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The Center of a Quantum Group, Part 2

Goal: We wish to determine the center $Z(U)$ of the quantum group $U = U_q(\mathfrak{g})$.

Approach: Proceed similarly to the classical case, where one determines the center of the universal enveloping algebra of a semisimple Lie algebra. We follow the presentation in Jantzen's book "Lectures on Quantum Groups" (hence forth refered to as Jantzen).

Where we are going...

Using the bialgebra pairing on $U^{\leq 0} \times U^{\geq 0}$, we construct a non-degenerate invariant pairing $\langle\ \rangle : U \times U \to \mathbb{K}(q^{1/2})$, which gives an embedding $U \hookrightarrow U^*$.

Then, in U^*, we construct U-invariants using the quantum trace of finite dimensional representations and show these lie in the image of the embedding; since they are invariant, their preimages will be invariant. A general fact of Hopf algebras is invariance w.r.t. adjoint rep is the same as being central. Then, we show that the images of these central elements under Ψ span $(U^0_{ev})^W$ as a $\mathbb{K}$ vector space.

Recall From Last Time...

The Main Theorem

- Recall $U_0 = U^0 \oplus \bigoplus_{\nu > 0} U^-_{-\nu} U^0 U^+_{\nu}$ gives a projection map $\overline{\pi} : U_0 \to U^0$.
- Recall the algebra isomorphism $\gamma_{-\rho} : U^0 \to U^0$ uniquely given by $\gamma_{-\rho}(K_\mu) = q^{-(\rho,\mu)} K_\mu$ $\forall \mu \in \Phi$.

Def. (Harish-Chandra homomorphism)

$\psi : U_0 \longrightarrow U^0$ is defined as

$$\psi = \gamma_{-\rho} \circ \overline{\pi}.$$

Recall $U^0_{ev} = \bigoplus_{\mu \in 2\Phi \cap 2\Lambda} \mathbb{k} K_\mu$ is a W-stable subalg of U^0.

(***) Main Theorem

Suppose $\mathbb{k}$ is a field of characteristic 0 and $q \in \mathbb{k}^\times$ is transcendental over the prime subfield $\mathbb{Q} \subset \mathbb{k}$. Then, the restriction

$$\psi|_{Z(U)} = \gamma_{-\rho} \circ \overline{\pi}|_{Z(U)} : Z(U) \longrightarrow U^0 \text{ is an isomorphism}$$

onto its image, which is $(U^0_{ev})^W$.

Henceforth, the restrictions of ψ and $\overline{\pi}$ to the center shall be denoted as ψ and $\overline{\pi}$, resp.

Last time, we showed Ψ is injective and $\Psi(Z(U)) \subseteq (U^0_{ev})^W$. We began the proof of surjectivity by constructing a bialgebra pairing $(\,,\,): U^{\leq 0} \times U^{\geq 0} \to \mathbb{K}$.

Thm. 6 There exists a unique bialgebra pairing $(\,,\,): U^{\leq 0} \times U^{\geq 0} \to \mathbb{K}$ s.t. $\forall\, \alpha, \beta \in \Pi$:

i.) $(K_\mu, K_\nu) = q^{-(\mu,\nu)}$, $(F_\alpha, E_\beta) = -\delta_{\alpha\beta}(q_\alpha - q_\alpha^{-1})^{-1}$

ii) $(K_\mu, E_\alpha) = (F_\alpha, K_\mu) = 0$

bialgebra pairing meaning $\forall x_1, x_2 \in U^{\leq 0}$, $y_1, y_2 \in U^{\geq 0}$

1) $(x_1 x_2, y_1) = (x_1 \otimes x_2, \Delta(y_1))$

2) $(x_1, y_1 y_2) = (\Delta(x_1), y_1 \otimes y_2)$

where Δ is the comultiplication on U.

