Chiral differential operators 2

1) Homomorphism $V_\kappa(\mathfrak{g}) \longrightarrow CDO(N_+) \otimes V_{\kappa(\mathfrak{m})}(\mathfrak{m})$

1.1) Homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}(N_+) \otimes \mathcal{U}(\mathfrak{m})$

Let G be a connected simple algebraic group, $P^+ \subset G$ be a parabolic subgroup with Levi decomposition $P^+ = M \ltimes N^+$.

Let $N^- \subset G$ be the opposite unipotent group & $P^- = M \ltimes N^-$ so that $G^\circ = N^+ P^- = P^+ N^-$ is an open affine subset of G. Our goal here is to recover the homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}(N_+) \otimes \mathcal{U}(\mathfrak{m})$ from Kenta's lecture in a somewhat different way.

Note that the action of G on G from the left gives rise to $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}(G)$. Then we have the restriction homomorphism $\mathcal{D}(G) \longrightarrow \mathcal{D}(G^\circ) = \mathcal{D}(N^+) \otimes \mathcal{D}(P^-)$. The image of the composition $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}(G^\circ)$ lies in the subalgebra of P^--invariants $(\mathcal{D}(N^+) \otimes \mathcal{D}(P^-))^{P^-} = \mathcal{D}(N_+) \otimes \mathcal{U}(\mathfrak{p}^-)$. Note that $\mathfrak{p}^- \twoheadrightarrow \mathfrak{p}^-/\mathfrak{n}^- \simeq \mathfrak{m}$, so we get the projection $\mathcal{U}(\mathfrak{p}^-) \longrightarrow \mathcal{U}(\mathfrak{m})$. Therefore we get a homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}(N^+)\otimes \mathcal{U}(\mathfrak{m})$

Exercise: Check that it coincides with the homomorphism constructed by Kenta.

1.2) Recap of CDO's.

Now let G be a connected algebraic group$/\mathbb{C}$, $\kappa \in S^2(\mathfrak{g}^*)^G$. We write $\kappa_{\mathfrak{g}} \in S^2(\mathfrak{g}^*)^G$ for the Killing form on $\mathfrak{g}$.

In part 1 of this notes we have defined a vertex algebra $CDO_\kappa(G)$ as $\mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_\kappa} \mathbb{C}[JG]$ with respect the action of $J\mathfrak{g} = \mathfrak{g}[[t]]$ on $\mathbb{C}[JG]$ from the left (by right invariant vector fields). We have vertex algebra embeddings

$$\mathbb{C}[JG] \hookrightarrow CDO_\kappa(G),\quad V_\kappa(\mathfrak{g}) \hookrightarrow CDO_\kappa(G)$$

Recall that $V_\kappa(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_\kappa} \mathrm{triv}$ and the 2nd embedding above is induced by $\mathrm{triv} \hookrightarrow \mathbb{C}[JG]$ (the constant functions). Note that the right action of JG on itself gives rise to an action of JG on $CDO_\kappa(G)$ by $\hat{\mathfrak{g}}_\kappa$-linear automorphisms. The subspace of invariants is exactly $\mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_\kappa} \mathbb{C}[JG]^{JG} = V_\kappa(\mathfrak{g})$.

In Section 1.4 of part 1 we have also produced a map $\iota: \mathfrak{g}t^{-1} \to CDO_k(G)$ that we claimed has the following two properties:

- It gives rise to a vertex algebra homomorphism $V_{-k-k_{\mathfrak{g}}}(\mathfrak{g}) \longrightarrow CDO_k(G)$
- The image commutes w. that of $V_k(\mathfrak{g})$.

