SOERGEL BIMODULES, HECKE ALGEBRAS, AND KAZHDAN-LUSZTIG BASIS

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Abstract. These are the notes for a talk at the MIT-Northeastern seminar for graduate students on category $\mathcal{O}$ and Soergel bimodules, Fall 2017.

Contents

1. Introduction

The main goal of this talk is to explain Soergel's approach to Kazhdan-Lusztig's conjecture [KL79]. This conjecture expresses the multiplicities of simple objects in standard ones in the principal block $\mathcal{O}_0$ of category $\mathcal{O}$ in terms of the values of certain polynomials in $\mathbb{Z}[v^{\pm 1}]$ at $v = 1$. These polynomials arise from Hecke algebras - certain algebras $\mathcal{H}$ over $\mathbb{Z}[v^{\pm 1}]$ with the basis indexed by the elements of a Weyl group W and relations deforming those of $\mathbb{Z}[W]$. The transition matrix from the standard basis to a certain basis (called Kazhdan-Lusztig's basis) is uni-triangular with non-diagonal entries in $v\,\mathbb{Z}_{\geq 0}[v]$. The matrix coefficients evaluated at $v = 1$ give the multiplicities of simple objects in standard ones in the principal block $\mathcal{O}_0$ of category $\mathcal{O}$. The precise formulations are given in Theorem 2.3.

The first proof was provided independently by Beilinston-Bernstein in [BB81] and Brylinski-Kashiwara in [BK81], using the machinery of D-modules and perverse sheaves in the beginning of 1980-s. A decade later Soergel in [Soe90] and [Soe92] suggested a different approach via bimodules over the polynomial ring $R = \mathbb{R}[\mathfrak{h}]$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The independent proof using Soergel's ideas was completed recently by Elias and Williamson in [EW].

The structure of the notes is as follows. In Section 2 we recall the generalities on Hecke algebras associated with finite Weyl groups, introduce the Kazhdan-Lusztig basis and verify its existence and uniqueness. The pivotal point of this section is the statement of Theorem 2.3 (known as the Kazhdan-Lusztig conjecture).

Soergel's approach to the conjecture starts to unravel in Section 3, culminating in Soergel's categorification theorem. In Section 5 we explain the connection of Bott-Samelson modules and bimodules to cohomology and equivariant cohomology of Bott-Samelson varieties.

2. Hecke Algebras

Definition 2.1. Let (W, S) be a Weyl group. The Hecke algebra $\mathcal{H}$ is the algebra over the ring $\mathbb{Z}[v^{\pm 1}]$ with the generators given by the symbols $\{H_s | s \in S\}$ and relations

$$\begin{cases} H_s^2 = (v^{-1} - v)H_s + 1 \Leftrightarrow (H_s + v)(H_s - v^{-1}) = 0 \ \forall s \in S & \text{(quadratic relations)} \\ \underbrace{H_t H_s H_t \dots}_{m_{st}} = \underbrace{H_s H_t H_s \dots}_{m_{st}} \ \forall s, t \in S & \text{(braid relations).} \end{cases} \tag{2.1}$$

For any element $x \in W$ and a reduced expression $x = s_{i_1} \dots s_{i_k}$, define $H_x := H_{s_{i_1}} \dots H_{s_{i_k}}$. We set H_e to be the unit.

Remark 2.1. As any two reduced expressions of an element $x \in W$ can be obtained from one another by a sequence of braid moves, the element H_x does not depend on the choice of a reduced expression of x.

Remark 2.2. The elements $\langle H_x \rangle_{x \in W}$ generate $\mathcal{H}$ as $\mathbb{Z}[v^{\pm 1}]$-module. One can show that they form a basis.

Exercise 2.1. Check that $H_s^{-1} = H_s + v - v^{-1}$. Therefore, H_x is invertible for any $x \in W$.

There is a ring involution τ on $\mathcal{H}$, given by $\tau : v \mapsto v^{-1}$ and $\tau : H_x \mapsto \overline{H_x} := H_{x^{-1}}^{-1}$.

Definition 2.2. Let w_1, w_2 be in W. Then $w_1 \prec w_2$ in the *Bruhat order* if $w_2 = s_{\beta i_k} \dots s_{\beta_{i_1}} w_1$ and $\ell(s_{\beta_{i_{k-j}}} \dots s_{\beta_{i_1}} w_1) > \ell(s_{\beta_{i_{k-j-1}}} \dots s_{\beta_{i_1}} w_1)$ for all $j \in \{1, \dots, k-1\}$, where $s_{\beta i_j}$ are some (not necessarily simple) reflections in W.

Proposition 2.1. *There exists a basis (the Kazhdan-Lusztig basis) $\langle b_x \rangle_{x \in W}$ of $\mathcal{H}$ uniquely characterized by two properties:*

$$\begin{gathered} \tau(b_x) = b_x; \\ b_x = H_x + \sum_{y \in W, y \prec x} c_{x,y} H_y, \end{gathered} \tag{2.2}$$

where each $c_{x,y} \in v\mathbb{Z}[v]$.

Remark 2.3. As the transition matrix from $\{H_x\}_{x\in W}$ to $\{b_x\}_{x\in W}$ is upper-triangular with 1's on the diagonal, the elements $\{b_x\}_{x\in W}$, indeed, form a basis of $\mathcal{H}$.

Definition 2.3. The polynomials $p_{y,x} := v^{\ell(x)-\ell(y)}c_{x,y}$ are called the *Kazhdan-Luzstig polynomials.*

Exercise 2.2. The elements $b_s := \{H_s + v\}_{s\in S}$ are self-dual with respect to τ. Check that $b_s^2 = (v+v^{-1})b_s$.