Lemma 9 $\forall \mu, \nu \in \mathbb{Z}_{\geq 0}\Pi$, $\mu \neq \nu$

$\Rightarrow (x, y) = 0$ $\forall x \in U^-_{-\mu}, y \in U^+_\nu$

Let's workout an example: if $\alpha \in \Pi$

<u>Ex.</u> $$(F_\alpha^n, E_\alpha^n) = (-1)^n q_\alpha^{n(n-1)/2} \frac{[n]_\alpha!}{(q_\alpha - q_\alpha^{-1})^n}$$

where $$[n]_\alpha = \frac{q_\alpha^n - q_\alpha^{-n}}{q_\alpha - q_\alpha^{-1}}.$$

<u>Pf.</u>

For $n=0$, it is obvious.

For $n=1$, $(F_\alpha, E_\alpha) = -(q_\alpha - q_\alpha^{-1})$ is clear from the prop.

For $n>1$, $$(F_\alpha^n, E_\alpha^n) = (F_\alpha^{n-1} \otimes F_\alpha, \Delta(E_\alpha^n)) \quad (*)$$

An inductive computation shows
$$\Delta(E_\alpha^n) = \sum_{i=0}^n q_\alpha^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_\alpha E_\alpha^{n-i} K_\alpha^i \otimes E_\alpha^i$$

In $(*)$, only the $i=1$ term survives by Lemma 9. So:
$$\begin{aligned}(F_\alpha^n, E_\alpha^n) &= (F_\alpha^{n-1} \otimes F_\alpha, \ q_\alpha^{n-1}[n]_\alpha E_\alpha^{n-1} K_\alpha \otimes E_\alpha) \\ &= q_\alpha^{n-1}[n]_\alpha (F_\alpha^{n-1}, E_\alpha^{n-1} K_\alpha)(F_\alpha, E_\alpha) \\ &= q_\alpha^{n-1}[n]_\alpha \cancel{q^{-(0,\alpha)}} (F_\alpha^{n-1}, E_\alpha^{n-1}) (F_\alpha, E_\alpha)\end{aligned}$$

Now, apply induction to deduce
$$= (-1)^n q_\alpha^{n(n-1)/2} \frac{[n]_\alpha!}{(q_\alpha - q_\alpha^{-1})^n}.$$

5.2.2) Non degeneracy of $(,)$.

Prop. 11 The restriction of $(,)$ to $U^-_{-\mu} \times U^+_\mu$ $(\mu \in Q_{\geq 0}\Pi)$ is non-degenerate $\forall \mu \in Q\Phi$.

Pf. (sketch)

Because $\dim U^-_{-\mu} = \dim U^+_\mu$ (by using the involution ω on U, given by $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$, $\omega(K_\alpha) = K_\alpha^{-1}$), it suffices to check that if $\exists y \in U^-_{-\mu}$ s.t. $\forall x \in U^+_\mu$ $(y, x) = 0$, then $y = 0$.

We induct on the partial ordering on $Q\Phi$. Clearly the claim is true for $\mu = 0$, as $U_0^- = U_0^+ = k$ and $(1,1) = 1$.

Now, suppose the restriction to $U^-_{-\nu} \times U^+_\nu$ is non-deg for all $0 \leq \nu < \mu$. Suppose $y \in U^-_{-\mu}$ is in the left radical of the form. Then, $\forall x \in U^+_{\mu-\alpha}$, $E_\alpha x$ and $x E_\alpha \in U^+_\mu$, so

$$(y, xE_\alpha) = 0 = (y, E_\alpha x).$$

Need to introduce some new elements to proceed:

<u>Def.</u> For $y \in U^-_{-\mu}$ $(\mu \in \mathbb{Z}\Phi)$, define

$r_\alpha(y)$, $r'_\alpha(y) \in U^-_{-\mu+\alpha}$ $(\alpha \in \Pi)$

as follows:

- Recall

$$\Delta(F_\beta) = F_\beta \otimes K_\beta^{-1} + 1 \otimes F_\beta \quad (\beta \in \Pi)$$
$$\Delta(K_\theta) = K_\theta \otimes K_\theta \quad (\theta \in \mathbb{Z}\Phi)$$

- Then, for $y \in U^-_{-\mu}$, one can deduce

$$\Delta(y) \in \bigoplus_{0 \le \nu \le \mu} U^-_{-\nu} \otimes U^-_{-(\mu-\nu)} K_\nu^{-1}.$$

