Benjamin Scmidt.

$\mathcal{A}$-abelian cat. w. enough injectives

$\mathcal{C}^b(\mathcal{A})$ -bounded complexes

$C^+(\mathcal{A})$ - complexes bounded below.

Derived cat-y (Verdier-Grothedieck)

$\mathcal{D}^b(\mathcal{A})$ - add formal inverses of quasi-isomorphisms to $\mathcal{C}^b(\mathcal{A})$

$\mathcal{D}^+(\mathcal{A})$ - . - . -

Lem: $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(E, F[n]) = \mathrm{Ext}^n(E,F)$

$\mathcal{D}^b(\mathcal{A})$ is triang. cat-y.

$T = \{A \to B \to C \to A[1]\}$ -distinguished triangles

Axioms: (1) $A \xrightarrow{id} A \to 0 \to A[1]$

$\{T\}$ is closed under isomorphisms.

$\forall\ A \to B$ is incl. into triangle

[$\{T\}$ is obtained as follows: $A \to B$ map of complexes, $C^\cdot = A^\cdot[1] \oplus B$
$$d_C = \begin{pmatrix} d_A[1] & 0 \\ f[1] & d_B \end{pmatrix}]$$

(2) $A \to B \to C \to A[1]$ is in $T \Leftrightarrow B \to C \to A[1] \to B[1]$ is in $\{T\}$

(3)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed, "\exists"'] & A[1] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

(4) Octahedron axiom

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed, "\exists"'] & A[1] \arrow[d, equal] \\ A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, dashed, "\exists"'] & A[1] \arrow[d] \\ 0 \arrow[r] \arrow[d] & F \arrow[r, "id"] \arrow[d] & F \arrow[r] \arrow[d, dashed, "\exists"'] & 0 \arrow[d] \\ A[1] \arrow[r] & B[1] \arrow[r] & D[1] \arrow[r] & A[2] \end{tikzcd}$$

$D \to E \to F \to D[1]$

For $A \hookrightarrow B \hookrightarrow C$ get exact $0 \to B/A \to C/A \to C/B \to 0$

$0 \to A \to B \to C \to 0 \quad (A, B, C \in \mathcal{A})$

$\Updownarrow$

$A \to B \to C \to A[1]$

given by el-t of $\mathrm{Ext}^1(C, A)$ presented by B

Derived functors:

$\mathcal{B}$ - another abelian cat-y $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ - left exact functor

$A \in \mathcal{A} \rightsquigarrow$ inj-ve resol-n $I^\bullet$

$R\mathcal{F}(A) = \mathcal{F}(I^\bullet) \in D^+(\mathcal{B})$

Can extend this to all $E^\bullet \in D^b(\mathcal{A})$ via Cartan-Eilenberg resolution

$\rightsquigarrow R\mathcal{F}: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ - exact functor (mapping distinguished triangles to distinguished triangles)

Coherent sheaves: X - smooth proj-ve var-y$/k$

Want to study $D^b(X) := D^b(\mathrm{Coh}\, X)$

Problem 1: have neither enough injectives nor enough projectives

Sol-n: $Q\mathrm{Coh}(X)$ has enough injectives

Problem 2: inj-ve resolutions may be infinite

Problem 3: need to deal w. left derived functors

A1: $D^b_{Coh}(Q\mathrm{Coh}(X)) \subset D^b(Q\mathrm{Coh}(X))$ - full subcat. w. coherent cohomology

Prop: Every complex in $D^b(X)$ is q.isom. to a complex, where all terms are coherent

Proof: $0 \to E^n \xrightarrow{d^n} E^{n-1} \to \ldots \to E^m \to 0 \quad E^i \in Q\mathrm{Coh}(X)$

Exer: $\mathcal{G} \twoheadrightarrow \mathcal{F} \in \mathrm{Coh}(X)$

$\mathcal{F}' \in \mathrm{Coh}(X) \rightsquigarrow$ then use induction to produce a quasi-isom complex inside $E^\bullet$

A2: not a problem for functors we care about

$\mathcal{F}: \mathrm{Coh}(X) \to \mathcal{A} \rightsquigarrow R\mathcal{F}: D^b(X) \to D^+(\mathcal{A})$?

Image is in $D^b(\mathcal{A})$ in the following cases:

(i) $R\mathrm{Hom}(E^\bullet, -): D^b(X) \to D^b(k\text{-Vec})$

- $\mathcal{R}\mathcal{H}om(E^\bullet, \cdot) : D^b(X) \to D^b(X) \rightsquigarrow (E^\bullet)^\vee = R\mathcal{H}om(E^\bullet, \mathcal{O}_X)$
- $R\Gamma(X, \cdot) : D^b(X) \to D^b(k\text{-Vec})$
- $f: X \to Y \rightsquigarrow Rf_* : D^b(X) \to D^b(Y)$ – can do f proper

A3: Sometimes flat/locally free resolutions do the job

- $f: X \to Y \rightsquigarrow Lf^* : D^b(Y) \to D^b(X)$
- $E^\bullet \otimes^L \cdot : D^b(X) \to D^b(X)$

