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The Center of a Quantum Group, Part 1

Goal: We wish to determine the center $Z(U)$ of the quantum group $U = U_q(\mathfrak{g})$.

Approach: Proceed similarly to the classical case, where one determines the center of the universal enveloping algebra of a semisimple Lie algebra. We follow the presentation in Jantzen's book "Lectures on Quantum Groups" (hence forth refered to as Jantzen).

1) Notation for Semisimple Lie Algebras

- $\mathbb{K}$ a field of characteristic 0
- $q \in \mathbb{K}^\times$ transcendental over prime subfield $\mathbb{Q} \subset \mathbb{K}$ (in particular, not a root of unity).
- $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$ of rank ℓ
- $U(\mathfrak{g})$ its universal enveloping algebra
- $\mathfrak{h}$ a Cartan subalgebra
- Φ the set of roots w.r.t. $\mathfrak{h}$
- Π a system of simple roots, $\Pi = \{\alpha_1, \alpha_2, \ldots \alpha_\ell\}$
- Φ^+ the set of positive roots w.r.t. Π
- $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ triangular decomposition
- W the Weyl group generated by simple reflections acting on $\mathfrak{h}^*$ (and hence on $\mathfrak{h}$, $U(\mathfrak{h})$).

- $(\,,\,)$ W-invariant inner product such that short roots have length 2.
- Λ weight lattice spanned by fundamental weights $\omega_1, \omega_2, \ldots \omega_\ell$
- $\Lambda^+ = \{\lambda \in \Lambda : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Pi\}$ the dominant integral weights
- ρ the Weyl vector, given by $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{n} \omega_i$
- s_α the reflection given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

2) Recalling the classical case

2.1) The Harish Chandra Homomorphism

Before we determine the center $Z(U)$, we briefly recall how one can determine the center $Z(U(\mathfrak{g}))$, which will serve as our blue print.

- Let $\{e_i, h_i, f_i\}$ denote the $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ associated to $\alpha_i \in \Pi$. The $\mathbb{Z}\Phi$ gradation on $\mathfrak{g}$ induces a $\mathbb{Z}\Phi$ gradation on $U(\mathfrak{g})$.

- The degree 0 piece $U(\mathfrak{g})_0$ contains $U(\mathfrak{h})$ and $Z(U(\mathfrak{g}))$ is a subalgebra.

  Furthermore, $U(\mathfrak{g})\mathfrak{n}^+ \cap U(\mathfrak{g})_0 = \mathfrak{n}^- U(\mathfrak{g}) \cap U(\mathfrak{g})_0$ is a two sided ideal of $U(\mathfrak{g})$ s.t.
  $U(\mathfrak{g})_0 = U(\mathfrak{h}) \oplus \underline{U(\mathfrak{g})\mathfrak{n}^+ \cap U(\mathfrak{g}_0)}$.

  Hence, we get an algebra hom $\pi : U(\mathfrak{g})_0 \to U(\mathfrak{h})$ given by projection.

- Let $\gamma : U(\mathfrak{h}) \to U(\mathfrak{h})$ denote the algebra hom defined by $h_i \mapsto h_i - 1$. $\forall i$, extended to $U(\mathfrak{h})$ as an algebra hom. If $\lambda \in \mathfrak{h}^*$, we can extend it to an algebra hom $\lambda : U(\mathfrak{g}) \to \mathbb{C}$ (by abuse of notation) by $\lambda(h_{i_1} h_{i_2} \cdots h_{i_r}) = \lambda(h_{i_1})\lambda(h_{i_2}) \cdots \lambda(h_{i_r})$, and $\lambda(1) = 1$.
  Then, $(\lambda+\rho)(h_i - 1) = (\lambda+\rho)(h_i) - (\lambda+\rho)(1) = \lambda(h_i) + \rho(h_i) - 1 = \lambda(h_i)$

Def. Harish Chandra Homomorphism

$\psi : U(\mathfrak{g})_0 \to U(\mathfrak{h})$ is defined as $\psi = \gamma \circ \overline{\pi}$

It follows $(\lambda+\rho)(\varphi(z)) = \lambda(\overline{\pi}(z)) \quad \forall z \in Z(U(\mathfrak{g}))$

Since $\mathfrak{h}$ is an abelian Lie algebra, $U(\mathfrak{h}) = S(\mathfrak{h})$, and can hence be viewed as polynomial functions on $\mathfrak{h}^*$, given by evaluation. Since W acts on $\mathfrak{h}$, we have $S(\mathfrak{h})^W := \{ h \in S(\mathfrak{h}) : wh = h \ \forall w \in W \}$.

