The Center of a Quantum Group, Part 1

Goal: We wish to determine the center $\mathcal{Z}(U)$ of the quantum group $U = U_q(\mathfrak{g})$.

Approach: Proceed similarly to the classical case, where one determines the center of the universal enveloping algebra of a semisimple Lie algebra. We follow the presentation in Jantzen’s book “Lectures on Quantum Groups” (hence forth referred to as Jantzen).
1) Notation for Semisimple Lie Algebras

- $\mathbb{K}$ a field of characteristic 0
- $q \in \mathbb{K}^\times$ transcendental over prime subfield $\mathbb{Q} \subset \mathbb{K}$ (in particular, not a root of unity)
- $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$ of rank $\ell$
- $U(\mathfrak{g})$ its universal enveloping algebra
- $\mathfrak{h}$ a Cartan subalgebra
- $\Phi$ the set of roots w.r.t $\mathfrak{h}$
- $\Pi$ a system of simple roots, $\Pi = \{ \alpha_i, \alpha_j, \ldots, \alpha_k \}$
- $\Phi^+ \subseteq \Phi$ the set of positive roots w.r.t. $\Pi$
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z} \oplus \mathfrak{z}^+$ triangular decomposition
- $W$ the Weyl group generated by simple reflections acting on $\mathfrak{h}^*$ (and hence on $\mathfrak{h}$, $U(\mathfrak{g})$).

- $(,)$ $W$-invariant inner product such that short roots have length 2.
- $\Lambda$ weight lattice spanned by fundamental weights $\omega_1, \omega_2, \ldots$
- $\Lambda^+$ the set of positive roots $\Pi \subseteq \Lambda^+$
- $\Pi^+ = \{ \alpha \in \Lambda^+ : <\alpha, \alpha> \in \mathbb{Z}_+ \}$ the dominant integral weights
- $\mathfrak{g}$ the Weyl vector, given by $g = 1/2 \sum \alpha = \sum_{\alpha \in \Pi^+} \omega_i$
- $S_\alpha$ the reflection given by $S_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. 
2) Recalling the Classical Case

2.1) The Harish Chandra Homomorphism

Before we determine the center \( Z(U) \), we briefly recall how one can determine the center \( Z(U(\mathfrak{g})) \), which will serve as our blueprint.

- Let \( \{e_i, h_i, f_i\} \) denote the \( sl_2 \)-triple in \( \mathfrak{g}_J \) associated to \( \alpha_i \in \Pi \). The \( \mathbb{Z}_2 \)-gradation on \( \mathfrak{g}_J \) induces a \( \mathbb{Z}_2 \)-gradation on \( U(\mathfrak{g}_j) \).

- The degree 0 piece \( U(\mathfrak{g}_j)_0 \) contains \( U(\mathfrak{h}) \) and \( Z(U(\mathfrak{g}_j)) \) is a subalgebra.

Furthermore, \( U(\mathfrak{g})_\mathbb{Z}_2^+ \cap U(\mathfrak{g})_0 = \mathfrak{z} - U(\mathfrak{g}) \cap U(\mathfrak{g})_0 \) is a two sided ideal of \( U(\mathfrak{g}) \) s.t.

\[
U(\mathfrak{g})_0 = U(\mathfrak{h}) \otimes U(\mathfrak{g})_\mathbb{Z}_2^+ \cap U(\mathfrak{g})_0.
\]

Hence, we get an algebra hom \( \overline{\Pi} : U(\mathfrak{g})_0 \to U(\mathfrak{h}) \) given by projection.

- Let \( \lambda : U(\mathfrak{h}) \to U(\mathfrak{g}) \) denote the algebra hom defined by \( h_i \mapsto h_i \). If \( \lambda \) is extended to \( U(\mathfrak{g}) \) as an algebra hom. If \( \lambda \) is extended to \( U(\mathfrak{g}) \) as an algebra hom, then \( \lambda(h_i) = \lambda(h_i) \).

Then, \( (A + I)(h_i - 1) = (A + I)(h_i) = (A + I)(1) = \lambda(h_i) + g(h_i) - 1 = \lambda(h_i) \).
Def. Harish Chandra Homomorphism

\[ \psi : U(\mathfrak{g})_0 \rightarrow U(\mathfrak{h}) \] is defined as \[ \psi = \gamma_0 \pi \]

It follows \( (A + g)(\varphi(z)) = \pi(\pi(z)) + z \in \pi(U(\mathfrak{g})) \)