Properties: $Rf_*(Lf^* E^\bullet \otimes F^\bullet) \simeq E^\bullet \otimes Rf_* F^\bullet$

$$R\mathcal{H}om(E^\bullet, F^\bullet \otimes^L G^\bullet) = R\mathcal{H}om(E^\bullet \otimes^L (G^\bullet)^\vee, F^\bullet)$$

$$R\mathcal{H}om(Lf^* \mathcal{E}, \mathcal{F}) = R\mathcal{H}om(\mathcal{E}, Rf_* \mathcal{F})$$

Thm (Grothendieck-Verdier duality)

$f: X \to Y$ – morphism of smooth varieties, $\dim f := \dim X - \dim Y$

f is domin

$\omega_f = \omega_X \otimes f^* \omega_Y^*$

Then $\forall\, F^\bullet \in D^b(X), E^\bullet \in D^b(Y)$

$$f_* \mathcal{H}om(F^\bullet, Lf^*(E^\bullet) \otimes \omega_f[\dim f]) \simeq \mathcal{H}om(Rf_* \mathcal{F}, E)$$

From now on, all functors are derived

<u>Flat base change</u>

$$\begin{tikzcd} X \times_Z Y \arrow[r, "v"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X \arrow[r, "u"'] & Z \end{tikzcd}$$

f proper, u flat

Then $u^* f_* \mathcal{F}^\bullet \simeq g_* v^* \mathcal{F}^\bullet$

<u>Trace</u> $R\mathcal{H}om(E^\bullet, E^\bullet) \longrightarrow \mathcal{O}_X$, $E^\bullet \in D^b(X)$

May assume all E^i are locally free so $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \subset \bigoplus_{i \in \mathbb{Z}} \mathcal{H}om(E^i, E^i)$ $\rightsquigarrow tr_{E^i} : \mathcal{H}om(E^i, E^i) \longrightarrow \mathcal{O}_X$. then just embed $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$ to the direct sum and take direct sum of homomorphisms.

Fourier-Mukai transforms

X, Y-smooth (projectives)

$$\begin{tikzcd} & X\times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd} \qquad \mathcal{P}\in D^b(X\times Y) \text{ kernel}$$

Def: $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$

$$E \mapsto p_*(q^*E\otimes \mathcal{P})$$

Ex: $f_* = \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(X) \to D^b(Y)$

(f^* - same but Y & X swapped)

Proof: $\iota : X \xrightarrow{\sim} \Gamma_f \hookrightarrow X\times Y$

$$\Phi_{\mathcal{O}_{\Gamma_f}}(E^\bullet) = p_*(q^*E\otimes \iota_*\mathcal{O}_X) \underset{\text{proj. f-la}}{=} p_*\iota_*(\iota^*q^*E\otimes \mathcal{O}_X) \underset{p\circ\iota = f,\ q\circ\iota = id}{=} f_*E$$

Ex 2: $\Phi_{\mathcal{O}_\Delta} = id$

FM transforms have left & right adj-s

$\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$

$\mathcal{P}_R = \mathcal{P}^\vee \otimes p^*\omega_Y[\dim X]$, $\mathcal{P}_L = \mathcal{P}^\vee\otimes q^*\omega_X[\dim X]$

Prop: $\Phi_{\mathcal{P}_L}$ is left adj-t to $\Phi_{\mathcal{P}}$ & $\Phi_{\mathcal{P}_R}$ is right adjoint.

Proof: left adj-t part:

? $q \leftrightarrow p$ $\quad Hom(\Phi_{\mathcal{P}_L}(F), E) = Hom(p_*(q^*(F)\otimes \mathcal{P}_L), E) \overset{\text{GV dual}}{=} Hom(\mathcal{P}_L\otimes q^*\mathcal{F}, p^*E\otimes q^*\omega_X[\dim X])$

$= Hom(\mathcal{P}^\vee\otimes \cancel{q^*\omega_X[\dim X]}\otimes q^*\mathcal{F}, p^*E\otimes \cancel{q^*\omega_X[\dim X]})$

$= Hom(\mathcal{P}^\vee\otimes q^*\mathcal{F}, p^*E) = Hom(q^*\mathcal{F}, p^*E\otimes\mathcal{P}) = Hom(\mathcal{F}, q_*(p^*(E)\otimes \mathcal{P}))$ $\square$

Composition of FM transforms:

$\mathcal{P}\in D^b(X\times Y)$, $Q\in D^b(Y\times Z) \rightsquigarrow \mathcal{R} = \pi_{XZ*}(\pi_{XY}^*\mathcal{P}\otimes \pi_{YZ}^*(Q))$

Prop-n: $\Phi_Q\circ\Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{R}}$.