Now we present the proof of Proposition 2.1.

Proof. We first show the existence of a basis, satisfying the required properties, arguing by induction on the Bruhat order. Thus, we set $b_e := 1$ and $b_s := H_s + v$ for $s \in S$ (these are self-dual due to Exercise 2.2) and suppose that b_w exist for $w \prec x$. It is direct to verify that

$$b_sH_x = \begin{cases} H_{sx} + vH_x, \ell(x) < \ell(sx) \\ H_{sx} + v^{-1}H_x, \ell(x) > \ell(sx). \end{cases} \tag{2.3}$$

Next, to find b_x, we use that there exists $s \in S$, such that $sx \prec x$. Hence, using the assumption that b_{sx} exists and formulas (2.3), one can conclude that $b_sb_{sx} = H_x + \sum\limits_{y\prec x} h_yH_y$ for some $h_y \in \mathbb{Z}[v]$ (the containment follows from the existence of $b_{sx} = H_{sx} + \sum\limits_{z\prec sx} h_zH_z$ with $h(z) \in v\,\mathbb{Z}[v^{\pm1}]$, formulas (2.3) show that the degrees of monomials in h_y are at most one less than the degrees of monomials in polynomials h_z it is derived from), i.e. some of the h_y's might have constant terms. However, subtracting $\sum\limits_{y\prec x} h_y(0)b_y$, we obtain the element b_x, which is fixed by τ (as a $\mathbb{Z}$-linear combination of fixed elements), whose coefficients are polynomials in $v\mathbb{Z}[v]$.

Now we show that b_x is unique. Indeed, if we have two elements $c = H_x + \ldots$ and $c' = H_x + \ldots$, both satisfying (2.2), then $c - c'$ is also stable under τ and $c - c' \in \sum\limits_{y\in W, y\prec x} v\mathbb{Z}[v]H_y$. Now the result follows from Lemma 2.2 below. □

Lemma 2.2. *If* $h \in \sum\limits_{y\in W} v\mathbb{Z}[v]H_y$ *and* $\tau(h) = h$, *then* $h = 0$.

Proof. Let z be one of the maximal elements (in the Bruhat order) in the expression of h in the lemma, i.e. we can write

$$h = p_zH_z + \sum_{y\not\succeq z} p_yH_y,$$

for some polynomials p_z and p_y's in $v\mathbb{Z}[v]$. Now $H_z \in b_z + \sum\limits_{f\prec z} \mathbb{Z}[v^{\pm1}]b_f$ (for some τ-invariant b_f's, the existence of which was already established). Hence, $\tau(H_z) \in b_z + \sum\limits_{f\prec z} \mathbb{Z}[v^{\pm1}]b_f \subset H_z + \sum\limits_{f\prec z} \mathbb{Z}[v^{\pm1}]H_f$. But then $\tau(h) = h$ implies $\tau(p_z) = p_z$, and we obtain a contradiction with the assumption $p_z \in v\mathbb{Z}[v]$. □

Example 2.1. Let us find the Kazhdan-Lusztig basis for the dihedral group $W = \langle s, t\rangle$ with $s^2 = t^2 = e$ and $\underbrace{sts\ldots}_{m} = \underbrace{tst\ldots}_{m}$. Clearly, $b_e = H_e, b_t = H_t + v$ and $b_s = H_s + v$. Next, $b_s b_t = H_{st} + v(H_s + H_t) + v^2$ satisfies the conditions 2.2, so we put $b_{st} = b_s b_t$, similarly, $b_{ts} = b_t b_s$. Using formulas (2.3), we find $b_s b_{ts} = H_{sts} + v(H_{st} + H_{ts}) + vH_s^2 + v^2(H_t + 2H_s) + v^3$. As $vH_s^2 = v((v^{-1} - v)H_s + 1) = -v^2 H_s + H_s + v$, we set $b_{sts} = b_s b_{ts} - b_s = H_{sts} + v(H_{st} + H_{ts}) + vH_s^2 + v^2(H_t + H_s) + v^3$. In general,

$$b_w = H_w + \sum_{x \prec w} v^{\ell(w)-\ell(x)} H_x. \tag{2.4}$$

Indeed, assume that 2.4 holds for $b_{w'}, w' \prec w$. Then, either $sw \prec w$ or $tw \prec w$. Arguing similarly to the proof of Proposition 2.1, one can easily verify the formula for b_w (w.l.o.g. assume $w' = sw \prec w$):

$$b_s b_{w'} = H_w + vH_{w'} + \sum_{x \prec w', x = t\ldots} v^{\ell(w')-\ell(x)} H_{sx} + v^{\ell(w')-\ell(x)+1} H_x + \sum_{x \prec w', x = s\ldots} v^{\ell(w')-\ell(x)} H_{sx} + v^{\ell(w')-\ell(x)-1} H_x, \tag{2.5}$$

which is $b_w + b_{tw'}$.

Remark 2.4. In particular, the Weyl groups of types A_2, B_2 and G_2 are dihedral for $m = 3, 4$ and 6. Hence, the Kazhdan-Lusztig basis is given by Example 2.1.

The following result and subsequent remark were conjectured in [KL79] and are proved by now. Theorem 2.3 is known as the Kazhdan-Lusztig conjecture.