Since \( \mathfrak{h} \) is an abelian Lie algebra, \( U(\mathfrak{h}) = S(\mathfrak{h}) \),
and can hence be viewed as polynomial functions on \( \mathfrak{h}^* \), given by evaluation. Since \( W \) acts on \( \mathfrak{h} \), we have \( S(\mathfrak{h})^W = \{ 1 \} \) by \( \mathfrak{h} \cdot W \).
The restriction of $\psi$ to $Z(U(\mathfrak{g}))$ is an algebra isomorphism $Z(U(\mathfrak{g})) \rightarrow S(\mathfrak{h})^W$. If we only state the key steps:

1. First, it is clear it is an algebra homomorphism.
2. Second, we need to check injectivity.

- Can view $U(\mathfrak{h}) = S(\mathfrak{h})$ as polynomial functions on $\mathfrak{h}$, given by evaluation. So from above:
  $$\psi^2(2g)(\mathfrak{h}) = (A_0)(\psi^2(2g)) = 2(\pi(2 g)) = \pi(2g)(2)$$

- If $z \in Z(U(\mathfrak{g}))$, $z$ acts on the Verma module as a scalar as $h \cdot z \mathbf{v}_\lambda = z(h) \mathbf{v}_\lambda = \pi(h)z \mathbf{v}_\lambda = 2(h)z \mathbf{v}_\lambda \Rightarrow z \mathbf{v}_\lambda = \chi_\lambda(z) \mathbf{v}_\lambda$ for some algebra hom $\chi_\lambda : Z(U(\mathfrak{g})) \rightarrow \mathbb{C}$, called a character.

- Since $z - \pi(2g) \in U(\mathfrak{g})^+ \cap Z(U(\mathfrak{g}))$ annihilates $\mathbf{v}_\lambda$.
  Therefore, $\chi_\lambda(z) = \chi(z(\pi(2g))) = \psi(2g)(A_0)$. 

So if \( \psi(z) = 0 \), \( z \) acts as 0 on every finite dimensional module \( \Rightarrow z = 0 \) (the proof of this is very similar to the analogous statement for quantum groups proved last time).

3. Third, we need to show that the image of \( \psi \) lies in \( S(U)W \).

- For \( w \in W \), let \( \omega \cdot z = w (A + p) - z \); defines dot action of \( W \).

If \( f_{a_i} \) simple r.t. \( (\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \), then \( M(\lambda) \) has a maximal vector \( f_{(\lambda, \alpha_i^\vee + 1)}v_\lambda \) of weight \( \lambda + \lambda - (\lambda, \alpha_i^\vee + 1)\alpha_i \).

(this is a consequence of viewing \( M(\lambda) \) as an \( sl_2 \)-module of the \( sl_2 \) triple \( \frac{1}{2} f_{a_i}, h_i, e_i \).

Hence, there is an embedding \( M(\lambda - (\lambda, \alpha_i^\vee + 1)\alpha_i) \hookrightarrow M(\lambda) \).

This implies \( \chi_{S_{\alpha_i} \cdot \lambda} = \chi_\lambda \). In particular, if \( \lambda \) is dominant integral, a short induction argument (omitted) shows \( \chi_\omega \cdot \lambda = \chi_\lambda \) \( \forall \omega \in W \).
Therefore, \( \bar{\Pi}(z)(\lambda) = x_{\lambda}(z) = x_{w \cdot \lambda}(z) = \bar{\Pi}(z)(w \cdot \lambda) \),

so the two polynomial functions \( \bar{\Pi}(z)(w \cdot \lambda) \) and \( \bar{\Pi}(z)(w \cdot \lambda) \) agree on \( \Lambda^+ \), a Zariski dense subset of \( \mathbb{C}^* \). Hence, they agree on all of \( \mathbb{C}^* \), so for all \( z \in \mathbb{C}^* \), we have

\[
\bar{\Pi}(z)(\lambda) = \bar{\Pi}(z)(w \cdot \lambda)
\]

\[
\Rightarrow \quad \Psi(z)(\lambda \cdot \mu) = \Psi(z)(w \cdot \lambda \cdot \mu)
\]

writing \( \mu = z^{-g} \cdot \lambda \):

\[
\Rightarrow \quad \Psi(z)(\mu) = \Psi(z)(w \mu)
\]

Therefore, \( \Psi \) is a map from \( \mathbb{C}(U\mathfrak{g}) \) to \( S(\mathfrak{g}) \).

4. Finally, one needs to show surjectivity.

This is somewhat long, so I will be brief.

Let:

- \( \mathcal{P}(\mathfrak{g}) = S(\mathfrak{g}^*) \) be polynomial functions on \( \mathfrak{g}^* \)
- \( \Theta: \mathcal{P}(\mathfrak{g}) \to \mathcal{P}(\mathfrak{h}) \) be given by restriction,

is an algebra homomorphism.

- \( \mathcal{A}(\mathfrak{g}) \) be the group generated by \( \{ \exp a \cdot x : x \text{ nilpotent} \} \)
$G$ acts on $p(g)$ by $(g.p)(y) = p(g.y)$ for $p \in p(g)$, $y \in \mathfrak{g}$, $g \in G$.

