AFFINE WEYL GROUPS

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1. AFFINE DYNKIN DIAGRAMS

Let \( g \) be a simple finite dimensional Lie algebra. Let \( D \) be the corresponding Dynkin diagram. Let \( W \) be the Weyl group of \( g \). Consider the lattice \( \Lambda^\vee_r \subset \mathfrak{h} \) generated by the simple coroots \( \alpha^\vee_i \). We can form the semi-direct product \( W^\vee_a := W \rtimes \Lambda^\vee_r \). Note that it acts on \( \mathfrak{h} \) by affine transformations.

This group will be called the affine Weyl group of \( g \). For \( \nu \in \Lambda^\vee_r \) we denote by \( t_\nu \in W^\vee_a \) the corresponding element of \( W^\vee_a \).

Example 1.1. Consider the case \( g = \mathfrak{sl}_2 \). In this case we have \( W = S_2 \) and \( \Lambda^\vee_r = 2\mathbb{Z} \subset \mathfrak{h} = \mathbb{C} \), where we identify \( \alpha^\vee \) with 2. The element \( (12) \in S_2 \) acts via \( x \mapsto -x \). Then \( W^\vee_a \) consists of transformations of the form \( x \mapsto \pm x + 2k \) for \( k \in \mathbb{Z} \). Note that it is the Coxeter group with simple reflections \( s_0, s_1 \) given by \( s_1(x) = -x, s_0(x) = 2 - x \).

2. ALCOVES

The affine action of \( W^\vee_a \)-action on \( \mathfrak{h} \) preserves the real form \( \Lambda^\vee_r \) spanned by the coroots.

Definition 2.1. By an affine root hyperplane in \( \Lambda^\vee_r \) we mean a hyperplane of the form \( \langle \alpha, \cdot \rangle = n \) for a root \( \alpha \) and \( n \in \mathbb{Z} \). By an open alcove we mean a connected component of \( \Lambda^\vee_r \) with all affine root hyperplanes removed. By an alcove we mean the closure of an open alcove, this is a simplex. The fundamental alcove \( A^+ \) is one given by \( \langle \alpha_i, \cdot \rangle \geq 0, i = 1, \ldots, r \) and \( \langle \alpha_0, \cdot \rangle \geq -1 \), where \( \alpha_0 \) denotes the minimal negative root of \( g \).

Example 2.2. For \( g = \mathfrak{sl}_2 \), the affine root hyperplanes are integers (we view \( \alpha^\vee \) as 2 so \( \alpha_0 = -\alpha = -1 \)). The alcoves are the intervals of the form \([n, n+1]\), where \( n \) is an integer. The fundamental alcove is \([0, 1]\).

Exercise 2.3. The \( W^\vee_a \)-action permutes affine root hyperplanes, hence alcoves.
Proposition 2.4. \( W^\vee a \) in its action on \( \Lambda_\mathbb{K}^\vee \) coincides with the group generated by reflections along affine root hyperplanes. In particular, \( W^\vee a \) is a Coxeter group.

Proof. The reflection along \( \langle \alpha, \cdot \rangle = n \) is \( x \mapsto x - (\langle \alpha, x \rangle - n)\alpha^\vee \), it lies in \( W^\vee a \). This equality also easily shows that \( t_\alpha^\vee \) lies in the group generated by reflections. Hence we see that the two groups of affine transformations coincide. \( \square \)

Corollary 2.5. \( W^\vee a \) permutes the alcoves simply transitively.

Let \( s_1, \ldots, s_r \) denote the reflections along the corresponding walls of the fundamental alcove. These are the simple reflections in \( W^\vee a \).

Let us give a formula for the length function \( \ell: W^\vee a \to \mathbb{Z}_{\geq 0} \) (for \( u \in W^\vee a \), the length \( \ell(u) \) is the number of affine root hyperplanes separating \( A^+ \) and \( u(A^+) \)) in terms of our initial presentation.

Proposition 2.6. For \( w \in W, \nu \in \Lambda_\mathbb{K}^\vee \) we have

\[
\ell(wt_\nu) = \sum_{\alpha \in \Delta_+} |\langle \alpha, \nu \rangle| + \sum_{\alpha \in \Delta_-, w(\alpha) \in \Delta_-} |1 + \langle \alpha, \nu \rangle|.
\]

Example 2.7. For \( g = sl_2 \) we have

\[
l((s_0s_1)^n) = 2n = |\langle \alpha, n\alpha^\vee \rangle| = l(t_{n\alpha^\vee}),
\]

\[
l(s_1(s_0s_1)^n) = |2n + 1| = |1 + \langle \alpha, n\alpha^\vee \rangle| = l(s_1t_{n\alpha^\vee}).
\]

3. Extended affine Weyl group

Let \( \Lambda^\vee \) be the coweight lattice of \( g \), in particular \( \Lambda_\mathbb{K}^\vee \subset \Lambda^\vee \).

Definition 3.1. The extended affine Weyl group \( \tilde{W}^\vee a \) is \( W \rtimes \Lambda^\vee \).

The group \( \tilde{W}^\vee a \) contains \( W^\vee a \) as a normal subgroup and still acts on \( \Lambda_\mathbb{K}^\vee \) by affine transformations.

Example 3.2. Consider the case \( g = sl_2 \). In this case we have \( \Lambda^\vee = \mathbb{Z} \) and \( W^\vee a \) is an index 2 subgroup of \( \tilde{W}^\vee a \).

The \( \tilde{W}^\vee a \) still permutes alcoves. The stabilizer of \( A^+ \) is naturally identified with \( \Lambda^\vee / \Lambda_\mathbb{K}^\vee \). Since \( W^\vee a \) acts on the set of alcoves simply transitively, we have

\[
\tilde{W}^\vee a = \Lambda^\vee / \Lambda_\mathbb{K}^\vee \rtimes W^\vee a.
\]

Note that we can still extend the notion of length to \( \tilde{W}^\vee a \) with the same geometric meaning as before. For \( \gamma \in \Lambda^\vee / \Lambda_\mathbb{K}^\vee, u \in W^\vee a \), we have \( \ell(\gamma u) = \ell(u) \).