Note that this homomorphism makes $CDO_k(G)$ into a $\hat{\mathfrak{g}}_{-k-k_{\mathfrak{g}}}$-module. Informally, the following claim holds b/c both left & right invariant vector fields form bases in $Vect(G)$.

$$\mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}1}^{\hat{\mathfrak{g}}_{-k-k_{\mathfrak{g}}}} \mathbb{C}[JG] \xrightarrow{\sim} CDO_k(G) \qquad (1).$$

Note that we can construct a natural vertex algebra structure on the left hand side (compare to Sec 1.1 in part 1). (1) becomes a vertex algebra isomorphism.

1.3) Decomposition of $CDO_k(G^{\circ})$

Now we want to emulate the construction from Sec 1.1 in the affine setting. The notation is as in Sec 1.1.

Recall (Sec 1.2 in part 1) that we have a localization $CDO_\kappa(G^\circ)$ of $CDO_\kappa(G)$. Note that $\mathbb{C}[JG^\circ] \subset CDO_\kappa(G^\circ)$ decomposes as $\mathbb{C}[JN^+] \otimes \mathbb{C}[JP^-]$.

Our goal now is to establish an analog of the decomposition $D(G^\circ) = D(N^+) \otimes D(P^-)$. First, we need analogs of subalgebras $D(N^+), D(P^-)$ in $CDO_\kappa(G^\circ)$. Consider two vertex subalgebras in $CDO_\kappa(G^\circ)$.

- The subalgebra generated by $\mathbb{C}[JN^+]$ & xt^{-1} for $x \in \mathfrak{n}^+$. As a subspace, it coincides with $\mathrm{Ind}_{\mathfrak{n}((t))}^{\mathfrak{n}^+((t))} \mathbb{C}[JN^+]$ and as a vertex algebra it is $CDO(N^+)$.

- The subalgebra generated by $\mathbb{C}[JP^-]$ & $l(yt^{-1})$ for $y \in \mathfrak{p}^-$. As a subspace it coincides w. $\mathrm{Ind}_{\mathfrak{p}^-[[t]] \oplus \mathbb{C}\mathbb{1}}^{\hat{\mathfrak{p}}^-_{-\kappa-\kappa_\mathfrak{g}}} \mathbb{C}[JP^-]$ (for the right $J\mathfrak{p}^-$-action of $\mathbb{C}[JP^-]$). So, as a vertex algebra, it is $CDO_{\kappa'}(P^-)$, where $\kappa' \in S^2(\mathfrak{p}^{-*})^{P^-}$ is such that

$$-\kappa' - \kappa_{\mathfrak{p}^-} = (-\kappa - \kappa_\mathfrak{g})|_{\mathfrak{p}^-} \Leftrightarrow \kappa' = \kappa|_{\mathfrak{p}^-} + \kappa_\mathfrak{g}|_{\mathfrak{p}^-} - \kappa_{\mathfrak{p}^-}$$

Note that κ' is lifted from $\mathfrak{m}$ via $\mathfrak{p}^- \to \mathfrak{m}$ &

$$\kappa'(x,y) = \kappa(x,y) + tr_{\mathfrak{n}^+}(ad(x)ad(y)).$$

equivalently κ' is lifted from $(\kappa - \kappa_c(\mathfrak{g}))|_\mathfrak{m} + \kappa_c(\mathfrak{m})$. We conclude that the subalgebra in question is $CDO_{k'}(P^-)$.

Exercise: $CDO(N^+)$ & $CDO_{k'}(P^-)$ commute.

So we get a vertex algebra homomorphism

$$CDO(N^+) \otimes CDO_{k'}(P^-) \longrightarrow CDO_k(G^\circ)$$

Premium exercise: This homomorphism is an isomorphism. Hints: it is surjective b/c the generators, $\mathbb{C}[JG^\circ]$ & $\mathfrak{g}t_i^{-1}$ lie in the image (note that left-invariant vector fields can be expressed via right invariant ones and vice versa). To show it's injective check that $CDO(N)$ has no nontrivial vertex algebra ideals, & more generally, every vertex algebra ideal in $CDO(N^+) \otimes ?$ is the product of $CDO(N^+)$ & a vertex algebra ideal in ?