Theorem 2.3. *(Kazhdan-Lusztig conjecture) The multiplicity $[P(x \cdot 0) : \triangle(y \cdot 0)]$ is given by the specialization of $c_{x,y}$ at $v = 1$ (using BGG-reciprocity $[\triangle(y \cdot 0)] : L(x \cdot 0)]$ equals $c_{x,y}|_{v=1}$ as well).*

Remark 2.5.
(1) The polynomials $h_{y,x}$ from 2.2 are in $\mathbb{Z}_{\geq 0}[v]$.
(2) If we write $b_x b_y = \sum \mu^z_{x,y} b_z$, then $\mu^z_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

3. Soergel bimodules

Let (W, S) be a Coxeter system. For any two simple reflections $s, t \in S$, the order of the element $st \in W$ will be denoted by $m_{st} \in \{2, 3, \ldots \infty\}$.

Definition 3.1. An *expression* of $w \in W$ is a word $\underline{w} = s_{i_1} \ldots s_{i_k}$. The expression $\underline{w}$ is called *reduced* if $\ell(w) = k$.

Next, we fix a vector space $\mathfrak{h}$ over $\mathbb{R}$, s.t. there exist subsets of linearly independent elements $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ with the following properties:

$$\alpha_s(\alpha_t^\vee) = -2\mathrm{Cos}(\frac{\pi}{m_{st}})\ \forall s, t \in S \tag{3.1}$$

$$s \cdot v = v - \alpha_s^\vee(v)\alpha_s\ \forall s \in S, v \in \mathfrak{h}. \tag{3.2}$$

We choose $\mathfrak{h}$ of minimal dimension with the above properties. Let $R = \mathbb{R}[\mathfrak{h}]$ be the coordinate ring of $\mathfrak{h}$. We define the grading on R by setting $\deg(\alpha) = 2$ for any $\alpha \in \mathfrak{h}^*$. In case W is a Weyl group, $\mathfrak{h}_{\mathbb{R}}$ is a real part of the Cartan subalgebra, the α_s's are the roots and $\alpha_t^\vee$'s are the coroots. The augmentation ideal (ideal of nonconstant polynomials) of R will be denoted by R^+.

We consider the abelian category of finitely generated graded R-bimodules. All morphisms preserve the grading (in other words, are homogeneous of degree 0).

Definition 3.2. For any simple reflection $s \in S$ set $B_s := R \otimes_{R^s} R(1)$. We denote by (n) the shift of grading by the corresponding number, i.e. $R \otimes_{R^s} R(1)$ means that the degree of $1 \otimes 1$ is -1, etc. The *Bott-Samelson bimodule* associated to an expression $\underline{w} = s_1 \dots s_m$ is $BS(\underline{w}) = B_{s_1} \otimes_R \dots \otimes_R B_{s_m} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_m}} R(n)$.

By the *Bott-Samelson* module we will understand $BS(\underline{w}) \otimes_R \mathbb{R}$.

Definition 3.3. The operator $R \to R$ given by $\partial_s(r) := \frac{r - s(r)}{2\alpha_s}$ is called the *Demazure operator*. Notice that ∂_s is R^s-linear.

Exercise 3.1. The elements $c_{id} := 1 \otimes 1$ and $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$ (of degrees -1 and 1) form a basis of B_s as a left (or right) R-module. One has relations

$$c_s r = r c_s \tag{3.3}$$

$$r c_{id} = c_{id} s(r) + \partial_s(r) c_s, \tag{3.4}$$

Remark 3.1. In general, one can check, that the elements $c_{\underline{\epsilon}} := c_{\epsilon_{i_1}} \otimes \dots \otimes c_{\epsilon_{i_k}}$, where $\underline{\epsilon} = s_{i_1} \dots s_{i_k}$ runs through all subexpressions of $\underline{w}$ form a basis of $BS(\underline{w})$ as a left (or right) R-module.

Notation 3.1. Henceforth we abbreviate

$$B_{s_{i_1}} \dots B_{s_{i_k}} := B_{s_{i_1}} \otimes_R \dots \otimes_R B_{s_{i_k}}$$

We provide an example of an easy calculation of the product of two Bott-Samelson bimodules.

Example 3.1. Using, $R = R^s \oplus R^s \alpha_s = R^s \oplus R^s(-2)$ (the equality of B_s-bimodules), we write

$$B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R = R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R =$$
$$= B_s(1) \oplus B_s(-1),$$

which is analogous to the relation

$$b_s^2 = (v + v^{-1}) b_s$$

in $\mathcal{H}$ (see Exercise 2.2).

Lemma 3.1. *In Example 5.1 ($W = A_2$), the Bott-Samelson bimodule $BS(\underline{s_1 s_2 s_1})$ decomposes into the direct sum $B_{s_1 s_2 s_1} \oplus B_{s_1}$, where $B_{s_1 s_2 s_1} = R \otimes_{R^W} R(3)$ is the submodule generated by $1 \otimes 1 \otimes 1$.*

Proof. Let us verify this decomposition. The main ingredient of the proof is to produce a nontrivial idempotent of degree 0 in $\mathrm{End}(BS(\underline{s_1 s_2 s_1}))$. For this we define some morphisms between bimodules:

$$m_s \in \mathrm{Hom}(B_s, R) : p \otimes q \mapsto pq$$

$$m_s^a \in \mathrm{Hom}(R, B_s) : 1 \mapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s$$

$$j_s \in \mathrm{Hom}(B_sB_s, B_s) : p \otimes h \otimes q \mapsto p\partial_s(h) \otimes q$$

$$j_s^a \in \mathrm{Hom}(B_s, B_sB_s) : p \otimes q \mapsto p \otimes 1 \otimes q.$$