<u>Orlov thm:</u>

Conj: All exact functors $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ are FM transforms

Wrong: Rizzardo - Van den Bergh '16

Thm (Orlov) X, Y-smooth, proj-ve, $F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is fully faithful & exact, F admits left & right adj-s. Then $\exists!\, \mathcal{P} \in \mathcal{D}^b(X \times Y)$ st

$F \simeq \Phi_{\mathcal{P}}$

Rmk: (1) F has left adj-t $\Leftrightarrow$ F has right adj-t - thx Grothendieck-Verdier duality: $H = G(\cdot \otimes \omega_X^\vee[-\dim X] \otimes \omega_Y[\dim Y])$

+ Serre dual-s

(2) Bondal - Van den Bergh: adjoints exist (if F is fully faithful?)

Proof (ideas)

Convolutions $A_m \xrightarrow{d_m} A_{m-1} \to \dots \to A_1 \xrightarrow{d_0} A_0$ complex in $\mathcal{D}^b(X)$

Assume

$$\begin{tikzcd}
A_m \arrow[rr, "d_m"] \arrow[dr, "="] & & A_{m-1} \arrow[rr] \arrow[dr] & & \cdots & & A_1 \arrow[rr, "d_0"] \arrow[dr] & & A_0 \arrow[dr, "d_0"] & \\
& A_m \arrow[ur] & & C_{m-1} \arrow[ll, "{[1]}"] \arrow[ur] & & \cdots & & C_1 \arrow[ll] \arrow[ur] & & C_0 \arrow[ll]
\end{tikzcd}$$

- convolution - existence & uniqueness not automatic

Ex: $A_m, \dots, A_0 \in \mathrm{Coh}(X) \Rightarrow C_0 \simeq (A_m \to A_{m-1} \to \dots \to A_0)$

Rmk: (i) Convolutions are stable under exact functors

(ii) Convolutions exist & unique $\mathrm{Hom}(A_i, A_j[r]) = 0$ $i > j$, $r < 0$

here we'll use full

(iii) Morphisms of complexes (w. condition on Ext's)

$\rightsquigarrow$ morphism between convolutions

Broad idea: Assume $F = \Phi_{\mathcal{E}}$ Need to recover $\mathcal{E}$

$\mathcal{E}' = p_{13}^* \mathcal{O}_\Delta \otimes p_{24}^* \mathcal{E}$ $\qquad \in \mathcal{D}^b(X \times X \times X \times Y)$

$\iota : x \mapsto (x, x)$, $\mathcal{O}_\Delta = \iota_* \mathcal{O}_X$

$\rightsquigarrow \Phi_{\mathcal{E}'} : \mathcal{D}^b(X \times X) \to \mathcal{D}^b(X \times Y)$

Lem1: $X \times X \times Y$ & proj-ns q_{ij}, q_i

(i) $\Phi_{\mathcal{E}'}(\mathcal{F}) = (q_{13})_* (q_{12}^* \mathcal{F} \otimes q_{23}^* \mathcal{E})$

(ii) $\Phi_{\mathcal{E}'}(\mathcal{O}_\Delta) = \mathcal{E}$ $\qquad$ (iii) $\Phi_{\mathcal{E}'}(A \boxtimes B) = A \boxtimes \Phi_{\mathcal{E}}(B)$

flat base change

Proof (i): $\mathcal{P}_{\mathcal{E}'}(F) = (p_{34})_*(p_{13}^* \iota_* \mathcal{O}_X \otimes p_{24}^* \mathcal{E} \otimes p_{12}^* \mathcal{F})$

$= (p_{34})_*(j_* q_1^* \mathcal{O}_X \otimes p_{24}^* \mathcal{E} \otimes p_{12}^* \mathcal{F})$

$= (p_{34})_* j_*(q_1^* \mathcal{O}_X \otimes j^* p_{24}^* \mathcal{E} \otimes j^* p_{12}^* \mathcal{F})$

$= (q_{13})_*(q_{12}^* \mathcal{E} \otimes q_{23}^* \mathcal{F})$

$$\begin{array}{ccc} X \times X \times Y & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X \times X \times X \times Y & \longrightarrow & X \times X \end{array}$$

$\square$

Boundedness: $F : D^b(X) \to D^b(Y)$ exact w. adjoints $\Rightarrow \mathcal{F}$ is bounded (uniform bound on left & right ends of $\mathcal{F}(\circ)$)

Lem $\mathcal{L} \in Coh(X)$ ample $E \in Coh(X)$, $P_i = \mathcal{L}^{\otimes i}$

$$\to A_{-1}^{\oplus k_1} \to A_0^{\oplus k_0} \to E$$

$$A_{-i} = P_i$$

Rmk: $m \gg 0$, $(A_m^{\oplus k_m} \xrightarrow{\delta} \dots \to A_0^{\oplus k_0}) \to E$

$K_m = \ker \delta \Rightarrow S_m \simeq K_m[m+1] \oplus E$

$(A_i \boxtimes B_i \to \dots \to A_0 \boxtimes B_0) \to \mathcal{O}_\Delta \to 0$

$\downarrow F$

$A_i \boxtimes F(B_i) \to \dots \to A_0 \boxtimes F(B_0))$

convolution can be shown to split into the product

$G_m = \mathcal{E}_m \oplus \mathcal{F}_m$

candidate for kernels ...

$\square$