Therefore, $\exists$ $r_\alpha(y)$, $r_\alpha'(y) \in U^-_{-(\mu-\alpha)}$ s.t.

$$\Delta(y) = \underbrace{y \otimes K_\mu^{-1}}_{\in U^-_{-\mu} \otimes U^-_0} + \underbrace{\sum_{\alpha \in \Pi} r_\alpha(y) \otimes F_\alpha K_{\mu-\alpha}^{-1}}_{\in U^-_{-(\mu-\alpha)} \otimes U^-_{-\alpha}} + \cdots + \underbrace{1 \otimes y}_{\in U^-_0 \otimes U^-_{-\mu}}.$$

$$\Delta(y) = \underbrace{1 \otimes y}_{\in U^-_0 \otimes U^-_{-\mu}} + \underbrace{\sum_{\alpha \in \Pi} F_\alpha \otimes r_\alpha'(y) K_\alpha^{-1}}_{\in U^-_{-\alpha} \otimes U^-_{-(\mu-\alpha)}} + \cdots + \underbrace{y \otimes K_\mu^{-1}}_{\in U^-_{-\mu} \otimes U^-_0}.$$

Example: $r_\alpha(F_\beta) = \delta_{\alpha\beta} = r'_\alpha(F_\beta)$ $(\alpha, \beta \in \Pi)$

A direct computation shows a Leibniz style property:

- $r_\alpha(yy') = q^{(\alpha,\mu)} y r_\alpha(y') + r_\alpha(y) y'$
- $r_\alpha'(yy') = y r_\alpha'(y') + q^{(\alpha,\mu')} r_\alpha'(y) y'$

$\left(\begin{array}{l} y \in U^-_{-\mu} \\ y' \in U^-_{-\mu'} \end{array} \right)$

We need these elements for the following lemma:

<u>Lemma 12)</u> $\forall \mu \in \mathbb{Z}_{\geq 0}\Pi$, $y \in U^-_{-\mu}$, $x \in U^+_\mu$,

$$(y, E_\alpha x) = (F_\alpha, E_\alpha)(r_\alpha(y), x)$$
$$(y, x E_\alpha) = (F_\alpha, E_\alpha)(r_\alpha'(y), x).$$

<u>Pf.</u> Left as an exercise. $\square$

Now, coming back to prop 11.

By the inductive hypothesis, nondegeneracy $\Longrightarrow r_\alpha(y) = r_\alpha'(y) = 0$.

<u>Lemma 13</u> $\forall \alpha \in \Pi$, $y \in U^-_{-\mu}$ $(\mu \in \mathbb{Z}_{\geq 0}\Pi)$.

(*) $E_\alpha y - y E_\alpha = (q_\alpha - q_\alpha^{-1})^{-1} (K_\alpha r_\alpha(y) - r_\alpha'(y) K_\alpha^{-1})$

<u>Pf.</u>

Left as an exercise. $\square$

Coming back to Prop 11, Lemma 13 implies that for y in the left radical of $(,)$,

$$E_\alpha y = y E_\alpha \quad \text{as} \quad r_\alpha(y) = r_\alpha'(y) = 0.$$

Lemma 14) If $y \in U^-_{\mu}$ and commutes with all E_α $(\alpha \in \Pi)$, then $y = 0$.

Pf. We claim such an element acts on every finite-dim representation by 0, from which $y=0$ is clear. Indeed, on any irreducible module M, y acts on a lowest weight vector v as 0 and E_α's acting on v generate M, so y commuting with $E_\alpha \Rightarrow y$ annihilates M.

Now, applying Lemma 14) proves Prop 11 $\square$.

5.2.3) The Invariant Pairing

Def Let M be a U-module and $(\, , \,): M \times M \to \mathbb{K}$ a bilinear form on M. $(\, , \,)$ is invariant if the induced map $M \otimes M \to \mathbb{K}$ is a homomorphism of U-modules ($\mathbb{K}$ viewed as trivial module).