Notice, that the morphisms m_s and m_s^a have degree 1, while the the degree of the morphisms j_s and j_s^a is -1. Next, let us introduce $e := -m_{s_2}^a j_{s_1}^a j_{s_1} m_{s_2} \in \mathrm{End}(B_{s_1}B_{s_2}B_{s_1}) : B_{s_1}B_{s_2}B_{s_1} \xrightarrow{m_{s_2}} B_{s_1}B_{s_1} \xrightarrow{j_{s_1}} B_{s_1} \xrightarrow{j_{s_1}^a} B_{s_1}B_{s_1} \xrightarrow{m_{s_2}^a} B_{s_1}B_{s_2}B_{s_1}$ and claim that e is an idempotent. Indeed, this follows from the equality $j_{s_1}m_{s_2}m_{s_2}^a j_{s_1}^a \in \mathrm{Hom}(B_{s_1}, B_{s_1}) : B_{s_1} \xrightarrow{j_{s_1}^a} B_{s_1}B_{s_1} \xrightarrow{m_{s_2}^a} B_{s_1}B_{s_2}B_{s_1} \xrightarrow{m_{s_2}} B_{s_1}B_{s_1} \xrightarrow{j_{s_1}} B_{s_1} = -id$ shown below:

$$p \otimes q \overset{j_{s_1}^a}{\mapsto} p \otimes 1 \otimes q \overset{m_{s_2}^a}{\mapsto} p \otimes (\alpha_{s_2} \otimes 1 + 1 \otimes \alpha_{s_2}) \otimes q \overset{m_{s_2}}{\mapsto} 2(p \otimes \alpha_{s_2} \otimes q) \overset{j_{s_1}}{\mapsto} -p \otimes q.$$

The last transition follows from the equality $s_1(\alpha_{s_2}) = \alpha_{s_2} + \alpha_{s_1}$ and Definition 3.3. Hence, e is a projector. As the first two maps in the definition of e are surjective and the last - injective and the chain of maps is $B_{s_1}B_{s_2}B_{s_1} \to B_{s_1}B_{s_1} \to B_{s_1} \to B_{s_1}B_{s_1} \to B_{s_1}B_{s_2}B_{s_1}$, we see that e is the projector onto B_{s_1}. Next, the morphism $1 - e$ is a projection as well, so, $B_{s_1}B_{s_2}B_{s_1} = im(e) \oplus im(1 - e)$. We first show that $B_{s_1}B_{s_2}B_{s_1}$ is generates by two elements $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes x_1 \otimes 1 \otimes 1$, where $R = \mathbb{R}[x_1, x_2, x_3, x_4]$. Indeed, as $(x_1 + x_2)(1 \otimes 1 \otimes 1 \otimes 1) = 1 \otimes (x_1 + x_2) \otimes 1 \otimes 1$ (as $(x_1 + x_2)$ is invariant under s_1), we have $1 \otimes x_2 \otimes 1 \otimes 1$ and, thus $1 \otimes (x_1 - x_2) \otimes 1 \otimes 1 = 1 \otimes \alpha_{s_1} \otimes 1 \otimes 1$ is in the submodule, generated by $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes x_1 \otimes 1 \otimes 1$. Next, $1 \otimes x_1 \otimes 1 \otimes 1 = 1 \otimes 1 \otimes x_1 \otimes 1$ and $1 \otimes 1 \otimes x_3 \otimes 1 = (1 \otimes 1 \otimes 1 \otimes 1)x_3$ and $1 \otimes 1 \otimes (x_1 - x_2) \otimes 1$, thus, $1 \otimes 1 \otimes (x_2 - x_3) \otimes 1 = 1 \otimes 1 \otimes \alpha_{s_2} \otimes 1$ are in the submodule as well. Similarly can be shown that the submodule contains $1 \otimes \alpha_{s_1} \otimes \alpha_{s_2} \otimes 1$ and, therefore, by Remark 3.1 generates the module.

The calculations above, in particular, show that $\dim(\mathbb{R} \otimes_R B_{s_1}B_{s_2}B_{s_1} \otimes_R \mathbb{R}) = 2$. The fact that $\dim(\mathbb{R} \otimes_R B_{s_1}B_{s_2}B_{s_1} \otimes_R \mathbb{R}) = 2$ implies that there are only two indecomposable summands in the decomposition of $BS_{\underline{s_1s_2s_1}}$.

Now define a map of R-bimodules

$$\gamma : R \otimes_{R^W} R(3) \to BS_{\underline{s_1s_2s_1}}$$

by

$$p \otimes q \mapsto p \otimes 1 \otimes 1 \otimes q.$$

Since as left R-modules $BS_{\underline{s_1s_2s_1}} \cong R(-3) \oplus R(-1)^{\oplus 3} \oplus R(1)^{\oplus 3} \oplus R(3)$ (see Remark 3.1) and $R \otimes_{R^W} R(3) \cong R(-3) \oplus R(-1)^{\oplus 2} \oplus R(1)^{\oplus 2} \oplus R(3)$, $im(1 - e)$ and $R \otimes_{R^W} R(3)$ have the same graded dimensions as vector spaces over $\mathbb{R}$, it suffices to show that the map γ is surjective. As $(1 - e)(1 \otimes 1 \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1$ (follows from $\partial_s(1) = 0$), and $1 \otimes 1 \otimes 1 \otimes 1 \in im(\gamma)$ as well, it suffices to show that $im(1 - e)$ is generated by $1 \otimes 1 \otimes 1 \otimes 1$. For this we need to show that the submodule, generated by $1 \otimes 1 \otimes 1 \otimes 1$ contains $im(1 - e)(1 \otimes x_1 \otimes 1 \otimes 1)$ First we compute $-e(1 \otimes x_1 \otimes 1 \otimes 1)$:

$$1 \otimes x_1 \otimes 1 \otimes 1 \overset{m_{s_2}}{\mapsto} 1 \otimes x_1 \otimes 1 \overset{j_{s_1}}{\mapsto} \frac{1}{2}(1 \otimes 1) \overset{j_{s_1}^a}{\mapsto} \frac{1}{2}(1 \otimes 1 \otimes 1) \overset{m_{s_2}^a}{\mapsto} \frac{1}{2}(1 \otimes (x_2 - x_3) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (x_2 - x_3) \otimes 1).$$