The crux is the following theorem of Chevalley:

Thm. (Restriction Thm)

$\Omega$ maps $p(\mathfrak{g})^G$ isomorphically onto $p(\mathfrak{h})^\mathfrak{w}$.

The proof of Chevalley's theorem involves looking at traces of finite dimensional representations.

After using the Killing form to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and $\mathfrak{h}$ with $\mathfrak{h}^*$, this gives an isomorphism

$$8(\mathfrak{g})^G \rightarrow 8(\mathfrak{h})^\mathfrak{w}$$

Finally, it can be shown $8(\mathfrak{g})^G$ is isomorphic as a vector space to $Z(U(\mathfrak{g}))$. 

$\Box$
3) **Notation for Quantum Groups**

- \( U = U_q(\mathfrak{g}) \) is the quantized enveloping algebra over \( \mathfrak{g} \) with parameter \( q \).

Recall it is defined by generators and relations

\[
U_q(\mathfrak{g}) = \langle E_a, F_a, K_a, K_a^{-1} : a \in \Pi \rangle / \text{relations}
\]

Relations, \( \Pi \) is \( \Pi \):

1. \( K_a K_a^{-1} = K_a^{-1} K_a = 1 \)
   \( K_a K_b = K_b K_a \)

2. \( K_a F_b K_a^{-1} = q^{(a,b)} F_b \)

3. \( K_a F_b K_a^{-1} = q^{-(a,b)} F_b \)

4. \( [E_a, F_b] = \delta_{ab} \frac{K_a - K_a^{-1}}{q_a - q_a^{-1}} \) where \( q_a = q^{(a,b)} \)

5. **quantized Serre relations**

- For \( \lambda, \mu \in \mathbb{Z} \), \( \Pi = \sum_{a \in \Pi} \alpha_a \), \( K_\lambda := \prod_{a \in \Pi} K_a^{\alpha_a} \). Clearly \( K_\lambda K_\mu = K_{\lambda + \mu} \)

\[
\text{for } \lambda, \mu \in \mathbb{Z} \]

- \( U^+ \) = subalgebra generated by \( E_a \)

- \( U^- \) = subalgebra generated by \( F_a \)

- \( U^0 \) = (commutative) subalgebra generated by \( K_a, K_a^{-1} \)

- \( Z(U) \) = center of \( U \).

- \( U \) admits a \( \mathbb{Z} \) grading where \( \deg E_a = \alpha \), \( \deg F_a = -\alpha \), \( \deg K_a = \deg K_a^{-1} = 0 \)

Let \( U = \{ x \in U : \deg x = 0 \} \) (\( \mathbb{D} \subseteq \mathbb{Z} \)).
• Notice for \( u \in U_0 \), \( K_2 u K_2^{-1} = q^u \) \( (1, 0) \) \( (1, 0) \in \mathbb{Z} \).

Since, \( q \) not a root of unity \( \Rightarrow U_0 = \bigoplus_{x} U \bigcup \bigcup_{x} \bigcup_{u} \bigcup_{v} U \cdot q^{x} \cdot q^{x} \cdot q^{x} \cdot q^{x} \cdot q^{x} \) \( \forall x \in \mathbb{Z} \).

\( U^2 \) = subalg generated by \( E_{x}, k_{x}^{\pm 1} \) \( (x \in \mathbb{F}) \)
\( U \subseteq \) subalg generated by \( E_{x}, k_{x}^{\pm 1} \)

• \( W \) acts on \( U^0 \) by \( W k_{x} = k_{x} W, \forall x \in \mathbb{Z} \).

• \( U^0 := \bigoplus_{x \in \mathbb{Z} \cap \mathbb{N}} K_{x} \), a subalgebra of \( U^0 \).

This is stable under \( W \).

• \( (U^0)^W := \bigoplus_{x \in U^0} : \omega u = u \forall \omega \in W \).
1. Description of the Center of a Quantum Group

1.1) A very short outline

Just proved that $\mathcal{Z}(U_h(\mathfrak{g})) = S(h)^W$. What can we say about $\mathcal{Z}(U)$, the center of the quantum group?

Turns out that we can produce an algebra isomorphism $\Upsilon: \mathcal{Z}(U) \to (U_0^0)^W$ which is what we will construct in this lecture.

Our approach will be similar in nature to the proof we produced, but with some modifications. Step 1 to show $\Upsilon$ is an algebra homom and Step 2 to show $\Upsilon$ is injective go through almost identically. Step 3 to show that the image is $(U_0^0)^W$ will require some modification.