Remark: We'll need equivariance properties of

$$CDO(N^+) \otimes CDO_{k'}(P^-) \xrightarrow{\sim} CDO_k(G^\circ) \tag{2}$$

First note that JP^- acts from the right & the isomorphism is equivariant by the construction. Also $\mathbb{J}P^+$ acts from the left: $\mathbb{J}M$ acts diagonally, while $\mathbb{J}N^+$ acts on the first factor only. (2) is $\mathbb{J}P^+$-equivariant as well.

1.4) Parabolic free field realization map

Using (2) & its equivariance properties we are ready to construct a "parabolic free field realization map"

$$V_k(\mathfrak{g}) \longrightarrow CDO(N^+) \otimes V_{k'}(\mathfrak{m})$$

Namely, consider the inclusion $V_k(\mathfrak{g}) \hookrightarrow CDO_k(G)$ (from the left) and compose with the inclusion $CDO_k(G) \hookrightarrow CDO_k(G^\circ)$. The image is contained in the $\mathbb{J}P^-$-invariants. Note that the action of $\mathbb{J}P^-$ on $CDO(N^+)$ is trivial. And the invariants in $CDO_{k'}(P^-)$ is $V_k(\mathfrak{p}^-)$ (see Sec 1.2). So we get an inclusion $V_k(\mathfrak{g}) \hookrightarrow CDO(N^+) \otimes V_{k'}(\mathfrak{p}^-)$. Now note that we have a vertex algebra epimorphism $V_{k'}(\mathfrak{p}^-) \twoheadrightarrow V_{k'}(\mathfrak{m})$ induced by $\mathfrak{p}^- \twoheadrightarrow \mathfrak{m}$. So we get a vertex algebra homomorphism

$$V_k(\mathfrak{g}) \hookrightarrow CDO(N^+) \otimes V_{k'}(\mathfrak{p}^-) \twoheadrightarrow CDO(N^+) \otimes V_{k'}(\mathfrak{m})$$

Exercise: $V_\kappa(\mathfrak{g}) \longrightarrow CDO(N^+)\otimes V_{\kappa'}(\mathfrak{m})$ is JP^+-equivariant.

1.5) Homomorphism $\mathfrak{z}(V_\kappa(\mathfrak{g})) \longrightarrow \mathfrak{z}(V_{\kappa(\mathfrak{m})}(\mathfrak{m}))$

Note that $\mathfrak{z}(V_\kappa(\mathfrak{g})) = V_\kappa(\mathfrak{g})^{JG}$. Consider the inclusion

$$V_\kappa(\mathfrak{g})^{JG} \hookrightarrow V_\kappa(\mathfrak{g})^{JP^+} \qquad (3)$$

Thx to Exercise in Sec 1.4, the parabolic free field realization map restricts to

$$V_\kappa(\mathfrak{g})^{JP^+} \longrightarrow \left(CDO(N^+)\otimes V_{\kappa'}(\mathfrak{m})\right)^{JP^+} \qquad (4)$$

We claim that the target of (4) is $\mathfrak{z}(V_{\kappa'}(\mathfrak{m}))$. First, consider the invariants of JN^+. It's $V(\mathfrak{n}^+)\otimes V_{\kappa'}(\mathfrak{m})$. The target of (4) is the JM-invariants in the latter vertex algebra. Note that $Z(M)^\circ \subset M \subset JM$ acts trivially on the 2nd factor, while the invariants in the $V(\mathfrak{n}^+)$ is $\mathbb{C}$ for weight reasons. So the target of (4) is $V_{\kappa'}(\mathfrak{m})^{JM} = \mathfrak{z}(V_{\kappa'}(\mathfrak{m}))$. Composing (3) & (4) we get a required map $\mathfrak{z}(V_\kappa(\mathfrak{g})) \longrightarrow \mathfrak{z}(V_{\kappa'}(\mathfrak{m}))$.

