Chiral differential operators 1

0) Motivation

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Let G be a simple algebraic group $/\mathbb{C}$, $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}$ be a triangular decomposition, $N^-, H, N\subset G$ be the corresponding subgroups. In Daishi's talk we have seen a homomorphism

$$\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}(N)\otimes\mathcal{U}(\mathfrak{h}) \qquad (1)$$

In Zeyu's talk an affine analog of this homomorphism was introduced: a vertex algebra homomorphism

$$V_\kappa(\mathfrak{g}) \longrightarrow M_{\mathfrak{g}}\otimes V_{\kappa-\kappa_c}(\mathfrak{h}) \qquad (2)$$

No proof was given, in fact, the proof provided in the book was technical and upleasant. In this note we begin to sketch an alternative approach, which constructs (2) using the affine analog of (1) based on the vertex algebra analogs of $\mathcal{D}(G)$ – the algebras of chiral differential operators.

1) Chiral differential operators

1.1) Construction

Let G be a connected algebraic group, and κ be a G-invariant symmetric bilinear form on $\mathfrak{g}$ (possibly zero). We can form the central extension $\hat{\mathfrak{g}}_\kappa$ just as for semisimple algebras.

Note that $\mathfrak{g}[[t]] = J\mathfrak{g}$ acts on $\mathbb{C}[JG]$ (in fact, in two different ways - via right & left invariant vector fields; we are now interested in the former action). So we can form the induced module

$$CDO_\kappa(G) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_\kappa} \mathbb{C}[JG]$$

where $\mathbb{1}$ acts by 1.

We need to equip $CDO_\kappa(G)$ with a vertex algebra structure. Note that $\mathbb{C}[JG] \hookrightarrow CDO_\kappa(G)$ (via $f \mapsto 1 \otimes f$) & $\mathfrak{g}t^{-1} \hookrightarrow CDO_\kappa(G)$ (via $x \mapsto x \otimes 1$). We set $|0\rangle = 1 \otimes 1$.

Exercise: $\exists!$ linear operator T on $CDO_\kappa(\mathfrak{g})$ s.t.

- T acts on $\mathbb{C}[JG]$ by the natural derivation of this algebra (that is a part of its vertex algebra structure).

- for $x \in \hat{\mathfrak{g}}_k$ & $F \in CDO_k(\mathfrak{g})$ we have

$$T(xF) = (-\partial_t x)F + x(TF).$$

We will explain how to define the fields $Y(f,z)$ & $Y(xt^{-1},z)$ for $f \in \mathbb{C}[JG]$ & $x \in \mathfrak{g}$. To define $Y(xt^{-1},z)$ is easy: these fields make sense for any smooth $\hat{\mathfrak{g}}_k$-module – including for $CDO_k(G)$. To define $Y(f,z)$ is more tricky. First, observe that $\mathbb{C}[JG]$ is a commutative vertex algebra. We want it to be a vertex subalgebra, which defines $Y(f,z)$ on $\mathbb{C}[JG]$. To extend $Y(f,z)$ to $CDO_k(G)$ we note that $CDO_k(G)$ is generated by $\mathbb{C}[JG]$ as a $\hat{\mathfrak{g}}_k$-module. So it's enough to specify the commutator $[Y(xt^{-1},z), Y(f,w)]$. By formula (2.3-7) in Frenkel's book in a vertex algebra this commutator has to equal

$$\sum_{n\geq 0} \frac{1}{n!} Y((xt^{-1})_{(n)}f, w)\, \partial_w^n \delta(z-w) \qquad (3)$$

Note that $(xt^{-1})_{(n)} = xt^n$ and so $(xt^{-1})_{(n)}f$ is already defined (b/c $\mathbb{C}[JG]$ is a $J\mathfrak{g}$-module). So (3) uniquely specifies $Y(f,z)$.

Exercise: Use the strong reconstruction theorem to show that $CDO_k(G)$ is indeed a vertex algebra.

A reference for this section is: S. Arkhipov, D. Gaitsgory "Differential operators and the loop group via chiral algebras."

1.2) Localization & filtration.

First, we comment on the localization. Let $G^\circ \subset G$ be an open affine (say, principal) subset. Then $J\mathfrak{g}$ still acts on $\mathbb{C}[JG^\circ]$ so we can form the induced module

$$\mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_k} \mathbb{C}[JG^\circ]$$

Just as in the previous section this induced module carries a natural vertex algebra structure. The resulting vertex algebra will be denoted by $CDO_k(G^\circ)$. Note that there is a vertex algebra embedding $CDO_k(G) \hookrightarrow CDO_k(G^\circ)$. The target should be thought of as the localization of $CDO_k(G)$.