In step 4 to show surjectivity, we need to modify heavily; the trace needs to be replaced by quantum trace, and more drastically we must work around using $\delta$. (Remark: this shows bijectivity.)
4.2) The (Quantum) Harish Chandra Homomorphism

For \( \lambda \in \Lambda \), we can produce an algebra hom \( \mathcal{U}^0 \to \mathcal{U}^0 \) which by abuse of notation we also denote \( \lambda \) defined by \( \lambda(k_\mu) := q(\nu, \mu) \quad \forall \mu \in \mathcal{Z} \Phi \),

which is well defined as \( k_\mu \) is a basis of \( \mathcal{U}^0 \) and is an algebra hom because

\[
\lambda(k_{\mu+\nu}) = q^{(\lambda, \mu+\nu)} = q^{(\lambda, \mu)} q^{(\lambda, \nu)} = \lambda(k_\mu) \lambda(k_\nu).
\]

Furthermore, if \( \lambda, \lambda' \in \Lambda \),

\[
(\lambda + \lambda')(k_\mu) = q^{(\lambda + \lambda', \mu)} = q^{(\lambda, \mu)} q^{(\lambda', \mu)}
\]

\[
= \lambda(k_\mu) \lambda'(k_\mu).
\]

Now, if \( \lambda \in \Lambda \), let \( \nu_\lambda \) denote the algebra hom \( \mathcal{U}^0 \to \mathcal{U}^0 \) given by

\[
\nu_\lambda(k_\mu) = \lambda(k_\mu) k_\mu = q^{(\lambda, \mu)} k_\mu
\]

(\( \mu \in \mathcal{Z} \Phi \)). Notice \( \nu_{\lambda + \lambda'} = \nu_\lambda \circ \nu_{\lambda'} \).
Recall $U_0 = \{ u \in U \mid \deg u = 0 \}$

$= \{ u \in U \mid \exists \mu, \nu : u = \mu \nu \}$

So clearly $Z(U) \subset U_0$.

The isomorphism $U^0 \otimes U^0 \otimes U^\perp \rightarrow U$ means we can write $U_0 = U^0 \oplus \bigoplus_{u \geq 0} U^0 U^0 U^\perp$. Hence, denote the projection $U_0 \rightarrow U^0$ as $\overline{\Pi}$ (similar to classical case).

Exercise: check $\overline{\Pi}$ is an algebra hom.

Def. (Harish-Chandra homomorphism)

$\psi : U_0 \rightarrow U^0$ is defined as

$\psi = \chi_{-g} \circ \overline{\Pi}$. 
4.3) The Main Theorem

\textbf{(****) Main Theorem}

Suppose \( K \) is a field of characteristic 0 and \( q \in K^* \) is transcendental over the prime subfield \( \mathbb{Q} \subset K \). Then the restriction

\[ \psi = \left. \frac{1}{2} \cdot \Pi \right|_{Z(0)} : Z(0) \to U^0 \]

is an isomorphism onto its image, which is \( (U^0_{ev})^W \).

From now on we shall just write \( \psi \) to denote the restriction of \( \psi \) to the center, and similarly for \( \Pi \).
5) **Proof of the Main Thm.**

5.1) **Outline**

We will spend the rest of the lecture proving this theorem.

Let's outline what we'll do.

1) First, show that \( \Psi \) is an injective algebra homomorphism from \( \mathbb{Z}(U) \) to \( (U^0_{ev})^{\mathbb{W}} \). This encompasses steps 1 to 3 in the proof of the classical result above.

Then, we need to replace step 4 in that proof to show surjectivity.

To do so:

2) We need to construct elements in \( \mathbb{Z}(U) \) whose images under \( \Psi \) span \( (U^0_{ev})^{\mathbb{W}} \). To do this, we turn to the
dual space $U^*$, which is naturally a $U$-mod
dual to the adjoint representation $U$, and
construct a nondegenerate, invariant pairing
on $U$, which gives an embedding $U \hookrightarrow U^*$.
Then, in $U^*$, we construct $U$-invariants using the
quantum trace of finite dimensional representations,
and show these lie in the image of the embedding.
Since they are invariant, their preimages will be invariant.
A general fact of Hopf algebras is invariance w.r.t. adjoint
rep is the same as being central. Then,
we show that the images of these central elements under $\Psi$ span $U$. 

Rmk. for this talk, we have assumed char $\mathbf{k} = 0$
and $q$ is transcendental over $\mathbf{Q} \subset \mathbf{k}$.
This can be weakened to requiring $q$ not be
a root of unity.
5.1) Showing \( \Psi \) is injective and its image is in \((U^0_{eu})^w\).