By definition:

$$(um, m') = (m, S(u)m') \quad \forall u \in U \quad \forall m, m' \in M.$$

Lemma 15 There is a well-defined $\mathbb{K}$-bilinear map

$$\langle \, , \, \rangle : U \times U \to \mathbb{K}(q^{1/2})$$

with

$$\langle (yK_\nu)K_\lambda x, (y'K_{\nu'})K_{\lambda'}x' \rangle$$
$$\parallel$$
$$(y', x)(y, x') q^{(2\rho, \nu)} (q^{1/2})^{-(\lambda, \lambda')}$$

where $x \in U^+_\mu$, $x' \in U^+_{\mu'}$, $y \in U^-_{-\nu}$, $y' \in U^-_{\nu'}$, $\lambda, \lambda' \in \mathbb{Z}\Phi$, $\mu, \mu', \nu, \nu' \in \mathbb{Z}_{\geq 0}\Pi$

Pf. The multiplication map $U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U$ is an isomorphism of vector spaces whose restriction gives an isomorphism:

$$U^-_{-\nu} K_\nu \otimes U^0 \otimes U^+_\mu \xrightarrow{\sim} U^-_{-\nu} K_\nu U^0 U^+_\mu$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U^-_{-\nu} U^0 U^+_\mu$$

since K_ν is a unit in U^0. Therefore,

$$\bigoplus_{\mu,\nu \in \mathbb{Z}_{\geq 0}\Pi} U^-_{-\nu} K_\nu \otimes U^0 \otimes U^+_\mu \xrightarrow{\sim} U, \text{ so}$$

we can extend $(,)$ in the way described. $\square$

Since $(,)$ restricted to $U^-_{-\nu} \times U^+_\mu$ is zero unless $\mu = \nu$ by lemma 9:

$$\langle U^-_{-\nu} U^0 U^+_\mu , U^-_{-\nu'} U^0 U^+_{\mu'} \rangle = 0 \text{ unless } \begin{matrix} \mu = \nu' \\ \nu = \mu' . \end{matrix}$$

Recall the adjoint representation: for $u \in U$, we define

$$\mathrm{ad}\, u : U \to U \quad \text{by}$$

if $\Delta(u) = \sum a_i \otimes a_i'$, then $(\mathrm{ad}\, u)(v) = \sum a_i v S(a_i')$.

Thm 16 $\langle\, ,\rangle$ is invariant i.e. $\forall u, w, w' \in U$,

$$\langle (\mathrm{ad}\, u) w, w' \rangle = \langle w, \mathrm{ad}\, S(u) w' \rangle.$$

Pf.

It suffices to check on $u = F_\alpha, K_\theta$, or E_α ($\alpha \in \Pi$, $\theta \in \mathbb{Z}\Phi$), and on

$$w = y K_\nu K_\lambda x \qquad w' = y' K_{\nu'} K_{\lambda'} x'$$

($\lambda, \lambda' \in \mathbb{Z}\Phi$, $\nu, \mu, \nu', \mu' \in \mathbb{Z}_{\geq 0}\Pi$, $x \in U_\mu^+$, $x' \in U_{\mu'}^+$, $y \in U_{-\nu}^-$, $y' \in U_{-\nu'}^-$).

- For $u = K_\theta$, orthogonality relation implies the form is degree 0, which gives the result.
- For $u = E_\alpha$, first recall that

$$(\mathrm{ad}\, E_\alpha) t = E_\alpha t - K_\alpha t K_\alpha^{-1} E_\alpha \qquad \forall t \in U$$

$$\Rightarrow (\mathrm{ad}\, E_\alpha) w = (\mathrm{ad}\, E_\alpha)(y K_\nu K_\lambda x)$$
$$= E_\alpha (y K_\nu K_\lambda x) - K_\alpha (y K_\nu K_\lambda x) K_\alpha^{-1} E_\alpha$$

$$= (E_\alpha y - y E_\alpha) K_\nu K_\lambda x + y E_\alpha K_\nu K_\lambda x$$
$$- q^{(\mu-\nu,\alpha)} y K_\nu K_\lambda x E_\alpha$$