1.6) Formulas

Now we discuss how to write formulas for

$$V_\kappa(\mathfrak{g}) \xrightarrow{ffr_{P^+}} CDO(N^+) \otimes V_{\kappa'}(\mathfrak{m})$$

In short, we can write some kind of formulas for the images of xt^{-1}, yt^{-1} where $x \in \mathfrak{n}^+$ and $y \in \mathfrak{m}$.

By the construction, $ffr_{P^+}(xt^{-1}) = xt^{-1} \otimes 1$, where in the r.h.s. we abuse the notation and write xt^{-1} for the image of this element in $CDO(N^+)$ under the natural map $V(\mathfrak{n}^+) \longrightarrow CDO(N^+)$. We then can express the elements $xt^{-1} \in CDO(N^+)$ via the constant vector fields similarly to Sec 1.3 of part 1. In the case when $P^+ = B^+$ we get formules as in Sec 4 of Zeyu's talk.

Now let's sketch how to compute the images of yt^{-1} w. $y \in \mathfrak{m}$. In the finite setting, we have $y \in \mathfrak{m} \mapsto y_{N^+} \otimes 1 + 1 \otimes y$, where $y_{N^+} \in Vect(N_+)$ is the image of y under the map corresponding to the adjoint action of M on N^+. Similarly, the image of yt^{-1} is $(yt^{-1})_{N^+} \otimes 1 + 1 \otimes yt^{-1}$, where $(yt^{-1})_{N^+}$ is obtained from y_{N^+} by replacing all coordinate functions a^*_α w. $a^*_{\alpha,0}$ and all constant vector fields a_α w. $a_{\alpha,-1}$ (cf. Zeyu's Section 4).

Premium exercise: Prove the claim in the previous sentence (hint: this is a computation in $V_\kappa(\mathfrak{p}^+) = \mathrm{Ind}_{\mathfrak{p}[[t]] \oplus \mathbb{C}1}^{\hat{\mathfrak{p}}_\kappa} \mathbb{C}[JP^+]$).

1.7) Transitivity

Let B^+ be a Borel in P^+ s.t. $B_M = B^+ \cap M$ is a Borel in M. Choose $H \subset B_M$. Let $\tilde{N}^+ = R_u(B^+)$, $\tilde{N}^- = R_u(B^-)$ (where B^- is the opposite Borel containing H), $N_M^\pm = \tilde{N}^\pm \cap M$ so that $\tilde{N}^+ \xleftarrow{\sim} N^+ \times N_M^+$ & $\tilde{N}^- \xleftarrow{\sim} N_M^- \times N^-$ (via the multiplication maps)

We have the following vertex algebra homomorphisms:

$$ffr_{B^+}: V_k(\mathfrak{g}) \longrightarrow CDO(\tilde{N}^+) \otimes V_{k-k_c}(\mathfrak{h})$$

$$ffr_{P^+}: V_k(\mathfrak{g}) \longrightarrow CDO(N^+) \otimes V_{k'}(\mathfrak{m})$$

$$ffr_{B_M^+}: V_{k'}(\mathfrak{m}) \longrightarrow CDO(N_M^+) \otimes V_{k-k_c}(\mathfrak{h})$$

Also note that we can identify $CDO(\tilde{N}^+)$ w. $CDO(N^+) \otimes CDO(N_M^+)$, thx to $\tilde{N}^+ \xleftarrow{\sim} N^+ \times N_M^+$, cf. Sec 1.3

The following claim is what we mean by the transitivity, cf. the end of Sec 1.1 in Kenta's talk.