- Recall the type I Verma module \( M(\lambda) = \bigoplus_{\mu \in \mathcal{P}} U_{\mu} \otimes U_{\mu}(k_\mu q^\mu) \)
gen

**Lemma**

a) Let \( \lambda \in \Lambda \). \( z \in Z(\mathfrak{u}) \) acts on \( M(\lambda) \) as a scalar, this scalar is \( \lambda(\overline{\pi}(z)) \).

b) \( \overline{\pi} \) is injective on \( Z(\mathfrak{u}) \).

**Proof**

a) Left as an exercise

b) If \( \overline{\pi}(z) = 0 \) for \( z \in Z(\mathfrak{u}) \), then \( z \) acts as 0 on all type I finite dimensional representations as these are quotients of \( M(\lambda) \) for some \( \lambda \).

Now apply the theorem from last time to deduce \( z = 0 \).

Recall if \( \lambda \in \Lambda \), \( \mathfrak{k}_z \) denotes the algebra hom \( \mathfrak{k}_z : U^0 \rightarrow U^0 \) given by

\[
\mathfrak{k}_z (k_\mu) = \mathfrak{z}(k_\mu) k_\mu = q^{(\lambda, \mu)} k_\mu
\]

(\( \mathfrak{z} \in \mathfrak{z} \)).
Recall the dot action of $W$ on $A$ given by $w \cdot A : = w (2 + g) - g$.

Recall $w W = W w$ defines an action of $W$ on $U$.

**Lemma 2** \[ \Psi(2(U)) \leq (U^0)^W \]

**Proof.** Let $z \in 2(U)$, and let $h = \Psi(z) = 2g \circ \pi_2(w)(z) \Rightarrow \Psi(h) = \pi_2(2(U))(z)$.

By Lemma 1, for $x \in A$, $z$ acts on $M(2)$ as the scalar $\lambda(\pi_2(2(U))(z)) = A(\Psi(h)) = (2 + g)(h)$

From last time, we saw that for $x \in \Pi$, if $<2x, \nu> = 0$, $f$ non-zero hom $M(S_2, 2) \rightarrow M(2)$, so $z$ must act as the same scalar on both. Hence

\[ (2 + g)(h) = (S_2 \cdot 2 + g)(h) = (S_2(2 + g))(h) = (2 + g)(S_2^{-1}(h)). \]

(*) also holds if $<2x, \nu> > 0$, if $<2x, \nu> = 1$.
then $s_0 \cdot \lambda = \lambda$ and \((\ast)\) follows. If
\[
\langle \alpha, \alpha \rangle < -1, \quad \text{then} \quad \langle s_0 \cdot \lambda, \alpha \rangle = 2 \frac{\langle s_0 (\alpha + \gamma), \alpha \rangle}{\langle \alpha, \alpha \rangle}
\]
\[
= -2 \langle \alpha + \gamma, \alpha \rangle - 2 \quad \text{by def of } \gamma \text{ and } \langle \alpha, \alpha \rangle
\]
\[
= -\langle \alpha, \alpha \rangle - 2 > 0, \quad \text{so } \ (\ast)
\]
applies to $s_0 \cdot \lambda$ and hence to $\lambda$.

Since simple reflections generate $W$, we deduce
\[
(\alpha + \gamma)(h) = (\alpha + \gamma)(wh) \quad \forall w \in W, \forall h, w \in V.
\]
\[
\Rightarrow \quad T(h) = T(wh) \quad \text{as } g < V.
\]

Finally, if $h - wh = \sum q_\mu K_\mu$, then $0 = \lambda(h - wh) = \sum q_\mu \xi^{(\gamma, \Delta)}$. Each
\[
\lambda \mapsto q_\lambda \xi^{(\gamma, \Delta)}
\]
is a character on $V$; these are distinct as $q_\lambda$ is not a root of unity, and it is known distinct characters are linearly independent $\Rightarrow q_\lambda = 0 \forall \mu \Rightarrow h - wh = 0 \Rightarrow q(2) \in (0^0) \, W$.

Recall $U^0 = \bigoplus_{\mu \in \omega} K_\mu$.  \[D\]
**Proposition 3.** The image of $\psi$ lies in $(U^0)^W$.

If $2 \in \mathcal{Z}(U)$, we already know that $\psi(2) \in (U^0)^W$. Hence, can write

$$\varphi(2) = \sum a_\mu K_\mu$$

where $a_\mu = a_\mu f$. Hence, we can write

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We want to show that $a_\mu \neq 0 \Rightarrow \mu \in 2\Lambda$. Since $\Pi$ is a basis of $2\Phi$, there exists a group hom $\sigma: 2\Phi \to 2\Phi$ such that $\sigma(x) = -x$ for $x \in \Pi$. Now, $\sigma$ induces an algebra automorphism $\tilde{\sigma}: U \to U$ by

$$\tilde{\sigma}(E_a) = \sigma(a) E_a = -E_a$$

$$\tilde{\sigma}(F_a) = F_a$$

$$\tilde{\sigma}(K_a) = \sigma(a) K_a = -K_a$$

Notice $\tilde{\sigma}$ preserves $\mathcal{Z}(U)$, $U^0$, $U^+$, $U^-$, grading on $U$. i.e.

$$\tilde{\sigma}(\mathcal{Z}(U)) = \mathcal{Z}(U), \quad \tilde{\sigma}(U^+) = U^+, \quad \tilde{\sigma}(U^-) = U^-$$

for $U \in 2\Phi$, $\tilde{\sigma}(U^0) = U^0$. It follows $\tilde{\sigma} \circ \Pi = \Pi \circ \tilde{\sigma}$.