2.2) The Harish-Chandra Isomorphism

<u>Thm</u>

The restriction of φ to $Z(U(\mathfrak{g}))$ is an algebra isomorphism $Z(U(\mathfrak{g})) \to S(\mathfrak{h})^W$.

<u>Pf.</u> we only state the key steps

1. First, it is clear it is an algebra homomorphism.
2. Second, we need to check injectivity.

- Can view $U(\mathfrak{h}) = S(\mathfrak{h})$ as polynomial functions on $\mathfrak{h}^*$ given by evaluation. So from above:

$$\varphi(z)(\lambda+\rho) = (\lambda+\rho)(\varphi(z)) = \lambda(\bar{\pi}(z)) = \bar{\pi}(z)(\lambda)$$

- If $z \in Z(U(\mathfrak{g}))$, z acts on the Verma module $(h \in \mathfrak{h})$.
  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}v_\lambda$ as a scalar as $h \cdot z \cdot v_\lambda = z \cdot h \cdot v_\lambda = \lambda(h) z v_\lambda$

$\Rightarrow$ $z v_\lambda = \chi_\lambda(z) v_\lambda$ for some algebra hom $\chi_\lambda : Z(U(\mathfrak{g})) \to \mathbb{C}$, called a <u>character</u>.

- Since $z - \bar{\pi}(z) \in U(\mathfrak{g})\mathfrak{n}_+$ annihilates v_λ,
  Therefore, $\chi_\lambda(z) = \lambda(\bar{\pi}(z)) = \varphi(z)(\lambda+\rho)$.

So if $\varphi(z) = 0$, z acts as 0 on every finite dimensional module $\Rightarrow z = 0$
(the proof of this is very similar to the analogous statement for quantum groups proved last time).

3. Third, we need to show that the image of φ lies in $S(\mathfrak{h})^W$.

- For $w \in W$, let $w \cdot \lambda := w(\lambda + \rho) - \rho$; defines <u>dot action</u> of W.
If $\exists \alpha_i$ simple s.t. $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$, then $M(\lambda)$ has a maximal vector $f_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} v_\lambda$ of weight $\lambda - (\langle \lambda, \alpha_i^\vee \rangle + 1)\alpha_i$

(this is a consequence of viewing $M(\lambda)$ as an $\mathfrak{sl}_2$-module of the $\mathfrak{sl}_2$ triple $\{f_i, h_i, e_i\}$.)

Hence, there is an embedding $\underbrace{M(\lambda - (\langle \lambda, \alpha_i^\vee \rangle + 1)\alpha_i)}_{= M(s_{\alpha_i} \cdot \lambda)} \hookrightarrow M(\lambda)$.

This implies $\chi_{s_{\alpha_i} \cdot \lambda} = \chi_\lambda$. In particular, if λ is dominant integral, a short induction argument (omitted) shows

$$\chi_{w \cdot \lambda} = \chi_\lambda \quad \forall w \in W.$$

- Therefore, $\bar{\Pi}(z)(\lambda) = \chi_\lambda(z) = \chi_{w\cdot\lambda}(z) = \bar{\Pi}(z)(w\cdot\lambda)$,

So the two polynomial functions $\bar{\Pi}(z)(-)$ and $\bar{\Pi}(z)(w\cdot -)$ agree on Λ^+, a Zariski dense subset of $\mathfrak{h}^*$. Hence, they agree on all of $\mathfrak{h}^*$, so for all $\lambda \in \mathfrak{h}^*$, $w \in W$

$$\bar{\Pi}(z)(\lambda) = \bar{\Pi}(z)(w\cdot\lambda)$$

$$\Rightarrow \varphi(z)(\lambda+\rho) = \varphi(z)(w\cdot\lambda+\rho)$$

writing $\mu = \lambda - \rho$:

$$\Rightarrow \varphi(z)(\mu) = \varphi(z)(w\mu)$$

Therefore, φ is a map from $Z(U(\mathfrak{g})) \to S(\mathfrak{h})^W$.

4. Finally, one needs to show surjectivity.

This is somewhat long, so I will be brief.

Let:

- $P(\mathfrak{g}) = S(\mathfrak{g}^*)$ = polynomial functions on $\mathfrak{g}$
- $\Theta : P(\mathfrak{g}) \to P(\mathfrak{h})$ be given by restriction, is an algebra homomorphism
- $G \subset \mathrm{Aut}\,\mathfrak{g}$ be the group generated by $\{\exp \mathrm{ad}\, x : x \text{ nilpotent}\}$

G acts on $P(\mathfrak{g})$ by $(g.p)(y) = p(g.y)$ $\forall p \in P(\mathfrak{g})$, $y \in \mathfrak{g}$, $g \in G$.

The crux is the following theorem of Chevalley:

<u>Thm.</u> (Restriction Thm)

Θ maps $P(\mathfrak{g})^G$ isomorphically onto $P(\mathfrak{h})^W$.