Proposition: The following diagram is commutative

$$\begin{tikzcd}
V_k(\mathfrak{g}) \arrow[r, "ffr_{P^+}"'] \arrow[d, "ffr_{B^+}"'] & CDO(N^+) \otimes V_{k'}(\mathfrak{m}) \arrow[d, "id \otimes ffr_{B_M^+}"] \\
CDO(\tilde{N}^+) \otimes V_{k-k_c}(\mathfrak{h}) & CDO(N^+) \otimes CDO(N_M^+) \otimes V_{k-k_c}(\mathfrak{h}) \arrow[l, "\sim"']
\end{tikzcd}$$

Sketch of proof:

Consider the inclusions

$$\tilde{N}^+B^- = N^+(N_M^+ H N_M^-)N^- \hookrightarrow N^+MN^- = N^+P^- \hookrightarrow G$$

They give rise to localization homomorphisms of vertex algebras $CDO_k(G) \hookrightarrow CDO_k(N^+P^-) \hookrightarrow CDO_k(\tilde{N}^+B^-)$ that, in turn give rise to inclusions

$$V_k(\mathfrak{g}) \hookrightarrow CDO_k(N^+P^-)^{\mathfrak{J}P^-} = \overbrace{CDO(N^+)\otimes V_{k'}(\mathfrak{p}^-)}^{(*)}$$

$$\hookrightarrow CDO_k(\tilde{N}^+B^-)^{\mathfrak{J}B^-} = CDO(\tilde{N}^+)\otimes V_{k-k_c}(\mathfrak{b}^-) = \tag{5}$$

$$\underbrace{CDO(N^+)\otimes\left(CDO(N_M^+)\otimes V_{k-k_c}(\mathfrak{b}^-)\right)}_{(**)}$$

The homomorphism from (*) to (**) is the tensor product of the identity on $CDO(\tilde{N}^+)$ & the homomorphism

$$CDO_{k'}(P^-)^{\mathfrak{J}P^-} = V_{k'}(\mathfrak{p}^-) \hookrightarrow CDO(N_M^+)\otimes V_{k-k_c}(\mathfrak{b}^-) \tag{6}$$
$$= CDO_{k'}(N^+\times N_M^+\times B^-)^{\mathfrak{J}B^-}$$

Using the epimorphisms $V_{k'}(\mathfrak{p}^-) \twoheadrightarrow V_{k'}(\mathfrak{m})$ & $V_{k-k_c}(\mathfrak{b}^-) \twoheadrightarrow V_{k-k_c}(\mathfrak{b}_M^-)$ from (6) we get:

$$CDO_{k'}(M)^{\mathfrak{J}M} = V_{k'}(\mathfrak{m}) \to CDO(N_M^+)\otimes V_{k-k_c}(\mathfrak{b}_M^-) = \tag{7}$$
$$CDO_k(N_M^+B_M^-)^{\mathfrak{J}B_M^-}$$

Composing (7) w. $id_{CDO(N_M^+)} \otimes [V_{k-k_c}(\mathfrak{b}_M^-) \twoheadrightarrow V_{k-k_c}(\mathfrak{h})]$ we recover $ffr_{B_M^+}$. So we get the following commutative diagram

$$\begin{tikzcd}
V_k(\mathfrak{g}) \arrow[r] \arrow[dr, "ffr_{p^+}"'] & CDO(N^+)\otimes V_{k'}(\mathfrak{b}^-) \arrow[d, "/\mathfrak{n}^-"] \arrow[dr, "(5)"] & \\
& CDO(N_+)\otimes V_{k'}(\mathfrak{m}) \arrow[dr, "id\otimes ffr_{B_M^+}"'] & CDO(N^+)\otimes CDO(N_M^+)\otimes V_{k-k_c}(\mathfrak{b}^-) \arrow[d, "/\tilde{\mathfrak{n}}^-"] \\
& & CDO(N^+)\otimes CDO(N_M^+)\otimes V_{k-k_c}(\mathfrak{h}) \arrow[d, "\wr"'] \\
& & CDO(\tilde{N}^+)\otimes V_{k-k_c}(\mathfrak{h})
\end{tikzcd}$$

To finish the proof it remains to note that the composition

$$\longrightarrow \searrow \downarrow \downarrow$$

is nothing else but ffr_{B^+} $\square$