EXERCISE SHEET. MICROLOCALIZATION

1. Completed Rees algebra

Let $\mathcal{A}$ be a $\mathbb{Z}$-filtered algebra. Set $R_h(\mathcal{A}) := \bigoplus_{i \in \mathbb{Z}} \mathcal{A}/h^i \subset \mathcal{A}[h^{\pm 1}]$.

a) Show that $R_h(\mathcal{A})$ is a graded $\mathbb{C}[h]$-subalgebra in $\mathcal{A}[h^{\pm 1}]$ (the degree is with respect to $h$). Identify $R_h(\mathcal{A})/(h)$ with $\text{gr} \mathcal{A}$ and $R_h(\mathcal{A})/(h - a)$ with $\mathcal{A}$ for $a \neq 0$.

We also consider the completed Rees algebra $R_h^\wedge(\mathcal{A})$, the $h$-adic completion of $R_h(\mathcal{A})$, so that $R_h^\wedge(\mathcal{A})$ is complete and separated in the $h$-adic topology and carries a $\mathbb{C}^\times$-action by $\mathbb{C}$-algebra automorphisms with $t.h = th$. This action is rational on all quotients mod $h^k$.

Now let $\mathcal{A}_h$ be a $\mathbb{C}[[h]]$-algebra that is complete and separated in the $h$-adic topology that comes equipped with a $\mathbb{C}^\times$-action by $\mathbb{C}$-algebra automorphisms that is rational on all quotients $\mathcal{A}_h/(h^k)$ and satisfying $t.h = th$. If $A := \mathcal{A}_h/(h)$ is commutative and finitely generated, we will call $\mathcal{A}_h$ a graded formal quantization of $A$. We define $\mathcal{A}_{h,\text{fin}}$ as the span of all elements $a \in \mathcal{A}_h$ with $t.a = t'a$ for some $i \in \mathbb{Z}$.

b) Prove that $\mathcal{A}_{h,\text{fin}}$ is a graded subalgebra of $\mathcal{A}_h$ that is dense in the $h$-adic topology and satisfies $\mathcal{A}_{h,\text{fin}}/(h) = A$.

c) Prove that $\mathcal{A}_{h,\text{fin}}/(h - 1)$ is a filtered quantization of $A$.

d) Prove that the maps $A \mapsto R_h(\mathcal{A})$ and $\mathcal{A}_h \mapsto \mathcal{A}_{h,\text{fin}}/(h - 1)$ are mutually inverse bijections between filtered quantizations and graded formal quantizations.

2. (Micro)Localization for formal quantizations

Let $\mathcal{A}_h$ be a formal quantization of $A$ (we do not require the presence of $\mathbb{C}^\times$-actions/gradings, $A$ is just required to be a finitely generated commutative algebra). We are going to sheafify $\mathcal{A}_h$ in the Zariski topology on $\text{Spec}(A)$.

a) Let $f \in A$ be a nonzero divisor and let $\hat{f} \in \mathcal{A}_k := \mathcal{A}_h/(h^k)$ be a lift of $f$. Show that $[\hat{f}, \cdot]^k = 0$ and deduce from here that every left fraction by $\hat{f}$ is also a right fraction. Show that the localization $\mathcal{A}_k[\hat{f}^{-1}]$ (defined by the same universality property as in the commutative case) makes sense and is independent of the choice of the lift. We will denote this localization by $\mathcal{A}_k[\hat{f}^{-1}]$.

b) Show that the algebras $\mathcal{A}_k[\hat{f}^{-1}]$ form an inverse system. Further show that $\mathcal{A}_h[\hat{f}^{-1}] := \varprojlim_{k \to \infty} \mathcal{A}_k[\hat{f}^{-1}]$ is a formal quantization of $A[\hat{f}^{-1}]$.

c) Establish a natural homomorphism $\mathcal{A}_h[\hat{f}^{-1}] \to \mathcal{A}_h[(fg)^{-1}]$.

d) Show that $\mathcal{A}_h$ naturally sheafifies to a sheaf $\mathcal{D}_h$ on $\text{Spec}(A)$. Show that $\Gamma(\mathcal{D}_h) = \mathcal{A}_h$.

Note that if $\mathcal{A}_h$ is graded, then $\mathcal{A}_h[\hat{f}^{-1}]$ is graded provided $f$ is $\mathbb{C}^\times$-semiinvariant. So we can get the microlocalization of $\mathcal{A}_{h,\text{fin}}/(h - 1)$ by taking the sheaf $\mathcal{D}_{h,\text{fin}}/(h - 1)$ that makes sense in the conical topology.

e) Work out the details.

f) Prove that $A[\hat{f}^{-1}]$ is a flat module over $\mathcal{A}$. 

3. **Coherent modules over formal quantizations**

Let $\mathcal{D}_h$ be a formal quantization of a Poisson scheme $X$. We say that a $\mathcal{D}_h$-module $M_h$ is coherent if it is complete and separated in the $h$-adic topology and $M_h/hM_h$ is a coherent sheaf on $X$.

a) Suppose that there is an open covering $X = \bigcup_i X^i$ such that $M_h|_{X^i}$ is coherent. Then $M_h$ is coherent.

Now suppose that $X$ comes with a $\mathbb{C}^\times$-action as before and $\mathcal{D} := \mathcal{D}_{h,fin}/(h - 1)$.

b) Show that a filtered $\mathcal{D}$-module $M$ is coherent with a good filtration if and only if $R^\wedge_h(M)$ is a coherent $\mathcal{D}_h$-module.

4. **Microlocalization of modules**

Let $\mathcal{A}$ be a filtered quantization of $A$ and $f \in A$. Our goal here is to define the localization functor $M \mapsto M[f^{-1}]$.

a) Assume that $M$ is equipped with a good filtration. Emulate the procedure in Exercise 2 to define $M[f^{-1}]$ and check that this space has a natural $\mathcal{A}[f^{-1}]$-module structure. Furthermore check that there is a natural isomorphism $\mathcal{A}[f^{-1}] \otimes_{\mathcal{A}} M \xrightarrow{\sim} M[f^{-1}]$ so that $M[f^{-1}]$ is independent of the choice of a filtration.

b) Check that the functor $M \mapsto M[f^{-1}]$ is exact.

c) Check that $M$ sheafifies in the conical topology on $X := \text{Spec}(A)$. 