The proof of Chevalley's theorem involves looking at traces of finite dimensional representations.

After using the Killing form to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and $\mathfrak{h}$ with $\mathfrak{h}^*$, this gives an isomorphism

$$S(\mathfrak{g})^G \to S(\mathfrak{h})^W$$

Finally, it can be shown $S(\mathfrak{g})^G$ is isomorphic as a vector space to $Z(U(\mathfrak{g}))$.

$\square$

3) Notation for Quantum Grps

- $U = U_q(\mathfrak{g})$ is the quantized enveloping algebra over $\mathbb{K}$ of $\mathfrak{g}$ with parameter q

Recall it is defined by generators and relations

$$U_q(\mathfrak{g}) = \langle E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi \rangle / \text{relations}$$

relations $\forall \alpha, \beta \in \Pi$:

1) $K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1$
$K_\alpha K_\beta = K_\beta K_\alpha$

2) $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta$

3) $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$

4) $[E_\alpha, F_\beta] = \delta_{\alpha\beta} \dfrac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$ where $q_\alpha = q^{(\alpha,\alpha)/2}$

5) quantized Serre relations

- For $\lambda \in \mathbb{Z}\Phi$, $\lambda = \sum_{\alpha \in \Pi} m_\alpha \alpha$, $K_\lambda := \prod_{\alpha \in \Pi} K_\alpha^{m_\alpha}$. Clearly $K_\lambda K_\mu = K_{\lambda+\mu}$ for $\lambda, \mu \in \mathbb{Z}\Phi$

- U^+ = subalgebra generated by E_α
U^- = subalgebra generated by F_α $(\alpha \in \Pi)$.
U^0 = (commutative) subalgebra generated by K_α, K_α^{-1}
$Z(U)$ = center of U.

- U admits a $\mathbb{Z}\Phi$ grading where $\deg E_\alpha = \alpha$, $\deg F_\alpha = -\alpha$, $\deg K_\alpha = \deg K_\alpha^{-1} = 0$

Let $U_\nu = \{ x \in U \mid \deg x = \nu \}$ $(\nu \in \mathbb{Z}\Phi)$.

- Notice for $u \in U_\nu$, $K_\lambda u K_\lambda^{-1} = q^{(\lambda,\nu)} u$ $\quad (\lambda, \nu \in \mathbb{Z}\Phi)$.
  Since, q not a root of unity $\Rightarrow$ $U_\nu = \{x \in U \mid K_\lambda x K_\lambda^{-1} = q^{(\lambda,\mu)}x$ $\forall \lambda \in \mathbb{Z}\Phi\}$.

- $U^{\geq 0}$ = subalg generated by $E_\alpha, K_\alpha^{\pm 1}$ $\quad (\alpha \in \Pi)$
  $U^{\leq 0}$ = subalg generated by $F_\alpha, K_\alpha^{\pm 1}$

- W acts on U^0 by $wK_\mu = K_{w\mu}$, $\mu \in \mathbb{Z}\Phi$.

- $U^0_{ev} := \bigoplus_{\mu \in \mathbb{Z}\Phi \cap 2\Lambda} \Bbbk K_\mu$, a subalgebra of U^0.
  This is stable under W.

- $(U^0)^W := \{u \in U^0 : wu = u \ \forall w \in W\}$

4) Description of the Center of a Quantum Group

4.1) A very short outline

Just proved that $Z(U(\mathfrak{g})) \cong S(\mathfrak{h})^W$. What can we say about $Z(U)$, the center of the quantum group?

Turns out that we can produce an algebra isomorphism $\varphi: Z(U) \longrightarrow (U^0_{ev})^W$, which is what we will construct in this lecture.

Our approach will be similar in nature proof we produced, but with some modifications. Step 1 to show φ is an algebra hom and step 2 to show φ is injective go through almost identically. Step 3 to show that the image is $(U^0_{ev})^W$ will require some modification. In step 4, to show surjectivity, we need to modify heavily; the trace needs to be replaced by quantum trace, and more drastically we must work around using $\mathfrak{g}$ (remark: this shows bijectivity).