Also clear \(\forall \lambda \in \Lambda\), $\tilde{\sigma} \circ \xi_\lambda = \xi_\lambda \circ \tilde{\sigma}$. Checking on $K_a$'s. Therefore,

$$\tilde{\sigma} \circ \psi = \tilde{\sigma} \circ \xi_\lambda \circ \pi_2(U_{\lambda 0}) = \xi_\lambda \circ \pi_2(U_{\lambda 0}) \circ \tilde{\sigma} = \psi \circ \tilde{\sigma}$$

Therefore, if $2 \in \mathcal{Z}(U)$ and $\psi(2) = \sum a_\mu K_\mu$, then

$$\psi(\tilde{\sigma}(2)) = \tilde{\sigma}(\psi(2)) = \sum a_\mu \tilde{\sigma}(K_\mu) = \sum a_\mu \sigma(a) K_\mu$$
$\sigma(2)$ is central so by Lemma 2, this sum lies in $(H)^W$.

Therefore, $a_m \sigma(m) = a_m \sigma(\omega \lambda) = a_m \sigma(m \lambda)$ for all $\lambda \in H$.

$s_m = 0 \Rightarrow \sigma(m) = \sigma(\omega \lambda)$ for all $\lambda \in W$, so if $m = s$ a simple, then

$1 = \sigma(m - s \lambda m) = \sigma(\langle m, \lambda \rangle) = (-1)^{\langle m, \lambda \rangle}$.

$\Rightarrow \langle m, \lambda \rangle$ is even for all $\lambda \in H \Rightarrow m \in 2W$.

Combining these statements, we have shown that

$\Psi$ is an embedding $\Psi: Z(H) \rightarrow (U_{eq})^W$.  

\[ \text{Therefore, } a_m \sigma(m) = a_m \sigma(\omega \lambda) = a_m \sigma(m \lambda) \text{ for all } \lambda \in H. \]

$s_m = 0 \Rightarrow \sigma(m) = \sigma(\omega \lambda)$ for all $\lambda \in W$, so if $m = s$ a simple, then

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Combining these statements, we have shown that

$\Psi$ is an embedding $\Psi: Z(H) \rightarrow (U_{eq})^W$.  

\[ \text{Therefore, } a_m \sigma(m) = a_m \sigma(\omega \lambda) = a_m \sigma(m \lambda) \text{ for all } \lambda \in H. \]
5.2) Showing that $\Psi$ is surjective on to $(U^0_e) W$

The remainder of the lecture will show $\Psi$ is surjective.

5.2.1) The Bialgebra Pairing

Let's recall the Hopf algebra structure $(\mu, \eta, \Delta, \varepsilon, S)$ on $U$.

* $\mu$ is the multiplication, $\eta$ is the unit,
* $\Delta: U \to U \otimes U$ is the comultiplication uniquely satisfying

\[
\Delta(E_x) = E_x \otimes 1 + K \circ E_x
\]

\[
\Delta(F_x) = F_x \otimes K^{-1} + 1 \otimes F_x
\]

\[
\Delta(K_x) = K_x \otimes K_x
\]

* $\varepsilon: U \to \#$ is the counit and algebra homomorphism uniquely satisfying $\varepsilon(E_x) = E_x = 0$, $\varepsilon(F_x) = 1$.

* $S: U \to U$ is the antipode, satisfying

\[
S(E_x) = -K_x E_x
\]

\[
S(F_x) = -F_x K_x, \quad S(K_x) = K^{-1}_x.
\]
In order to construct the invariant pairing on $U \times U$, we first need to construct a bialgebra pairing on $U \otimes U \otimes U$.