More generally, for a smooth variety X subject to some cohomology vanishing conditions there are sheaves of vertex algebras on X called chiral differential operators. For more details the readers are referred to:

F. Malikov: An introduction to Algebras of Chiral Differential operators.

Second, $CDO_k(G)$ is filtered by the degree of differential operator: this filtrations is the filtration on the induced module coming from the PBW filtration on $U(\hat{\mathfrak{g}}_k)/(\mathbb{1}-1)$. Let $CDO_k(G)_{\le m}$ be the deg m piece.

Exercise: Let $a \in CDO_k(G)_{\le m}$, $b \in CDO_k(G)_{\le \ell}$. Then

$$a_{(n)} b \in \begin{cases} CDO_k(G)_{\le m+\ell} & \text{if } n<0 \\ CDO_k(G)_{\le m+\ell-1} & \text{if } n \geq 0 \end{cases}$$

Hint: cf (3).

1.3) "Graded" unipotent groups.

Consider the case when N is a unipotent group & $k=0$. Suppose that $\mathbb{G}_m$ acts on N by automorphisms s.t. the induced grading on $\mathfrak{n}$ is positive: $t.\xi = t^i\xi$ for $\xi\in\mathfrak{n}\setminus\{0\} \Rightarrow i>0$. For example if $P\subset G$ is a standard parabolic, we can take $N=\operatorname{Rad}_u(P)$ and take $\mathbb{G}_m \curvearrowright N$ from the principal grading on $\mathfrak{g}$.

We identify N w. $\mathfrak{n}$ via the exponential map and choose a basis $y_1,\dots y_\ell\in\mathfrak{n}$ homogeneous for $\mathbb{G}_m\curvearrowright\mathfrak{n}$. Then $\mathbb{C}[JN]=\mathbb{C}[y^i_n]_{i=1,\dots\ell,\, n\leqslant 0}$, where $y^1,\dots y^\ell$ is the dual basis, & y^i_0 is the image of y^i under the embedding $\mathbb{C}[N]\hookrightarrow\mathbb{C}[JN]$. This is a negatively graded algebra. Note that $CDO(N)$ inherits the grading: $CDO(N)=\bigoplus\limits_{i\in\mathbb{Z}} CDO(N)[i]$ & the grading is compatible w. the algebra structure: if a has deg i & b has deg j, then $a_{(n)}b$ has deg $i+j$. We also note that $\mathbb{C}[JN]$ is negatively graded, while $V(\mathfrak{n})$ is positively graded.

Let $\partial_1,\dots\partial_\ell$ denote the constant vector fields on $\mathfrak{n}$ corresponding to $y_1,\dots y_\ell$. We write ${}^Ry_1,\dots {}^Ry_\ell$ for the right invariant vector fields on $N\rightarrow\mathfrak{n}$ corresponding to $y_1,\dots y_\ell$. Let

$f_{ij} \in \mathbb{C}[N]$ satisfy $\partial_i = \sum_{j=1}^{\ell} f_{ij}\, {}^R y_j$, this determines f_{ij}'s uniquely. Set

$$\partial_{i,-1} := \sum_{j=1}^{\ell} (f_{ij})_{(-1)}\, {}^R y_{j,-1}, \quad i = 1, \dots \ell. \tag{4}$$

Proposition: The following equalities hold

(i) $[Y(y_0^i, z), Y(y_0^j, w)] = 0$

(ii) $[Y(\partial_{i,-1}, z), Y(y_0^j, w)] = \delta(z-w)$.

(iii) $[Y(\partial_{i,-1}, z), Y(\partial_{j,-1}, w)] = 0$

Sketch of proof:

(i) follows b/c $\mathbb{C}[JN]$ is a commutative VA.