4.2) The (quantum) Harish Chandra Homomorphism

For $\lambda \in \Lambda$, we can produce an algebra hom $U^0 \to \mathbb{K}$ which by abuse of notation we also denote λ defined by
$$\lambda(K_\mu) := q^{(\lambda,\mu)} \quad \forall \mu \in \mathbb{Z}\Phi.$$
which is well defined as $\{K_\mu\}_{\mu \in \mathbb{Z}\Phi}$ is a basis of U^0 and is an algebra hom because
$$\lambda(K_{\mu+\nu}) = q^{(\lambda,\mu+\nu)} = q^{(\lambda,\mu)} q^{(\lambda,\nu)} = \lambda(K_\mu)\lambda(K_\nu).$$

Furthermore, if $\lambda, \lambda' \in \Lambda$,
$$(\lambda+\lambda')(K_\mu) = q^{(\lambda+\lambda',\mu)} = q^{(\lambda,\mu)} q^{(\lambda',\mu)}$$
$$= \lambda(K_\mu)\lambda'(K_\mu).$$

Now, if $\lambda \in \Lambda$, let γ_λ denote the algebra hom $\gamma_\lambda : U^0 \to U^0$ given by
$$\gamma_\lambda(K_\mu) = \lambda(K_\mu) K_\mu = q^{(\lambda,\mu)} K_\mu$$
$(\mu \in \mathbb{Z}\Phi)$. Notice $\gamma_{\lambda+\lambda'} = \gamma_\lambda \circ \gamma_{\lambda'}$.

Recall $U_0 = \{ u \in U \mid \deg u = 0 \}$
$= \{ u \in U \mid K_\mu u K_\mu^{-1} = u \ \forall \mu \in \mathcal{P} \}$.

So clearly $Z(U) \subset U_0$.

The isomorphism $U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U$ means we can write

multiplication

$U_0 = U^0 \oplus \bigoplus_{\nu > 0} U^-_{-\nu} U^0 U^+_\nu$. Hence, denote the

Projection $U_0 \to U^0$ as $\overline{\pi}$ (similar to classical case)

Exercise: Check $\overline{\pi}$ is an algebra hom.

<u>Def.</u> (Harish-Chandra homomorphism)

$\psi : U_0 \to U^0$ is defined as

$\psi = \gamma_{-\rho} \circ \overline{\pi}$.

4.3) The Main Theorem

(***) <u>Main Theorem</u>

Suppose $\mathbb{K}$ is a field of characteristic 0 and $q \in \mathbb{K}^\times$ is transcendental over the prime subfield $\mathbb{Q} \subset \mathbb{K}$. Then, the restriction

$$\Psi|_{Z(U)} = \gamma_{-\rho} \circ \overline{\Pi}|_{Z(U)} : Z(U) \longrightarrow U^0$$

is an isomorphism onto its image, which is $(U^0_{ev})^W$.

From now on we shall just write Ψ to denote the restriction of Ψ to the center, and similarly for $\overline{\Pi}$.

5) Proof of the Main Thm.

5.1) Outline

We will spend the rest of the lecture proving this theorem.

Let's outline what we'll do.

1) First, show that Ψ is an injective algebra homomorphism from $Z(U)$ to $(U^0_{ev})^W$. This encompasses steps 1 to 3 in the proof of the classical result above.

Then, we need to replace step 4 in that proof to show surjectivity. To do so:

2) We need to construct elements in $Z(U)$ whose images under Ψ span $(U^0_{ev})^W$. To do this, we turn to the

dual space U^*, which is naturally a U-mod dual to the adjoint representation U, and construct a nondegenerate, invariant pairing on U, which gives an embedding $U \to U^*$. Then, in U^*, we construct U-invariants using the quantum trace of finite dimensional representations, and show these lie in the image of the embedding; since they are invariant, their preimages will be invariant. A general fact of Hopf algebras is invariance w.r.t. adjoint rep is the same as being central. Then, we show that the images of these central elements under ψ span $(U^0_{ev})^W$.

Rmk: For this talk, we have assumed char $k = 0$ and q is transcendental over $\mathbb{Q} \subset k$. This can be weakened to requiring q not be a root of unity.

5.1) Showing Ψ is injective and its image is in $(U^0_{ev})^W$.

- Recall the type 1 Verma module $M(\lambda) = U / \sum_{\alpha\in\Pi} UE_\alpha + \sum_{\alpha\in\Pi} U(K_\alpha - q^{(\lambda,\alpha)})$ generated by a coset v_λ of 1.

Lemma 1

a) Let $\lambda \in \Lambda$. $z \in Z(U)$ acts on $M(\lambda)$ as a scalar this scalar is $\lambda(\overline{\pi}(z))$

b) $\overline{\pi}$ is injective on $Z(U)$.

Pf. a) Left as an exercise

b) If $\overline{\pi}(z)=0$ for $z \in Z(U)$, then z acts as 0 on all type 1 finite dimensional representations as these are a quotient of $M(\lambda)$ for some λ.

Now apply the theorem from last time to deduce $z=0$. $\square$

Recall if $\lambda \in \Lambda$, γ_λ denotes the algebra hom $\gamma_\lambda : U^0 \to U^0$ given by

$$\gamma_\lambda(K_\mu) = \lambda(K_\mu)K_\mu = q^{(\lambda,\mu)}K_\mu$$

$(\mu \in \mathbb{Z}\Phi)$.