**Definition.** Let $X, Y$ be two Hopf algebras. A pairing $\langle, \rangle : X \otimes Y \to K$ is a bialgebra pairing if

1. $(x(x_2, y)) = (x, \Delta(x_2), \Delta(y))$ and
2. $(x, y, y_2) = (\Delta(x), y, \Delta(y_2))$,

where $x, x_2 \in X$, $y, y_1, y_2 \in Y$, $\Delta$ and $\Delta'$ are the comultiplications on $X$, $Y$, respectively, and the form on $X \otimes X \otimes Y \otimes Y$ is defined by

$$\langle x \otimes x_1, y_1 \otimes y_2 \rangle = (x, x_2)(y_1, y_2).$$

**Motivation:** $A^* = \text{Alg}(A, A)$ is naturally an algebra by defining its multiplication as $\mu_{A^*} : A^* \otimes A^* \to A^*$ as $\mu_{A^*} = \Delta^* \circ (\Delta \otimes \Delta)^* : A^* \otimes A^* \to A^* \otimes A^* \to A^* \otimes A^* \to A^*$. Similarly for $B^*$. Therefore, the conditions in $(1)$, $(2)$ are the same as requiring $A \to B^*$ ($a \mapsto (a, i)$) and $B \to A^*$ ($b \mapsto (\cdot, b)$) be algebra homs, resp.
$U^{<0}$ and $U^{<0}$ are Hopf subalgebras of $U$. 

**Thm. 6**

There exists a unique bialgebra pairing 

$(\cdot, \cdot) : U^{<0} \times U^{<0} \rightarrow \mathbb{H}$ s.t. $\forall \alpha \beta \in \mathbb{H}$:

1. $(\alpha, \beta) = -q(\beta, \alpha)$, $(\alpha, \beta) = -\delta_{\alpha \beta} (2q_{\alpha} - q_{\alpha}^{-1})^{-1}$
2. $(\alpha, \beta) = (\alpha, \beta) = 0$

**Pr.**

Assume existence of such a form, uniqueness is immediate as relations 1) and 2) define it on the generators, from which the bialgebra property will show it is defined on the whole algebras. For example:

$$(\alpha F \beta, \alpha E \beta) = (\alpha \Delta \beta, \alpha \Delta E \beta)$$

$$= (\alpha \alpha \beta, \alpha \beta \beta)(\alpha \beta \alpha + k \alpha \beta \beta)$$

$$= (\alpha \Delta \beta, \alpha \beta \beta) = (\alpha, \beta) (\beta, 1) + ... + (\alpha, k \beta)(\beta, \beta)$$

then use (2) and it will be in terms of the form on generators.
Now, let's do existence.

- Recall that \( O^{\text{eo}} \) is the algebra with the same generators and relations as \( U^{\leq 0} \) except the quantum Serre relations.

\( O^{\text{eo}} \) is a bialgebra with the Hopf structure formulas recalled above, and \( U^{\text{eo}} \) is a bi-

- We will first define a bialgebra pairing on \( O^{\text{eo}} \times U^{20} \) as it is easier to define. Then, we show how it descends to \( U^{\text{eo}} \times U^{20} \).
• For \( \alpha \in \Pi \), define \( f^\alpha \in (U^{20})^* \) by
\[
\begin{align*}
\sum f^\alpha (F_{\alpha} \kappa \mu) &= - (q_{\alpha} - q^{-1}_{\alpha})^{-1} \quad \forall \mu \in \mathcal{P} \\
f^\alpha (U^{20}) &= 0 \quad \forall \alpha \neq \alpha \in \mathcal{P}
\end{align*}
\]

• For \( \lambda \in \Lambda \), define \( k^\lambda \in (U^{20})^* \) by
\[
k^\lambda (u \kappa \mu) = z(u) q^{-1}(\lambda/\mu) \quad (\mu \in \mathcal{P}, \lambda \in \mathcal{U})
\]
Note \( k^\lambda \) is an algebra hom,

**Lemma 7** \( F_{\alpha} \mapsto f^\alpha \), \( k_{\mu} \mapsto k^\lambda \) (\( \alpha \in \Pi \), \( \mu \in \mathcal{P} \))
defines an algebra hom
\[
\tilde{U} \xrightarrow{\cong} (U^{20})^*.
\]

**Pf.** Left as an exercise (but this is straightforward as \( \tilde{U} \xrightarrow{\cong} \) behaves like a free algebra, since \( F \)'s have no relations, and relations involve \( \kappa \mu \) are easy).

Therefore, this algebra hom induces a pairing
\((C_1): \; \widetilde{U} \leq 0 \times U^{20} \rightarrow \# \text{satisfying} \)

\((x_1, x_2, y) = (x_1 \otimes x_2, ay) \) \quad \forall x_1, x_2 \in U \\
\forall y \in U^{20}.

\underline{Lemma 8}

\((C_1): \; \widetilde{U} \leq 0 \times U^{20} \rightarrow \# \text{ also satisfies} \)

\((x_1, y, y_2) = (ay, x_1 \otimes y_2) \) \quad \forall x_1 \in \widetilde{U} \leq 0 \\
\forall y, y_2 \in U^{20}.