So, $(1-e)(1\otimes x_1\otimes 1\otimes 1) = 1\otimes x_1\otimes 1\otimes 1+\frac{1}{2}(1\otimes(x_2-x_3)\otimes 1\otimes 1+1\otimes 1\otimes(x_2-x_3)\otimes 1)$. Now we show that this element lies in the submodule generated by $1\otimes 1\otimes 1\otimes 1$. For this we write $1\otimes x_1\otimes 1\otimes 1 = \frac{1}{2}(1\otimes x_1\otimes 1\otimes 1+1\otimes 1\otimes x_1\otimes 1)$ and show that $\frac{1}{2}(1\otimes x_1\otimes 1\otimes 1+1\otimes(x_2-x_3)\otimes 1\otimes 1)$ is in the submodule (for $\frac{1}{2}(1\otimes 1\otimes x_1\otimes 1+1\otimes 1\otimes(x_2-x_3)\otimes 1)$ the computation is completely analogous):

$$\frac{1}{2}(1\otimes x_1\otimes 1\otimes 1+1\otimes(x_2-x_3)\otimes 1\otimes 1) = \frac{1}{2}(1\otimes(x_1+x_2-x_3)\otimes 1\otimes 1 = \frac{1}{2}(x_1+x_2-x_3)\otimes 1\otimes 1\otimes 1.$$

This concludes the proof. □

One should notice the resemblance between the decomposition $B_{s_1}B_{s_2}B_{s_1} = B_{s_1s_2s_1}\oplus B_{s_1}$ and the relation $b_{s_1}b_{s_2}b_{s_1} = b_{s_1s_2s_1}+b_{s_1}$ derived in Example 2.1.

Remark 3.2. More generally, it can be shown that if W is a dihedral group generated by simple reflections (s,t) and $\ell(w') < \ell(w)$, where $w' = sw$, then $B_sB_{w'} = B_w\oplus B_{tw'}$ (compare to (2.5)).

Definition 3.4. The category of *Soergel bimodules* $\mathcal{S}Bim$ is the full subcategory of $\mathbb{Z}$-graded R-bimodules, where the objects are the direct sums of direct summands of graded shifts of BS-bimodules. The morphisms are grading preserving morphisms of $R-R$-bimodules.

Similarly, we define the category of *Soergel modules* $\mathcal{S}Mod$ to be the full subcategory of $\mathbb{Z}$-graded left R-modules, where the objects are the direct sums of direct summands of graded shifts of BS- modules. The morphisms are grading preserving morphisms of R-bimodules.

Remark 3.3. Notice that $BS_{\underline{w_1}}BS_{\underline{w_2}} = BS_{\underline{w_1w_2}}$ implies that the category $\mathcal{S}Bim$ is closed w.r.t the tensor product. As $fg_1\otimes_{R^{s_{i_1}}}g_2\otimes_{R^{s_{i_2}}}\dots\otimes_{R^{s_{i_n}}}g_n = g_1\otimes_{R^{s_{i_1}}}g_2\otimes_{R^{s_{i_2}}}\dots\otimes_{R^{s_{i_n}}}g_nf$ for $f\in R^W$, every Soergel bimodule is actually an $R\otimes_{R^W}R$-module.

Definition 3.5. An additive category is said to be *Krull-Schmidt* if every object is isomorphic to a direct sum of indecomposable objects and such decomposition is unique up to isomorphism and permutation of summands.

Proposition 3.2. *The category of Soergel bimodules is Krull-Schmidt.*

Proof. We notice that the category $\mathcal{S}Bim$ is closed under taking direct summands (by its definition). Since the bimodule $\mathrm{Hom}_{R\otimes R}(M,N)$ between any two finitely generated graded bimodules M and N is graded and finitely generated, the degree 0 part is a finite-dimensional space. Thus, the additive category $\mathcal{S}Bim$ is closed under taking direct summands and has finite-dimensional Hom-spaces. It is a standard fact that such categories are Krull-Schmidt. □

Next we explain what we mean by the split Grothendieck group $K_0(\mathcal{S}Bim)$ of the category $\mathcal{S}Bim$. This is the abelian group generated by symbols $[B]$ for all objects $B\in\mathcal{S}Bim$ subject to the relations $[B] = [B']+[B'']$ whenever $B\cong B'\oplus B''$ in $\mathcal{S}Bim$. We make $K_0(\mathcal{S}Bim)$ into a $\mathbb{Z}[v^{\pm 1}]$-module via $v^i[M] = [M](i)$ and $[M]\in K_0(\mathcal{S}Bim)$. The tensor product on $\mathcal{S}Bim$ endows $K_0(\mathcal{S}Bim)$ with multiplication, thus, making it a $\mathbb{Z}[v^{\pm 1}]$-algebra. Moreover, $K_0(\mathcal{S}Bim)$ is a free $\mathbb{Z}[v^{\pm 1}]$ - module, whose basis consists of indecomposable objects (we take one up to a grading shift).

