j)}{\prod_{k\neq j}(\zeta_k - \zeta_j)}\, p_i$$

$n=2$

$$\det(z - T) = z^2 - \underbrace{\left(\frac{\zeta_1 - q\zeta_2}{\zeta_1 - \zeta_2}p_1 + \frac{\zeta_2 - q\zeta_1}{\zeta_2 - \zeta_1}p_2\right)}_{= T_1} z + \underbrace{p_1 p_2}_{= T_2}$$

$$\begin{cases} T_1 = a_1 + a_2 \\ T_2 = a_1 \cdot a_2 \end{cases}$$

(G, q)-opers: $\mathcal{F}_G$ – principal G-bundle over $\mathbb{P}^1$, $M_q : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto qz$

q-connection $A \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

U – Zariski dense open subset

Restrict A on $U \cap M_q^{-1}(U) = V$

$A(z) \in G(z)$

q-gauged transformations $\quad A(z) \mapsto g(zq)\, A(z)\, g(z)^{-1}, \qquad g(z) \in G(z)$

$$(\mathcal{F}_G, A) \leftrightarrow G(z)/\sim \text{gauge}$$

Def: A meromorphic (G,q)-oper on $\mathbb{P}^1$ is $(\mathcal{F}_G, A, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, A)$ is a q-connection, $\mathcal{F}_{B_-}$ is reduction of $\mathcal{F}_G$ to B_+:

$$A\big|_V : \mathcal{F}_G \to \mathcal{F}_G^{\,q} \text{ takes values in } B_+(\mathbb{C}[U]) \cdot c \cdot B_+(\mathbb{C}[U])$$

c - Coxeter element.

$$A(z) = n'(z) \prod_i \left(\phi_i(z)^{-\check{\alpha}_i} s_i\right) n(z), \qquad n', n \in N_+(z)$$

s_i simple refl.

$N_+(z) = [B_+, B_+](z)$

$\phi_i(z) \in \mathbb{C}[z]$

Def: A Miura (G,q)-oper is $(\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_+})$ is as above, and $\mathcal{F}_{B_-}$ - restriction of $\mathcal{F}_G$ on B_- that is preserved by A.

$\mathcal{F}_{B_+}, \mathcal{F}_{B_-}$ are in generic relative position at $x \in \mathbb{P}^1$ if

[FKS2]

$$\mathcal{F}_{G,x} \simeq G$$
$$\mathcal{F}_{B_+,x} = a B_+ \subset G$$
$$\mathcal{F}_{B_-,x} = b B_- \subset G$$

$a^{-1}b$ is in the Bruhat cell

$$B_- \backslash G / B_+ \simeq W_G$$
$$a^{-1}b \mapsto 1.$$

Def: Add singularities:

$$A(z) = n'(z) \prod_i \left(\Lambda_i(z)^{-\check{\alpha}_i} s_i\right) n(z)$$

Λ_i polynomials $\forall\, i = 1, \dots, r$

$$A(z) = \prod_i g_i(z)^{-\check{\alpha}_i} \exp \frac{\Lambda_i(z)}{g_i(z)} f_i$$

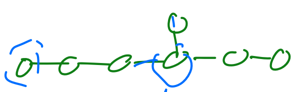
Def: Z-twisted: $\quad \exists\; g(z) \in G(z)$ s.t. $g(qz) A(z) g(z)^{-1} = Z = \prod_{i=1}^{r} \zeta_i^{-\check{\alpha}_i}$,

+ Miura $\quad B(z)$

$\zeta_i \in \mathbb{C}^\times$

$\zeta_i / \zeta_j \notin q^{\mathbb{Z}}$

Miura-Plücker (G,q)-opers



A_i

Let ω_i - ith fund. weight of G

V_i^- - irrep of G w.r.t. $-\omega_i$ - lowest weight w.r.t. B_-, $\quad v_{\omega_i}^-$ - lowest weight vector

L_i^- - Span$(v_{\omega_i}^-)$ $\quad$ vector $e_i v_{\omega_i}^-$ has weight $-\omega_i + \alpha_i$

$W_i = \mathrm{Span}\,(v^-_{\omega_i},\ e_i v^-_{\omega_i})$
$\cap$
V_i^-

Associated bundles:

$\mathcal{V}_i = \mathcal{F}_{B_-} \underset{B_-}{\times} V_i^-$

$\mathcal{W}_i = \mathcal{F}_{B_-} \underset{B_-}{\times} W_i$

$\mathcal{L}_i = \mathcal{F}_{B_-} \underset{B_-}{\times} L_i^-$

Let A_i – q-connections for $\mathcal{W}_i$, $i = 1 \dots r$

$(GL(2), q)$-opers: $A_i : \mathcal{L}_i \hookrightarrow \left(\mathcal{W}_i / \mathcal{L}_i\right)^q$

Def: A Z-twisted Miura-Plücker (G,q)-oper is a (G,q)-oper s.t. $\exists\ v(z) \in B_-(z)$:

$$A_i(z) = v(qz)\, Z\, v(z)^{-1}\Big|_{W_i} = v_i(qz)\, Z_i\, v_i(z)^{-1},$$

$v_i(z) = v(z)\big|_{W_i}$, $Z_i = Z\big|_{W_i}$

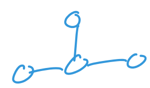
Th: $\{$space of Z-twisted Miura-Plücker (G,q)-opers$\}$

$\Updownarrow$

$\{$solutions of QQ-system for ${}^L G$:

$$\xi_i\, Q_i^+(qz)\, Q_i^-(z) - \tilde{\xi}_i\, Q_i^+(z)\, Q_i^-(qz) = \Lambda_i(z) \prod_{j<i} Q_j^+(z)^{-a_{ji}} \cdot \prod_{j>i} Q_j^+(qz)^{-a_{ji}}$$

$$\xi_i = \zeta_i \prod_{j<i} \zeta_j, \quad \tilde{\xi}_i = \zeta_i^{-1} \prod_{j>i} \zeta_j^{-1} \}$$



Bäcklund-like transformations

$$A^{(i)}(z) = e^{\mu_i(qz) e_i} A(z)\, e^{-\mu_i(z) e_i}, \qquad \mu_i(z) = \frac{\prod_{j \neq i} Q_+^j(z)^{-a_{ji}}}{Q_+^i(z)\, Q_-^i(z)}$$

$$A(z) = \prod_i g_i(z)^{-\check{\alpha}_i} \exp \frac{\Lambda_i(z)}{g_i(z)} f_i, \qquad g_i(z) = \zeta_i \frac{Q_i^+(qz)}{Q_i^+(z)}$$

QQ system: $Q_+^i(z) \mapsto Q_-^i(z)$

$Z \mapsto s_i(Z)$

We can produce a Bäcklund-type transform $\forall\ w \in W_G$

$\{Q_+^{i,w}\}_{\substack{i=1\dots r \\ w \in W_G}} \Rightarrow$ full QQ system

Assume that $w = C \Rightarrow$ everything is okay :)

Th: $\forall\ W_G$ - generic Z-twisted MP (G,q)-oper is a nondegenerate Z-twisted Miura (G,q)-oper.