AFFINE WEYL GROUPS

Contents

1. Affine Dynkin diagrams

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Let D be the corresponding Dynkin diagram. Let W be the Weyl group of $\mathfrak{g}$. Consider the lattice $\Lambda_r^\vee \subset \mathfrak{h}$ generated by the simple coroots $\alpha_i^\vee$. We can form the semi-direct product $W^{\vee a} := W \ltimes \Lambda_r^\vee$. Note that it acts on $\mathfrak{h}$ by affine transformations.

This group will be called the *affine Weyl group* of $\mathfrak{g}$. For $\nu \in \Lambda_r^\vee$ we denote by $t_\nu \in W^{\vee,a}$ the corresponding element of $W^{\vee,a}$.

Example 1.1. *Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. In this case we have $W = S_2$ and $\Lambda_r^\vee = 2\mathbb{Z} \subset \mathfrak{h} = \mathbb{C}$, where we identify $\alpha^\vee$ with 2. The element $(12) \in S_2$ acts via $x \mapsto -x$. Then $W^{\vee a}$ consists of transformations of the form $x \mapsto \pm x + 2k$ for $k \in \mathbb{Z}$. Note that it is the Coxeter group with simple reflections s_0, s_1 given by $s_1(x) = -x, s_0(x) = 2 - x$.*

2. Alcoves

The affine action of $W^{\vee a}$-action on $\mathfrak{h}$ preserves the real form $\Lambda_{\mathbb{R}}^\vee$ spanned by the coroots.

Definition 2.1. *By an affine root hyperplane in $\Lambda_{\mathbb{R}}^\vee$ we mean a hyperplane of the form $\langle \alpha, \cdot \rangle = n$ for a root α and $n \in \mathbb{Z}$. By an open alcove we mean a connected component of $\Lambda_{\mathbb{R}}^\vee$ with all affine root hyperplanes removed. By an alcove we mean the closure of an open alcove, this is a simplex. The fundamental alcove A^+ is one given by $\langle \alpha_i, \cdot \rangle \geqslant 0, i = 1, \ldots, r$ and $\langle \alpha_0, \cdot \rangle \geqslant -1$, where α_0 denotes the minimal negative root of $\mathfrak{g}$.*

Example 2.2. *For $\mathfrak{g} = \mathfrak{sl}_2$, the affine root hyperplanes are integers (we view $\alpha^\vee$ as 2 so $\alpha_0 = -\alpha = -1$). The alcoves are the intervals of the form $[n, n+1]$, where n is an integer. The fundamental alcove is $[0, 1]$.*

Exersise 2.3. *The $W^{\vee a}$-action permutes affine root hyperplanes, hence alcoves.*

Proposition 2.4. *$W^{\vee a}$ in its action on $\Lambda^\vee_{\mathbb{R}}$ coincides with the group generated by reflections along affine root hyperplanes. In particular, $W^{\vee a}$ is a Coxeter group.*

Proof. The reflection along $\langle \alpha, \cdot \rangle = n$ is $x \mapsto x - (\langle \alpha, x\rangle - n)\alpha^\vee$, it lies in $W^{\vee a}$. This equality also easily shows that $t_{\alpha^\vee}$ lies in the group generated by reflections. Hence we see that the two groups of affine transformations coincide. □

Corollary 2.5. *$W^{\vee a}$ permutes the alcoves simply transitively.*

Let $s_1, \ldots, s_r$ denote the reflections along the corresponding walls of the fundamental alcove. These are the simple reflections in $W^{\vee a}$.

Let us give a formula for the length function $\ell : W^{\vee a} \to \mathbb{Z}_{\geqslant 0}$ (for $u \in W^{\vee a}$, the length $\ell(u)$ is the number of affine root hyperplanes separating A^+ and $u(A^+)$) in terms of our initial presentation.

Proposition 2.6. *For $w \in W$, $\nu \in \Lambda^\vee_r$ we have*

$$\ell(wt_\nu) = \sum_{\alpha\in\Delta_+, w(\alpha)\in\Delta_+} |\langle \alpha, \nu\rangle| + \sum_{\alpha\in\Delta_+, w(\alpha)\in\Delta_-} |1 + \langle \alpha, \nu\rangle|.$$

Example 2.7. *For $\mathfrak{g} = \mathfrak{sl}_2$ we have*

$$l((s_0s_1)^n) = 2n = |\langle \alpha, n\alpha^\vee\rangle| = l(t_{n\alpha^\vee}),$$
$$l(s_1(s_0s_1)^n) = |2n+1| = |1 + \langle \alpha, n\alpha^\vee\rangle| = l(s_1 t_{n\alpha^\vee}).$$

3. Extended affine Weyl group

Let $\Lambda^\vee$ be the coweight lattice of $\mathfrak{g}$, in particular $\Lambda^\vee_r \subset \Lambda^\vee$.

Definition 3.1. *The extended affine Weyl group $\tilde{W}^{\vee a}$ is $W \ltimes \Lambda^\vee$.*

The group $\tilde{W}^{\vee a}$ contains $W^{\vee a}$ as a normal subgroup and still acts on $\Lambda^\vee_{\mathbb{R}}$ by affine transformations.

Example 3.2. *Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. In this case we have $\Lambda^\vee = \mathbb{Z}$ and $W^{\vee a}$ is an index 2 subgroup of $\tilde{W}^{\vee a}$*

The $\tilde{W}^{\vee a}$ still permutes alcoves. The stabilizer of A^+ is naturally identified with $\Lambda^\vee/\Lambda^\vee_r$. Since $W^{\vee a}$ acts on the set of alcoves simply transitively, we have

$$\tilde{W}^{\vee a} = \Lambda^\vee/\Lambda^\vee_r \ltimes W^{\vee a}.$$

Note that we can still extend the notion of length to $\tilde{W}^a$ with the same geometric meaning as before. For $\gamma \in \Lambda^\vee/\Lambda^\vee_r, u \in W^{\vee a}$, we have $\ell(\gamma u) = \ell(u)$.