by (Lemma 13)
$$= (q_\alpha - q_\alpha^{-1})^{-1} (K_\alpha r_\alpha(y) - r_\alpha'(y) K_\alpha^{-1}) K_\nu K_\lambda x$$
$$q^{(-\lambda-\nu,\alpha)} y K_\nu K_\lambda E_\alpha x - q^{(\mu-\nu,\alpha)} y K_\nu K_\lambda x E_\alpha$$

$$= \underbrace{(q_\alpha - q_\alpha^{-1})^{-1} \left(q^{-(\nu-\alpha,\alpha)} r_\alpha(y) K_{\lambda+\alpha} - r_\alpha'(y) K_{\lambda-\alpha}\right) K_\nu x}_{\in U^-_{-(\nu-\alpha)} U^0 U^+_\mu}$$

$$+ \underbrace{y K_\nu K_\lambda \left(q^{(-\lambda-\nu,\alpha)} E_\alpha x - q^{(\mu-\nu,\alpha)} x E_\alpha\right).}_{\in U^-_{-\nu} U^0 U^+_{\mu+\alpha}}$$

Now, if we look at $\langle \operatorname{ad} E_\alpha w, w' \rangle = \langle \operatorname{ad} E_\alpha w, y' K_{\nu'} K_{\lambda'} x' \rangle$,

(0) implies that for this to be nonzero, either

$\nu' = \mu$ or $\nu' = \mu + \alpha$

$\mu' = \nu - \alpha$ $\quad\quad$ $\mu' = \nu$

So we can restrict to these cases. A similar computation shows that $\langle w, \operatorname{ad} S(E_\alpha) w' \rangle$

is nonzero in these very same cases. In either case, a straightforward computation shows that $\langle \mathrm{ad}(F_\alpha)\,\omega, \omega' \rangle = \langle \omega, \mathrm{ad}\, S(F_\alpha)\, \omega' \rangle$.

Finally, a similar thing can be done for F_α, or one can use the involution ω on U and the following exercise:

<u>Exercise:</u> Show $\langle \omega \circ S(v), \omega \circ S(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in U$.

$\square$

We can now prove the nondegeneracy of the pairing:

Prop 17 For any $u \in U$, if $\forall w \in U$ $\langle w, u \rangle = 0$, then $u = 0$.

Pf. Recall
$$U = \bigoplus_{\nu,\mu \in \mathbb{Z}_{\geq 0}\Pi} U^-_{-\nu} U^0 U^+_\mu = \bigoplus_{\nu,\mu \in \mathbb{Z}_{\geq 0}\Pi} U^-_{-\nu} K_\nu U^0 U^+_\mu$$
as K_ν is a unit; since $\langle\ ,\ \rangle$ is 0 on
$$U^-_{-\nu} U^0 U^+_\mu \times U^-_{-\nu'} U^0 U^+_{\mu'} \quad \text{unless } \mu = \nu',\ \mu' = \nu$$
by

it suffices to show the prop for the restricted form to
$U^-_{-\nu} U^0 U_\mu^+ \times U_{-\mu} U^0 U_\nu$ $(\nu, \mu \in \mathbb{Z}_{\geq 0}\Pi)$.

Fix $\mu \in \mathbb{Z}_{\geq 0}\Pi$. $u_1^\mu, u_2^\mu, \dots, u^\mu_{r_\mu}$ be a basis of U^+_μ, and $w_1^\mu, w_2^\mu, \dots w^\mu_{r_\mu}$ be a basis of $U^-_{-\mu}$ s.t. $(w_i^\mu, u_j^\mu) = \delta_{ij}$, which exists by prop 11. Then
$$\left\{ (w_i^\nu K_\nu) K_\lambda u_j^\mu \ : \ \begin{array}{l} 1 \leq i \leq r_\nu \\ 1 \leq j \leq r_\mu \end{array} \quad \lambda \in \mathbb{Z}\Phi \quad \mu, \nu \in \mathbb{Z}_{\geq 0}\Pi \right\}$$
is a basis of $U^-_{-\nu} U^0 U^+_\mu$. Therefore, by definition:
$$\langle (w_h^\mu K_\mu) K_{\lambda'} u_l^\nu, (w_i^\nu K_\nu) K_\lambda u_j^\mu \rangle$$
$$= (w_i^\nu, u_l^\nu)(w_h^\mu, u_j^\mu)\, q^{(2\rho,\nu)} (q^{1/2})^{-(\lambda',\lambda)}$$
$$= \delta_{il} \delta_{hj}\, q^{(2\rho,\nu)} (q^{1/2})^{-(\lambda,\lambda')}$$