Recall the dot action of W on Λ given by
$w \cdot \lambda := w(\lambda+\rho) - \rho$.

Recall $w K_\mu = K_{w\mu}$ defines an action of W on U^0.
$(w \in W)$.

Lemma 2 $\quad \Psi(Z(U)) \subseteq (U^0)^W$.

Pf. Let $z \in Z(U)$, and let $h = \Psi(z)$
$= \gamma_{-\rho} \circ \overline{\pi}|_{Z(U)}(z) \Rightarrow \gamma_\rho(h) = \overline{\pi}|_{Z(U)}(z)$.

By Lemma 1, for $\lambda \in \Lambda$, z acts on $M(\lambda)$ as the scalar $\lambda(\overline{\pi}|_{Z(U)}(z)) = \lambda(\gamma_\rho(h))$
$$= (\lambda+\rho)(h)$$

From last time, we saw that for $\alpha \in \Pi$, if $\langle \lambda, \alpha^\vee \rangle \geq 0$, $\exists$ non-zero hom $M(s_\alpha \cdot \lambda) \to M(\lambda)$.
So z must act as the same scalar on both. Hence,

$$(*) \quad (\lambda+\rho)(h) = (s_\alpha \cdot \lambda + \rho)(h) = (s_\alpha(\lambda+\rho))(h)$$
$$= (\lambda+\rho)(s_\alpha^{-1}(h)).$$

$(*)$ also holds if $\langle \lambda, \alpha^\vee \rangle < 0$; if $\langle \lambda, \alpha^\vee \rangle = -1$,
($\in \mathbb{Z}$ by def)

then $s_\alpha \cdot \lambda = \lambda$ and (*) follows. If $\langle \lambda, \alpha^\vee \rangle < -1$, then $\langle s_\alpha \cdot \lambda, \alpha^\vee \rangle = \frac{2(s_\alpha(\lambda+\rho)-\rho, \alpha)}{(\alpha,\alpha)}$

$$= -\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} - 2$$ by def of ρ and W-invariance of $(\,,\,)$.

$$= -\langle \lambda, \alpha^\vee \rangle - 2 \geq 0,$$ so (*) applies to $s_\alpha \cdot \lambda$ and hence to λ.

Since simple reflections generate W, we deduce

$$(\lambda+\rho)(h) = (\lambda+\rho)(wh) \quad \forall w \in W \ \forall \lambda \in \Lambda.$$

$$\Rightarrow \lambda(h) = \lambda(wh) \text{ as } \rho \in \Lambda.$$

Finally, if $h - wh = \sum_\mu a_\mu K_\mu$, then $0 = \lambda(h-wh) = \sum_\mu a_\mu q^{(\lambda,\mu)}$. Each $\lambda \mapsto q^{(\lambda,\mu)}$ is a character on Λ; these are distinct as q is not a root of unity, and it is known distinct characters are linearly independent $\Rightarrow a_\mu = 0 \ \forall \mu \Rightarrow h - wh = 0 \Rightarrow \varphi(z) \in (U^0)^W$.

□

Recall $$U^0_{ev} = \bigoplus_{\mu \in \mathbb{Z}\Phi \cap 2\Lambda} \mathbb{C} K_\mu.$$

Proposition 3: The image of φ lies in $(U^0_{ev})^W$.

Pf. If $z \in Z(U)$, we already know that $\varphi(z) \in (U^0)^W$.
Hence, can write
$$\varphi(z) = \sum_{\mu \in \mathbb{Z}\Phi} a_\mu K_\mu$$
where $a_\mu = a_{w\mu}$ $\forall w \in W$.

We want to show that $a_\mu \neq 0 \Rightarrow \mu \in 2\Lambda$. Since Π is a basis of $\mathbb{Z}\Phi$, there exists a group hom $\sigma : \mathbb{Z}\Phi \to \{\pm 1\}$ s.t. $\sigma(\alpha) = -1$ $\forall \alpha \in \Pi$. Now, σ induces an algebra automorphism $\tilde{\sigma} : U \to U$ by

$$U(E_\alpha) = \sigma(\alpha) E_\alpha = -E_\alpha$$
$$U(F_\alpha) = F_\alpha$$
$$U(K_\alpha^{\pm 1}) = \sigma(\alpha) K_\alpha^{\pm 1} = -K_\alpha^{\pm 1}$$

Notice $\tilde{\sigma}$ preserves $Z(U)$, U^0, U^+, U^-, grading on U
i.e. $\tilde{\sigma}(Z(U)) = Z(U)$, $\tilde{\sigma}(U^+_\nu) = U^+_\nu$, $\tilde{\sigma}(U^-_\nu) = U^-_\nu$
for $\nu \in \mathbb{Z}\Phi$, $\tilde{\sigma}(U^0) = U^0$. It follows $\tilde{\sigma} \circ \pi = \pi \circ \tilde{\sigma}|_{U_0}$.