(ii): We have

$$[Y(\partial_{i,-1}, z), Y(y_0^j, w)] = \sum_{n \geq 0} \frac{1}{n!} Y(\partial_{i,(n)} y_0^j, w)\, \partial_w^n \delta(z-w).$$

Let's commute $\partial_{i,(n)} y_0^j$. This is the coefficient of z^{-n-1} in $Y(\partial_{i,-1}, z) y_0^j$. Thx to (4), we have

$$Y(\partial_{i,-1}, z) y_0^j = \sum_{h=1}^{\ell} :Y(f_{ih,0}, z)\, Y({}^R y_{h,-1}, z): y_0^j \tag{5}$$

Note that $f_{ih,(k)} y_0^j = 0$ for $k \geq 0$ b/c $\mathbb{C}[JN]$ is commutative

Also ${}^R y_{h,(k)} y_0^j = 0$ for $k > 0$. This is because

- $y_0^j \in \mathbb{C}[N] \subset \mathbb{C}[JN]$
- The operator ${}^R y_{h,(k)}$ is the action of $y_h t^k \in J\mathfrak{h}$. The subalgebra $tJ\mathfrak{h}$ acts by 0 on $\mathbb{C}[N]$ b/c the action of JN on $\mathbb{C}[N]$ factors through N.

So the r.h.s. of (5) simplifies to

$$\sum_{h=1}^{\ell} f_{ih,(-1)} \, {}^R y_{h,(0)} \, y_0^j = \partial_{i,(0)} y_0^j = \delta_{ij}.$$

The claim of (ii) follows.

(iii): This part is going to be a sketch.

Thanks to the filtration on $CDO(N)$ (see Sec 1.2), $\partial_{i(n)} \partial_j$ is in filtration degree 1. From (ii) it follows that $[Y(\partial_{i,-1}, z), Y(\partial_{j,-1}, w)]$ commutes w. $Y(y_0^h, u)$ $\forall\, i,j,h$. Recall that for usual differential operators, if a differential operator of order ≤ 1 commutes w. the multiplication by every function, it is of order 0. Similarly to this, we can show that $\partial_{i(n)} \partial_j \in \mathbb{C}[JN]$. But $\partial_{i(n)} \partial_j$ is in positive degree, while $\mathbb{C}[JN]$ is in a non-negative degree. It follows that $\partial_{i(n)} \partial_j = 0$ ($\forall\, i,j = 1, \ldots \ell$, $\forall\, n \geq 0$) implying (iii) $\square$

Remarks: 1) The conclusion of Proposition works for all unipotent groups.

2) Similarly to Proposition, one can consider left-invariant vector fields ${}^L y_i$, $i=1,\dots \ell$, on N (by convential ${}^L y_i + {}^R y_i$ is 0 at $1\in N$). Similarly to (4) one can define ${}^L y_{i,-1} \in CDO(N)$. The assignment $y_i t^{-1} \mapsto {}^L y_{i,-1}$ defines another embedding $V(\mathfrak{h}) \hookrightarrow CDO$ whose image commutes w. that of the initial embedding.

1.4) Embedding $V_{-\kappa-\kappa_{\mathfrak{g}}}(\mathfrak{g}) \hookrightarrow CDO_\kappa(G)$.

Our goal here is to produce, for suitable κ', an embedding

$$\iota: V_{\kappa'}(\mathfrak{g}) \longrightarrow CDO_\kappa(G)$$

whose image commutes w. that of $V_\kappa(\mathfrak{g})$:

$$[Y(xt^{-1},z), Y(\iota(yt^{-1}),w)]=0$$

In the case when G is unipotent (and graded), $\kappa=0$ and Remark 2 in Sec 1.3 gives such an embedding from $V(\mathfrak{g})$. In general, the embedding should also have to do with right invariant vector fields but the naive construction in Remark 2 doesn't work as the next example shows.

Example: Suppose $G = \mathbb{G}_m$ so that $\mathbb{C}[G] = \mathbb{C}[x^{\pm 1}]$. The vector field ${}^R y$ is $-x\frac{d}{dx}$ & ${}^L y = x\frac{d}{dx}$. But note that $V_k(\mathfrak{g})$ is not commutative if $k \neq 0$, instead

$$[Y(yt^{-1}, z), Y(-yt^{-1}, w)] = -k(y,y)\, \partial_w \delta(z-w).$$

So we need to find a function $f \in \mathbb{C}[J\mathbb{G}_m]$ with a property that $[Y(yt^{-1}, z), Y(F, w)] = -k(y,y)\, \partial_w \delta(z-w)$, then we can set $\iota(yt^{-1}) = -{}^R y_{-1} - F$. In other words, we want F w.