□

The remainder of the proof is left as an exercise (see next page for the answer).

Now, suppose $u \in U^-_{-\nu} U^0 U^+_{\mu}$ satisfies $\langle v, u \rangle = 0$ $\forall v \in U^-_{-\mu} U^0 U^+_{\nu}$.
Then, if $u = \sum_{i,j,\lambda} a_{ij\lambda} (v_i^{\nu} K_\nu) K_\lambda u_j^{\mu}$, then $\langle v, u \rangle = 0$ $\forall v$ means

$$0 = \langle (v_j^{\mu} K_\mu) K_{\lambda'} u_i^{\nu}, u \rangle$$

$$\Rightarrow 0 = \sum_{\lambda \in \mathbb{Z}\Phi} a_{ij\lambda} q^{(2\rho,\nu)} (q^{1/2})^{-(\lambda,\lambda')} \Rightarrow 0 = \sum_{\lambda \in \mathbb{Z}\Phi} a_{ij\lambda} (q^{1/2})^{-(\lambda,\lambda')}.$$

Each λ defines a character $\mathbb{Z}\Phi \longrightarrow k(q^{1/2})^{\times}$;
$$\lambda' \longmapsto (q^{1/2})^{-(\lambda,\lambda')}$$

different λ give different characters as $q^{1/2}$ is not a root of unity. By linear independence of these characters, we deduce $a_{ij\lambda} = 0$ $\forall i, j, \lambda \Rightarrow u = 0$.

$\square$

Prop. 18 $u \in U$ is invariant under the adjoint rep iff u is central.

Pf. This is a general fact for Hopf algebras — left as an exercise in the case of U.

Prop 19.
Let $\lambda \in \Lambda$ be dominant (i.e. $(\lambda, \alpha) \geq 0$ $\forall \alpha$ simple) s.t. $2\lambda \in \mathbb{Z}\Phi$. Then $\exists!$ $z_\lambda \in U$ s.t.
$$\langle u, z_\lambda \rangle = \mathrm{tr}_{L(\lambda)} u K_{2\rho}^{-1} \quad \forall u \in U.$$
Furthermore, z_λ is central.

Rmk: Since $U \hookrightarrow U^*$, $x \mapsto \langle \cdot, x \rangle$ is an injective U-mod hom from the adjoint rep to its dual $\langle \cdot, z_\lambda \rangle \in U^*$ being invariant means z_λ is invariant, and hence central by prop 18.

Pf.

If M is a finite dimensional U-module, $\forall f \in M^*$, $m \in M$, let the matrix coefficient $c_{f,m} \in U^*$ be given by $c_{f,m}(v) = f(vm)$. $(v \in U)$

We need the following lemma:

Lemma 20 Let M be a finite dimensional U-module (type 1) whose weights λ each satisfy $2\lambda \in \mathbb{Z}\Phi$. Then, for all $f \in M^*$, $m \in M$, $\exists!\, u_m \in U$ s.t. $c_{f,m}(v) = \langle v, u \rangle$ $\forall v \in U$.

Pf. uniqueness is a consequence of Prop 17.