Also clear $\forall \lambda \in \Lambda$, $\tilde{\sigma} \circ \gamma_\lambda = \gamma_\lambda \circ \tilde{\sigma}|_{U^0}$, by checking on K_μ's.
Therefore, for $z \in Z(U)$,
$$\tilde{\sigma} \circ \varphi = \tilde{\sigma} \circ \gamma_{-\rho} \circ \pi|_{Z(U(\mathfrak{g}))} = \gamma_{-\rho} \circ \pi|_{Z(U(\mathfrak{g}))} \circ \tilde{\sigma} = \varphi \circ \tilde{\sigma}$$

Therefore, if $z \in Z(U)$ and $\varphi(z) = \sum a_\mu K_\mu$, then
$$\varphi(\tilde{\sigma}(z)) = \tilde{\sigma}(\varphi(z)) = \sum a_\mu \tilde{\sigma}(K_\mu) = \sum_\mu a_\mu \sigma(\mu) K_\mu$$

$\tilde{\sigma}(z)$ is central so by Lemma 2 this sum lies in $(U^0)^W$.
Therefore, $a_\mu \sigma(\mu) = a_{w\mu}\sigma(w\mu) = a_\mu \sigma(w\mu)$ $\forall \mu \in \mathbb{Z}\Phi$, $w \in W$.

$a_\mu \neq 0 \Rightarrow \sigma(\mu) = \sigma(w\mu)$ $\forall w \in W$, so if $w = s_\alpha$ α simple, then, $1 = \sigma(\mu - s_\alpha \mu) = \sigma(\langle \mu, \alpha^\vee \rangle \alpha) = (-1)^{\langle \mu, \alpha^\vee \rangle}$.

$\Longrightarrow \langle \mu, \alpha^\vee \rangle$ is even $\forall \alpha \in \Pi \Rightarrow \mu \in 2\Lambda$.

$\square$

Combining these statements, we have shown that is φ is an embedding $\varphi : Z(U) \to (U^0_{ev})^W$.

5.2) Showing that Ψ surjects on to $(U^0_{ev})^W$

The remainder of the lecture will show Ψ is surjective.

5.2.1) The Bialgebra Pairing

- Let's recall the Hopf algebra structure $(\mu, \eta, \Delta, \varepsilon, S)$ on U.
  - μ is the multiplication, η is the unit.
  - $\Delta: U \to U \otimes U$ is the comultiplication uniquely satisfying $(\alpha \in \Pi)$

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$$
$$\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$$
$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

  - $\varepsilon: U \to \mathbb{K}$ is the counit and algebra hom uniquely satisfying $\varepsilon(E_\alpha) = \varepsilon(F_\alpha) = 0$, $\varepsilon(K_\alpha) = 1$.
  - $S: U \to U$ is the antipode, satisfying $S(E_\alpha) = -K_\alpha^{-1} E_\alpha$, $S(F_\alpha) = -F_\alpha K_\alpha$, $S(K_\alpha) = K_\alpha^{-1}$.

In order to construct the invariant pairing on $U \times U$, we first need to construct a bialgebra pairing on $U^{\leq 0} \times U^{\geq 0}$

Def. Let X, Y be two Hopf algebras. A pairing $(,) : X \times Y \to \mathbb{k}$ is a <u>bialgebra pairing</u> if (1) $(x_1 x_2, y) = (x_1 \otimes x_2, \Delta_Y(y))$ and

(2) $(x, y_1 y_2) = (\Delta_X(x), y_1 \otimes y_2)$,

where $x_1, x_2 \in X$, $y_1, y_2 \in Y$, Δ_X and Δ_Y are the comultiplications on X, Y, respectively, and the form on $X \otimes X \times Y \otimes Y$ is defined by

$$(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, x_2)(y_1, y_2).$$

<u>**Motivation:**</u>

A^* is naturally an algebra by defining its multiplication as

$$\mu_{A^*} = A^* \otimes A^* \hookrightarrow (A \otimes A)^* \xrightarrow{\Delta_A^*} A^*.$$

Similarly for B^*. Therefore, the conditions in (1), (2) are the same as requiring $A \to B^*$ $(a \mapsto (a, \cdot))$ and $B \to A^*$ $(b \mapsto (\cdot, b))$ be algebra homs, resp.

$U^{\leq 0}$ and $U^{\geq 0}$ are Hopf subalgebras of U.