We can now formulate the main theorem.

Theorem 3.3. *(Soergel's categorification theorem) There is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras $\mathcal{H} \to K_0(\mathcal{S}Bim)$, sending b_s to $[B_s]$.*

Corollary 3.4. *(Weak form of Soergel's categorification theorem). There exists a unique homomorphism of rings $c : \mathcal{H} \to K_0(\mathcal{S}Bim)$, s.t. $c(b_s) = B_s$.*

Proof. The quadratic relation was checked in Example 3.1 and the braid relations - in Lemma 3.1 (for the simply laced case) and stated in Remark 3.2 for the general case. The uniqueness of c is obvious, since it is defined on a generating set. $\square$

4. Soergel's categorification theorem

Next we would like to present the classification of indecomposable Soergel bimodules and give a prove of the main theorem. We will use the following proposition (see Section 4 of [Soe92]).

Proposition 4.1. *For two Soergel bimodules B_1, B_2, the canonical map $G : \operatorname{Hom}_{R\otimes R}(B_1, B_2) \otimes_R \mathbb{R} \to \operatorname{Hom}_R(B_1 \otimes_R \mathbb{R}, B_2 \otimes_R \mathbb{R})$ is an isomorphism.*

Corollary 4.2. *The map $\delta : M \mapsto M \otimes_R \mathbb{R}$ induces an embedding of indecomposable objects in $\mathcal{S}Bim$ into indecomposable objects in $\mathcal{S}Mod$.*

Proof. We first show that the image of an indecomposable module M is indecomposable. Indeed, if $\delta(M)$ would decompose as $M_1 \oplus M_2$ there would be a degree zero idempotent $e_{M_1} \in \operatorname{End}_R(M \otimes_R \mathbb{R})$, but since $\operatorname{End}_R(\delta(M)) \cong \operatorname{End}_{R\otimes R}(M) \otimes_R \mathbb{R}$, this implies the existence of a degree zero idempotent (there exists a lift - this is a standard fact, which can be shown by constructing the lifts modulo $(R_+)^n$ for every $n \in \mathbb{N}$ and $(R_+)^n End_{R\otimes R}(M)$ has no degree 0 elements for n large enough) $\widetilde{e} \in \operatorname{End}_{R\otimes R}(M) \otimes_R \mathbb{R}$, which (as $M \cong \widetilde{e}M \oplus (1-\widetilde{e})M$ and $\widetilde{e} \neq 1$) contradicts our assumption that M is indecomposable.

Next we check that δ maps non isomorphic indecomposables to non isomorphic ones. Assume the contrary and let $M_1, M_2 \in \mathcal{S}Bim$ be indecomposable and $\delta(M_1) \cong \delta(M_2) = \widetilde{M} \in \mathcal{S}Mod$. Then there exist a $\alpha \in \operatorname{Hom}_{R\otimes R}(M_1, M_2)$ and $\beta \in \operatorname{Hom}_{R\otimes R}(M_2, M_1)$, s.t. $G(\alpha \circ \beta)$ is invertible. The application of graded Nakayama's lemma implies $\alpha \circ \beta$ is invertible:

$$\alpha \circ \beta(M_2) + R_+ M_2 = M_2$$
$$R_+ \frac{M_2}{\alpha \circ \beta(M_2)} = \frac{M_2}{\alpha \circ \beta(M_2)}$$

gives $M_2 = \alpha \circ \beta(M_2)$. So $\alpha \circ \beta$ is surjective, hence invertible. $\square$

Theorem 4.3. *Each Bott-Samelson bimodule $BS(\underline{w})$ contains a unique indecomposable summand B_w which does not appear in $BS(\underline{x})$ for $x \prec w$ depends only on w, but not on the reduced expression.*

Proof. Recall from Dmytro's talk (Corollary 5.9) that, for a reduced $\underline{w}$, the module $BS(\underline{w}) \otimes_R \mathbb{R}$ contains a unique graded indecomposable summand, S_w, that does not appear in $BS(\underline{w}') \otimes_R \mathbb{R}$ for shorter $\underline{w}'$ and that depends only on w. In fact, in Dmytro's talk the claim was proved over $\mathbb{C}$ but one can show it holds over $\mathbb{R}$ as well. So $BS(\underline{w}) \otimes_R \mathbb{R} = S_w \oplus \bigoplus_{w'} S_{w'}(d_i)^{\oplus n_i}$ where the sum is taken over $w' \prec w$. Let $BS(\underline{w}) = B_1 \oplus \ldots \oplus B_k$ be the decomposition into indecomposables. By Corollary 4.2 , there is a unique index i (say i=1 to be definite) such that $B_1 \otimes_R \mathbb{R} = S_w$ and then $B_i \otimes_R \mathbb{R} \cong S_{w'}(d_{w'})$ for $i > 1$. We set $B_w := B_1$. Our claim follows from the induction on the length of $\underline{w}$ and Corollary 4.2.

□

Corollary 4.4. *The indecomposable Soergel bimodules are in bijection with the elements of $W \times \mathbb{Z}$.*

Proof. The result follows from the observation that grading shifts preserve indecomposability. □

The above results allow us to prove the main theorem (Theorem 3.3).