For existence:

Claim: Let $\mu, \nu \in \mathbb{Z}_{\geq 0}\Pi$, $\theta \in \mathbb{Z}\Phi$. Then for every bilinear pairing $U^-_{-\nu} \times U^+_\mu \to k$, $\exists$ $u \in U^-_{-\nu} K_\nu K_\theta U^+_\mu$ s.t. $\forall x \in U^+_\nu$, $y \in U^-_{-\nu}$, $\lambda' \in \mathbb{Z}\Phi$,

$$\langle (y K_\nu) K_{\lambda'} x, u \rangle = \phi(y, x) (q^{1/2})^{-(\theta, \lambda')}$$

Pf. $u = \sum \phi(v_j^\mu, u_i^\nu) q^{-(\beta, \mu)} (v_i^\mu K_\nu) K_\theta u_j^\mu$ will work (in the notation of the proof of prop 17). $\square$

It suffices to show existence with f, m as weight vectors in M^*, M with weights $-\lambda', \lambda \in \Lambda$ (λ', λ weights of M). Then it can be shown

$$c_{f,m}((yK_\mu) K_\eta x) = (q^{1/2})^{(\eta, 2\lambda + 2\nu)} f(yK_\mu x m)$$

$\forall \mu, \nu \in \mathbb{Z}_{\geq 0}\Pi$, $\eta \in \mathbb{Z}\Phi$, $x \in U^+_\mu$, $y \in U^-_{-\nu}$. Furthermore, because M is finite-dimensional, it has finitely many weights, so there are finitely many ν s.t. $U^+_\nu m \neq 0$.
Furthermore, $U^-_{-\mu} U^0 U^+_\nu m$ is in $\lambda+\nu-\mu$ weightspace, so $c_{f,m}(U^-_{-\mu} U^0 U^+_\nu) = 0$ unless $\lambda' = \lambda+\nu-\mu$.
Therefore, there finitely many pairs (μ,ν) s.t. $c_{f,m}(U^-_{-\nu} U^0 U^+_\mu) \neq 0$

Now, apply the claim to the function $(y,x) \mapsto f(yK_\mu xm)$ with $\theta = -2\lambda - 2\nu$ to get an element $u_{\mu\nu} \in U^-_{-\nu} U^0 U^+_\nu$ s.t. $\langle \omega, u_{\mu\nu} \rangle = c_{f,m}(v)$ $\forall v \in U^-_{-\mu} U^0 U^+_\nu$.
Then, $u_m = \sum_{\mu,\nu} u_{\mu\nu}$ does the trick by prop 17 (which is a finite sum by above remarks).

$\square$

Since trace is the sum of matrix coefficients, apply the lemma to $M = L(\lambda)$ to deduce $\exists z_\lambda$ s.t. $\mathrm{tr}_{L(\lambda)}(u K^{-1}_{2\rho}) = \langle u, z_\lambda \rangle$. z_λ is unique by prop 17 and central by the remark.

$\square$

Notation for lemma 21

z_λ as above lies in $Z(U) \subseteq U_0$

$$= \bigoplus_{\substack{\mu \geq 0 \\ \in \mathbb{Z}\Phi}} U^-_{-\mu} U^0 U^+_{\mu}, \quad \text{so } z_\lambda = \sum z_{\lambda,\mu}, \quad z_{\lambda,\mu} \in U^-_{-\mu} U^0 U^+_{\mu}.$$

(notice $z_{\lambda,0} = \pi(z_\lambda)$).

Lemma 21

Let z_λ be as in prop 18. Then

$$\varphi(z_\lambda) = \sum_{\nu \in \mathbb{Z}\Phi \cap 2\Lambda} \dim L(\lambda)_{-\nu/2} K_\nu$$

Pf

If $z_{\lambda,0} = \sum_{\nu \in \mathbb{Z}\Phi} a_\nu K_\nu$, then

$$(*) = \langle K_\mu, z_\lambda \rangle = \langle K_\mu, z_{\lambda,0} \rangle = \sum a_\nu (q^{1/2})^{-(\nu,\mu)}$$