Thm. 6

There exists a unique bialgebra pairing
$(,) : U^{\leq 0} \times U^{\geq 0} \longrightarrow \mathbb{K}$ s.t. $\forall\, \alpha, \beta \in \Pi$:

i.) $(K_\mu, K_\nu) = q^{-(\mu,\nu)}$, $(F_\alpha, E_\beta) = -\delta_{\alpha\beta}\,(q_\alpha - q_\alpha^{-1})^{-1}$

ii) $(K_\mu, E_\alpha) = (F_\alpha, K_\mu) = 0$

Pf.

Assuming existence of such a form, uniqueness is immediate as relations 1) and 2) define it on the generators, from which the bialgebra property will show it is defined on the whole algebras. For example

$$(F_\alpha F_\beta , E_\alpha E_\beta) \overset{(1)}{=} (F_\alpha \otimes F_\beta , \Delta(E_\alpha E_\beta))$$
$$= (F_\alpha \otimes F_\beta , (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)(E_\beta \otimes 1 + K_\beta \otimes E_\beta))$$
$$= (F_\alpha \otimes F_\beta , E_\alpha E_\beta \otimes 1 + \cdots + K_{\alpha+\beta} \otimes E_\alpha E_\beta)$$
$$= (F_\alpha , E_\alpha E_\beta)(F_\beta , 1) + \cdots + (F_\alpha , K_{\alpha+\beta})(F_\beta , E_\alpha E_\beta)$$

then use (2) and it will be in term of the form on generators.

Now, let's do existence.

- Recall that $\tilde{U}^{\leq 0}$ is the algebra with the same generators and relations as $U^{\leq 0}$ except the quantum Serre relations.

  $\tilde{U}^{\leq 0}$ is a bialgebra w.r.t. the Hopf structure formulas recalled above, and $U^{\leq 0}$ is a bialgebra quotient.

- We will first define a bialgebra pairing on $\tilde{U}^{\leq 0} \times U^{\geq 0}$ as it is easier to define. Then, we show how it descends to $U^{\leq 0} \times U^{\geq 0}$.

- For $\alpha \in \Pi$, define $f^\alpha \in (U^{\geq 0})^*$ by

$$\begin{cases} f^\alpha(F_\alpha K_\mu) = -(q_\alpha - q_\alpha^{-1})^{-1} & \forall \mu \in \mathbb{Z}\Phi \\ f^\alpha(U_\nu^{\geq 0}) = 0 & \forall \nu \neq \alpha \in \mathbb{Z}\Phi \end{cases}$$

- For $\lambda \in \Lambda$, defie $k^\lambda \in (U^{\geq 0})^*$ by

$$k^\lambda(u K_\mu) = \varepsilon(u) q^{-(\lambda,\mu)} \qquad \begin{pmatrix} \mu \in \mathbb{Z}\Phi \\ u \in U^+ \end{pmatrix}$$

Note k^λ is an algebra hom.

<u>Lemma</u>. 7 $F_\alpha \mapsto f^\alpha$, $K_\mu \mapsto k^\mu$ $\begin{pmatrix} \forall \alpha \in \Pi, \\ \mu \in \Phi \end{pmatrix}$

defines an algebra hom

$$\tilde{U}^{\leq 0} \longrightarrow (U^{\geq 0})^*.$$

<u>Pf</u>. Left as an exercise (but this is straightforward as $\tilde{U}^{\leq 0}$ behaves like a free algebra, since F_α's have no relations, and relations involving K_μ are easy). $\square$

Therefore, this algebra hom induces a pairing

$(\ ,\ ): \tilde{U}^{\leq 0} \times U^{\geq 0} \to \mathbb{k}$ satisfying

$$(x_1 x_2, y) = (x_1 \otimes x_2, \Delta(y)) \quad \forall x_1, x_2 \in \tilde{U}^{\leq 0}$$
$$\forall y \in U^{\geq 0}.$$

Lemma 8

$(\ ,\ ): \tilde{U}^{\leq 0} \times U^{\geq 0} \to \mathbb{k}$ also satisfies

$$(x, y_1 y_2) = (\Delta(x), y_1 \otimes y_2) \quad \forall x \in \tilde{U}^{\leq 0}$$
$$\forall y_1, y_2 \in$$

Pf.

We first claim that if $(x_1, y_1 y_2) = (\Delta(x_1), y_1 \otimes y_2)$ and $(x_2, y_1 y_2) = (\Delta(x_2), y_1 \otimes y_2)$ $\forall x_1, x_2 \in \tilde{U}_{\leq 0}$, $y_1, y_2 \in U_{\geq 0}$, then $(x_1 x_2, y_1 y_2) = (\Delta(x_1 x_2), y_1 y_2)$. This is a direct computation using Sweedler notation for the comultiplication, and the fact $(x_1 x_2, y) = (x_1 \otimes x_2, \Delta(y))$ by construction of the pairing — no other property of U is needed, so this is a general statement left as an exercise.