Proof. We choose one reduced expression $\underline{w} = \underline{s_1 \dots s_m}$ for every element $w \in W$, then it follows from Theorem 4.3 that the classes of the corresponding $BS(\underline{w})$'s form a basis of $K_0(\mathcal{S}Bim)$ (each $[BS(\underline{w})]$ contains the indecomposable $[B_w]$ as a summand with coefficient 1 and it is not hard to show by induction on the Bruhat order that there exists a $\mathbb{Z}[v^{\pm 1}]$-linear combination of $[BS(\underline{w'})], w' \prec w$ that, being subtracted from $[BS(\underline{w})]$, gives $[B_w]$). Then the corresponding elements $b_{s_1} \dots b_{s_m} \in \mathcal{H}$ are also a basis (again, using induction on the Bruhat order analogously to the proof of Proposition 2.1, we show that b_w is $b_{s_1} \dots b_{s_m}$ minus a $\mathbb{Z}[v^{\pm 1}]$-linear combination of $b_{s_{j_1}} \dots b_{s_{j_k}}$ for $w' = s_{j_1} \dots s_{j_k} \prec w$). □

5. Bott-Samelson varieties

Let G be a connected reductive complex algebraic group, $T \subset B \subset G$ a maximal torus and a Borel subgroup. Every element w in the Weyl group $W = N_G(T)/T$ has a reduced expression $\underline{w} = s_{i_1} \dots s_{i_\ell}$ with s_{i_j} - simple reflections. We also denote by P_{i_j} the minimal parabolic subgroup generated by B and a representative of $s_{i_j} \in W$ (the Lie algebra $p_{i_j} := \mathrm{Lie}(P_{i_j}) = \mathfrak{b} \oplus f_{i_j}$ with f_{i_j} - a negative simple root).

Definition 5.1. The *Schubert cell C_w* is the variety BwB/B and the *closed Schubert subvariety* $\overline{C_w}$ is $\overline{BwB}/B$.

Definition 5.2. The *Bott-Samelson variety $\mathcal{BS}_{\underline{w}}$* is defined as $P_{i_1} \times_B P_{i_2} \times_B \dots \times_B P_{i_\ell}/B$, where the action of $B^{\times \ell(w)}$ on $P_{i_1} \times P_{i_2} \times \dots \times P_{i_\ell}$ is given by $(b_1, \dots, b_\ell) \cdot (p_{i_1}, \dots, p_{i_\ell}) = (p_{i_1} b_1^{-1}, b_1 p_{i_2} b_2^{-1}, \dots, b_{\ell-1} p_{i_\ell} b_\ell^{-1})$.

As $P_{i_k}/B \simeq \mathbb{P}^1$, one can think of a Bott-Samelson variety as a tower of $\mathbb{P}^1$-bundles. In particular, these varieties are smooth. Moreover, there is a surjective map $\varphi : \mathcal{BS}_{\underline{w}} \to \overline{C_w}$

$$\varphi(p_{i_1}, \dots p_{i_\ell}) = p_{i_1} \dots p_{i_\ell},$$

which restricts to an isomorphism on the preimage of C_w. Hence this map is birational and, therefore, a resolution of singularities. However, the following example illustrates that even if $\overline{C_w}$ is smooth φ is not necessarily an isomorphism.

Example 5.1. We consider the element $s_1s_2s_1 \in S_3$, the corresponding Schubert variety $\overline{C_w}$ is the flag variety, hence, smooth. However, the map $\varphi : \mathcal{BS}_{s_1s_2s_1} \to \overline{C}_{s_1s_2s_1}$ is not an isomorphism. One reason is that the Poincare polynomial for the cohomology of $\mathcal{BS}_{s_1s_2s_1}$ is $(1+q^2)^3$, while the one for $\overline{C}_{s_1s_2s_1}$ is $1+2q^2+2q^4+q^6 = (1+q^2)(1+q^2+q^4)$ (as $\overline{C}_{s_1s_2s_1}$ is the flag variety GL_3/B). Alternatively, one can see, that the preimage of any point in relative position e or (12) with respect to the standard flag is isomorphic to $\mathbb{P}^1$.

Before stating and proving the result, explaining the connection of Bott-Samelson varieties and Bott-Samelson bimodules, we need some definitions.

Definition 5.3. The variety X is called *equivariantly formal* if $H^\bullet_T(X)$ (the equivariant cohomology with respect to the algebraic torus $T = (\mathbb{C}^*)^n$-action) is a free R-module.

From now on we assume that X is a complex projective algebraic variety, which is equivariantly formal and has finitely many T - fixed points and one-dimensional orbits.

Definition 5.4. The *moment graph* of a T-variety X is a triple (V, E, χ_E), where

- V is the set of vertices and is in bijection with the points of X^T;

- E is the set of edges and is in bijection with the one-dimensional orbits of T on X, each edge connects the two vertices of V, corresponding to the points in its closure;

- For any edge in E the closure of the corresponding orbit is isomorphic to $\mathbb{P}^1$ with two fixed points. The action of T on this $\mathbb{P}^1$ produces a character $T \to \mathbb{C}^*$. We denote the differential of this character by χ_E and put it as a label on the corresponding edge.

Theorem 5.1. (1) *There is an isomorphism of R - bimodules*

$$H^\bullet_T(\mathcal{BS}_{s_1\dots s_m}) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_m}} R$$

(2) $H^\bullet(\mathcal{BS}_{s_1\dots s_m}) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_m}} R \otimes_R \mathbb{R}$

Proof. We verify (1) using the localization theorem (Theorem 1.2.2. of [GKM98]), which says that if X is equivariantly formal and has finitely many T - fixed points and one-dimensional orbits, then there is an embedding $i^* : H^\bullet_T(X) \hookrightarrow H^\bullet(X^T) \otimes R$ (this embedding comes from the map $i : X^T \hookrightarrow X$ and becomes an isomorphism after tensoring with the quotient field $Quot(R)$, but we will not need this). Furthermore, one can give a precise description of the image of i^*.