But by prop 18, $(*) = \mathrm{tr}_{L(\lambda)}(K_\mu K_{2\rho}^{-1})$. Since $K_\mu K_{2\rho}^{-1} \in U^0$, $L(\lambda)$ decomposing into weight spaces tells us:

$$(*) = \sum_{\lambda' \in \Lambda} \dim L(\lambda)_{\lambda'} q^{(\lambda', \mu - 2\rho)} = \sum_{\lambda' \in \Lambda} \dim L(\lambda)_{\lambda'} q^{-2(\lambda',\rho)} (q^{1/2})^{(2\lambda',\mu)}$$

$$\Longrightarrow \quad a_\nu = \dim L(\lambda)_{-\nu/2} \, q^{(\nu,\rho)}$$

$$\Rightarrow \quad z_{\lambda,0} = \sum_{\nu \in \mathbb{Z}\Phi \cap 2\Lambda} \dim L(\lambda)_{-\nu/2} \, q^{(\nu,\rho)} K_\nu.$$

Hence,

$$\varphi(z_\lambda) = \gamma_{-\rho} \circ \pi(z_\lambda) = \gamma_{-\rho} \circ z_{\lambda,0} = \sum_{\nu \in \mathbb{Z}\Phi \cap 2\Lambda} \dim L(\lambda)_{-\nu/2} K_\nu.$$

$\square$

Finally, we can finish the proof of the main theorem!

Thm. The Harish-Chandra homomorphism Ψ is an isomorphism between $Z(U)$ and $(U^0_{ev})^W$.

Pf. We already proved injectivity, so we need to prove surjectivity.

Now that we have elements of the form $\Psi(z_\lambda)$ $= \sum_{\nu \in \mathbb{Z}\Phi \cap 2\Lambda} \dim L(\lambda)_{-\nu/2} K_\nu \in (U^0_{ev})^W$ by lemma 20, a standard argument (see Bourbaki Lie groups and Lie algebras, Ch 8) shows these span $(U^0_{ev})^W$ $(\lambda \in \Lambda^+$ and $2\lambda \in \mathbb{Z}\Phi)$.

(proof given on next page). $\square$

For any $\mu \in \mathbb{Z}\Phi$, let $W\mu$ denote the W-orbit of μ. Let

$$\mathrm{av}(\mu) = \sum_{\nu \in W\mu} K_\nu$$

By definition of $(U_{ev}^0)^W$, the collection $\{\mathrm{av}(\mu)\}$ where μ runs over the W orbit representatives in $\mathbb{Z}\Phi \cap 2\Lambda$ forms a basis of $(U_{ev}^0)^W$.

Each orbit contains exactly one weight that is dominant. Hence, $\{\mathrm{av}(-\mu) : \mu \in \mathbb{Z}\Phi \cap 2\Lambda \text{ dominant}\}$ is a basis for $(U_{ev}^0)^W$.

We want these $\mathrm{av}(-\mu)$ to be in the image of φ. We will show this inductively on height of the weight

- If $\mu = 0$, then $\mathrm{av}(0) = 1 = \varphi(1)$.
- Suppose all $\mathrm{av}(-\lambda)$ lie in the image for all $\lambda < \mu$. Since $\mu \in \mathbb{Z}\Phi \cap 2\Lambda$ is dominant, $\mu/2$ is dominant and satisfies the hypotheses of the preceding lemma.

Therefore, we can construct $z_{\mu/2} \in Z(U)$ s.t.

$$\varphi(z_{\mu/2}) = \sum_{\nu \in \mathbb{Z}\Phi \cap 2\Lambda} \dim L(\mu/2)_{-\nu/2} K_\nu = \sum_{\substack{2\nu \in \mathbb{Z}\Phi \\ \text{and } \nu \in \Lambda}} \dim L(\mu/2)_\nu K_{-2\nu}$$

$$= \mathrm{av}(-\mu) + \sum_{\substack{\nu < \mu/2,\ \nu \text{ dominant} \\ \nu \in \Lambda,\ 2\nu \in \mathbb{Z}\Phi}} \dim L(\mu/2)_\nu \, \mathrm{av}(-2\nu).$$

where we use the fact $L(\lambda)_\delta = L(\lambda)_{w\delta}$ $\forall \lambda \in \Lambda$ dominant, δ a weight, and $w \in W$, and $\dim L(\mu/2)_{\mu/2} = 1$.

Apply the induction hypothesis to get the claim. $\square$

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