$$yt^n . F = 0 \text{ if } n=0 \text{ or } \geq 2,\ yt.F = -k(y,y) \tag{6}$$

Recall that $\mathbb{C}[J\mathbb{G}_m] = \mathbb{C}[x_i \mid i \leq 0][x_0^{-1}]$, where the embedding $\mathbb{C}[\mathbb{G}_m] \hookrightarrow \mathbb{C}[J\mathbb{G}_m]$ is by $x \mapsto x_0$. The derivations yt^n w. $n \geq 2$ act by 0 on $\mathbb{C}[x_0^{\pm 1}, x_{-1}] \subset \mathbb{C}[J\mathbb{G}_m]$, compare to the proof of (ii) of Proposition in Sec 1.3. The derivation y sends $x_i \mapsto -x_i$ for $i = 0, -1$, while $yt.x_0 = 0$, $yt.x_{-1} = -x_{-1}$. So $F = k(y,y)\, x_0^{-1} x_{-1}$ satisfies (6).

We set $\iota(yt^{-1}) = y^L_{-1} - k(y,y)\, x_0^{-1} x_{-1}$.

Exercise: This defines an embedding $V_{-k}(Lie(\mathbb{G}_m)) \hookrightarrow CDO_k(G)$.

Now consider the general case. Let $k_{\mathfrak{g}}$ denote the Killing form on $\mathfrak{g}$, $k_{\mathfrak{g}}(x,y) = tr(ad(x)ad(y))$. We are going to sketch a vertex algebra embedding $\iota: V_{-k-k_{\mathfrak{g}}}(\mathfrak{g}) \hookrightarrow CDO_k(G)$ whose image commutes w. $V_k(\mathfrak{g})$ (generalizing what we have for $G = \mathbb{G}_m$, where $k_{\mathfrak{g}}=0$).

As in the case of $G=\mathbb{G}_m$, we construct ι as $\iota_1 + \iota_0$. Pick a basis $y_1, \dots y_\ell$ of $\mathfrak{g}$. To construct ι_1 we express the left invariant vector field ${}^{l}y_i$ as $\sum_{j=1}^{\ell} f_{ij}\, {}^{R}y_j$ and set

$$\iota_1(y_i t^{-1}) := \sum_{j=1}^{\ell} (f_{ij})_{(-1)}\, {}^{R}y_{j,-1}, \quad i=1,\dots \ell, \qquad (7)$$

cf. (4).

To define ι_0 on $\mathfrak{g}t^{-1} \subset \hat{\mathfrak{g}}_{-k-k_{\mathfrak{g}}}$ we need a natural map $\mathfrak{g} \longrightarrow \mathbb{C}[JG]$. Let T be the canonical derivation of $\mathbb{C}[JG]$. Consider the composition of T with the natural embedding $\mathbb{C}[G] \hookrightarrow \mathbb{C}[JG]$. This is a derivation $\mathbb{C}[G] \to \mathbb{C}[JG]$ and hence it factors through a $\mathbb{C}[G]$-linear map $\Omega^1(G) \longrightarrow \mathbb{C}[JG]$. For example, for $G = \mathbb{G}_m$ this map sends $\frac{dx}{x}$ to $x_0^{-1} x_{-1}$.

We can trivialize $\mathrm{Vect}(G) \xrightarrow{\sim} G \times \mathfrak{g}$ & $\Omega^1(G) \xrightarrow{\sim} G \times \mathfrak{g}^*$ using left-invariant vector fields. Next, we can view $\kappa' := -\kappa - \kappa_{\mathfrak{g}}$ as a map $\mathfrak{g} \to \mathfrak{g}^*$. We can extend this map to $\mathfrak{g} \longrightarrow \Omega^1(G)$ (to the constant differential forms). For example, for $G = \mathbb{G}_m$, this map sends y to $-\kappa(y,y)\frac{dx}{x}$. For ι_0 we take the composition

$$\mathfrak{g}t^{-1} \xrightarrow{t} \mathfrak{g} \longrightarrow \Omega^1(G) \longrightarrow \mathbb{C}[JG]$$

In particular, for $G = \mathbb{G}_m$ we recover the construction from the example.

The following result whose proof we omit was obtained by Arkhipov-Gaitsgory in their work quoted in the end of Section 1.1.

Theorem: The map $\iota = \iota_0 + \iota_1 : \mathfrak{g}t^{-1} \to CDO_\kappa(G)$ gives a vertex algebra embedding $V_{-\kappa-\kappa_{\mathfrak{g}}}(\mathfrak{g}) \hookrightarrow CDO_\kappa(G)$ whose image commutes w. that of $V_\kappa(\mathfrak{g})$.