By Theorem 1.2.2. of [GKM98] the T-equivariant cohomology of X are given by

$$\left\{ (f_x) \in \bigoplus_{x\in V} R_x \mid \chi_E \text{ divides } f_x - f_y \text{ whenever } x \text{ and } y \text{ are connected by the edge } E \right\}.$$

In our case, there are 2^m fixed points. These are the points $(p_{i_1}, \dots, p_{i_m}) \in \mathcal{BS}_{\underline{w}} = P_{i_1} \times_B P_{i_2} \times_B \dots \times_B P_{i_m}/B$ with each $p_{i_k} \in B$ or $s_{i_k}B$. Two vertices are connected, whenever the corresponding sequences of elements differ only in one entry. The example of the moment graph for $\mathcal{BS}_{\underline{s_1s_2s_1}}$ is given on Figure 5. Number the fixed points so that the corresponding vertices of the moment graph are arranged from bottom to top and from left to right on each row.

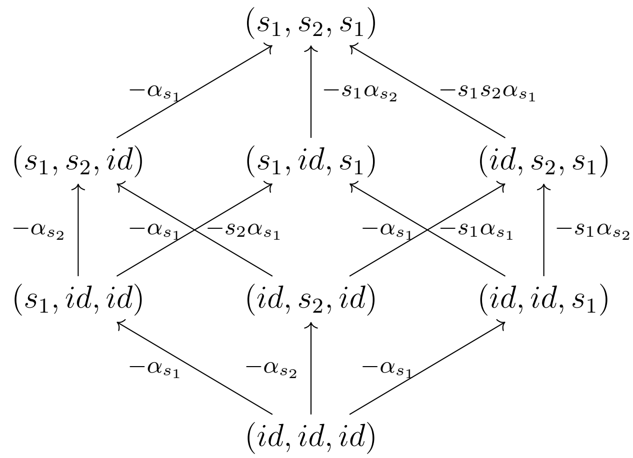

Figure 1. Moment graph of $BS(\underline{s_1s_2s_1})$

Define the map $\theta : BS_{\underline{w}} \to H^\bullet_T(\mathcal{BS}_{\underline{w}})$ as follows: θ maps the element $g \otimes f_1 \otimes \ldots \otimes f_m$ to $g(f_1 \ldots f_m, \ldots, s_{i_m}(f_m(s_{i_m-1}(f_{\ell-1} \ldots s_{i_1}(f_1))))) \in R^{2^{\ell(w)}}$, where the ith coordinate is given by the following procedure: apply the leftmost element of the sequence corresponding to the vertex with number i (i.e. id or s_1) to f_1, then the second element from the left to the product of f_2 with the result of the first application, etc. For instance, the image of $\theta : BS_{\underline{s_1s_2s_1}} \to H^\bullet_T(\mathcal{BS}_{\underline{s_1s_2s_1}})$ is given by

$$\theta(g \otimes f_1 \otimes f_2 \otimes f_3) = g(f_1f_2f_3, s_1(f_1)f_2f_3, s_2(f_1f_2)f_3, s_1(f_1f_2f_3), s_2s_1(f_1)s_2(f_2)f_3, s_1(s_1(f_1)f_2f_3),$$
$$s_1(s_2(f_1f_2)f_3), s_1(s_2(s_1(f_1)f_2)f_3)).$$

It remains to show that θ is an isomorphism. For this purpose, it is more convenient to rewrite $BS_{\underline{s_1\ldots s_m}}$ using the method of Example 3.1 multiple times (starting from the rightmost copy of R as we look at $BS_{\underline{s_1\ldots s_m}}$ a left R-module) as

$$\left(R \oplus \underbrace{R(-2) \oplus \ldots \oplus R(-2)}_{m} \oplus \ldots \oplus R(-2m)\right)(m).$$

Now the key observation is that we can recover the decomposition of $F = g \otimes f_1 \otimes \ldots \otimes f_m$ in terms of the basis provided in Remark 3.1 from the image of θ. As every $f \in R$ can be written as $f = \frac{s+1}{2}(f) + \frac{s-1}{2}(f)$, with $\frac{s+1}{2}(f) \in R^s$ and $\frac{s-1}{2}(f) \in R^s\alpha_s$, the coefficient of the basic element $c_{\underline{\epsilon}}$ in the decomposition of F equals to $g\frac{s_1 \pm 1}{2}\left(f_1 \ldots \frac{s_{m-1} \pm 1}{2}\left(f_{m-1}\frac{s_m \pm 1}{2}(f_m)\right) \ldots\right)$, where the sign is a minus, if $c_{\epsilon_k} = c_{s_k}$ and a plus in case $c_{\epsilon_k} = c_{id}$. The expression $g\frac{s_1 \pm 1}{2}\left(f_1 \ldots \frac{s_{m-1} \pm 1}{2}\left(f_{m-1}\frac{s_m \pm 1}{2}(f_m)\right) \ldots\right)$ is nothing else, but the sum of the monomials in $\theta(F)$ with coefficients equal to $\frac{\pm 1}{2^m}$.

The second assertion is a consequence of the basic fact that $H^\bullet(X) \cong H^\bullet_T(X)/(R^+ H^\bullet_T(X))$, whenever X is equivariantly formal (see remarks after Theorem 1.2.2. of [GKM98]). $\square$

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