Then, one can simply verify on the generators F_α, K_μ $(\alpha \in \Pi, \mu \in \mathbb{Z}\Phi)$ that
$$(K_\mu, y_1 y_2) = (\Delta(K_\mu), y_1 \otimes y_2)$$
and
$$(F_\alpha, y_1 y_2) = (\Delta(F_\alpha), y_1 \otimes y_2). \quad (*)$$

Let's do it for K_μ:

$$\begin{aligned}(K_\mu, y_1 y_2) &= K^\mu(y_1 y_2) = K^\mu(y_1) K^\mu(y_2) \\ &= (K_\mu, y_1)(K_\mu, y_2) = (K_\mu \otimes K_\mu, y_1 \otimes y_2) \\ &= (\Delta(K_\mu), y_1 \otimes y_2).\end{aligned}$$

For F_α, one can check $(*)$ first for $y_1 = E_{\beta_1} E_{\beta_2} \cdots E_{\beta_r} K_{\mu_1}$ and $y_2 = E_{\theta_1} E_{\theta_2} \cdots E_{\theta_s} K_{\mu_2}$ $(\beta_i, \theta_i \in \Pi, \mu_1, \mu_2 \in \mathbb{Z}\Phi)$ by a direct calculation. Then $(*)$ follows by linearity. The details are left to the reader.

$\square$

Combining lemmas 7 and 8, we have a bialgebra pairing $(\,,\,): \tilde{U}^{\leq 0} \times U^{\geq 0} \to \mathbb{K}$.

__Lemma 9__ $\forall \mu, \nu \in \mathbb{Z}_{\geq 0}\Pi$, $\mu \neq \nu$,

$\Rightarrow (x,y) = 0 \quad \forall x \in \tilde{U}^-_{-\mu},\ y \in U^+_\nu$

Pf. This is by construction of the form.

In order to show $(\ ,\ ) : \tilde{U}^{\leq 0} \times U^{\geq 0} \to \mathbb{K}$ descends to a form on $U^{\leq 0} \times U^{\geq 0}$, we need the left radical to contain the elements $(\forall \alpha \neq \beta \in \Pi)$:

$$U^-_{\alpha\beta} = \sum_{s=0}^{r_{\alpha\beta}} (-1)^s \begin{bmatrix} r_{\alpha\beta} \\ s \end{bmatrix}_\alpha F_\alpha^{r_{\alpha\beta}-s} F_\beta F_\alpha^s$$

$\in \tilde{U}$. $\qquad (r_{\alpha\beta} = 1 - \langle \beta, \alpha^\vee \rangle)$.

which define the quantum Serre relations. Indeed,

__Lemma 10__

$(U^-_{\alpha\beta}, x) = 0 \quad \forall x \in U^{\geq 0},\ \forall \alpha \neq \beta \in \Pi.$

Pf. By lemma 9, we need only look at the restriction of the pairing to $\tilde{U}^-_{-(r\alpha+\beta)} \times U^+_{r\alpha+\beta}$ $(\alpha\neq\beta)$, as $u^-_{\alpha\beta} \in \tilde{U}^-_{-(r\alpha+\beta)}$

$A_1 \times A_1$ is easy

Now, we will look at what happens when type $\mathfrak{g} = A_2$; the general case is left as an exercise.

There are two possibilities for $\alpha\neq\beta \in \Pi$ in this case, but by symmetry, it suffices to look at one. Picking one, we have $a_{\alpha\beta} = -1$, and

$$u^-_{\alpha\beta} = \sum_{s=0}^{2} (-1)^s \begin{bmatrix} 2 \\ s \end{bmatrix}_\alpha F_\alpha^{2-s} F_\beta F_\alpha^s = F_\alpha^2 F_\beta - [2]_\alpha F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2$$

and a basis for $U^+_{2\alpha+\beta}$ is

$$\{E_\alpha^2 E_\beta K_\mu,\ E_\alpha E_\beta E_\alpha K_\mu\}_{\mu\in\mathbb{Z}\Phi}$$

The K_μ's can effectively be ignored in the pairing, they only contribute a factor. A direct computation shows that $(u^-_{\alpha\beta}, b) = 0$ for each basis vector b. $\square$

Now, applying lemma 10, the bialgebra pairing on $\tilde{U}^{\leq 0} \times U^{\geq 0}$ descends to one on $U^{\leq 0} \times U^{\geq 0}$, also denoted as $(\ ,\ )$. This proves theorem 6. $\square$

References

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