\underline{Pf.}

We first claim that if \((x_1, y, y_2) = (\Delta(x_1), y_1 \otimes y_2)\) and \((x_2, y, y_2) = (\Delta(x_2), y_1 \otimes y_2)\) \quad \forall x_1, x_2 \in \widetilde{U} \leq 0, \; y, y_2 \in U^{20}, \; \text{then} \quad (x_1x_2, y, y_2) = (\Delta(x_1x_2), y_1 \otimes y_2)\)

This is a direct computation using Sweedler notation for the comultiplication, and the fact \((x, x_2, y) = (x \otimes x_2, \Delta(y))\) by construction of the pairing — no other property of \(U\) is needed, so this is a general statement left as an exercise.
Then, one can simply verify on the generators \( F_\alpha, K_m \) (at \( \pi, \mu \in \mathbb{D} \)) that

\[
(K_m, y_1 y_2) = (\Delta(K_m), y_1 \otimes y_2)
\]

and

\[
(F_\alpha, y_1 y_2) = (\Delta(F_\alpha), y_1 \otimes y_2). \quad (\star)
\]

Let's do it for \( K_m \):

\[
(K_m, y_1 y_2) = k^m(y_1 y_2) = k^m(y_1) k^m(y_2)
\]

\[
= (K_m, y_1)(K_m, y_2) = (k^m \otimes k^m, y_1 \otimes y_2)
\]

\[
= (\Delta(k^m), y_1 \otimes y_2).
\]

For \( F_\alpha \), one can check \((\star)\) first for \( y_1 = E_{\beta_1} E_{\beta_2} \ldots E_{\beta_q} K_m \) and \( y_2 = E_{\theta_1} E_{\theta_2} \ldots E_{\theta_s} K_{m_1} \)

\[
(\beta_i, \theta_i \in \Pi, \ m, m_1 \in \mathbb{D})
\]

by a direct calculation. Then \((\star)\) follows by linearity. The details are left to the reader. \( \square \)

Combining lemmas 7 and 8, we have a bialgebra pairing \((1)\): \( \mathcal{D} \otimes \mathcal{D} \to \mathbb{K} \).
Lemma 9. \( \forall \mu, \nu \in \mathbb{Z}_{20}, \mu \neq \nu \)

\[ \Rightarrow (x, y) = 0 \quad \forall x \in \mathbb{U}^\mu, \quad y \in \mathbb{U}^\nu \]

This is by construction of the form.

In order to show \( C_1 : J^\leq \times U^\geq \to I \) it descends to a form on \( U^\leq \times U^\geq \), we need the left radical to contain the elements \( (X + \beta \in \mathbb{U}) \):

\[
\mathbb{U}^\alpha_{\beta} = \sum_s (1 - \beta)^s \left[ \alpha_{\beta} \right] F^s F^s \beta^s
\]

\( s = 0 \)

which define the quantum commutation relations. Indeed,

Lemma 10

\[ (\mathbb{U}^\alpha_{\beta}, x) = 0 \quad \forall x \in \mathbb{U}^\geq, \quad \forall \alpha \neq \beta \in T. \]
If. By lemma 9, we need only look at the restriction of the pairing to \( \tilde{\mathbf{U}}^-(\alpha + \beta) \times \mathbf{U}^+_{\alpha + \beta} \).

\[ \mathbf{U}_{\alpha \beta} \in \tilde{\mathbf{U}}^-(\alpha + \beta) \]

Now, we will look at what happens when type \( \alpha_j = A_2 \); the general case is left as an exercise.

There are two possibilities for \( \alpha + \beta \in \mathbb{T} \) in this case, but by symmetry, it suffices to look at one. Picking one, we have \( \alpha \beta = -1 \), and

\[ \mathbf{U}_{\alpha \beta} = \sum_{S=0}^{2} C(S) F_s^2 F_\beta F_s F_\alpha = F_\alpha F_\beta - [2] F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2 \]

and a basis for \( \mathbf{U}^+_{2 \alpha + \beta} \) is

\[ \sum F_\alpha^2 F_\beta K_\alpha, F_\alpha F_\beta F_\alpha K_\alpha^3 \mu \in \mathbb{Z} \]

The \( K_\alpha \)'s can effectively be ignored in the pairing, they only contribute a factor. A direct computation shows that \( (\mathbf{U}_{\alpha \beta}, b) = 0 \) for each basis vector \( b \).

Now, applying lemma 10, the biadjoint pairing on \( \mathbf{g}_{\alpha} \mathbf{x} \times \mathbf{U}^+_{\alpha} \) descends to one on \( \mathbf{g}_{\alpha} \mathbf{x} \times \mathbf{U}^+_{\alpha} \) also denoted as \( C_1 \). This proves theorem 6. \( \square \)
References

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