§7.3 & 8.1: Screening operators, cont'd & describing the center of the affine vertex algebra

Review of the big picture thus far:

We have worked hard to give a "free field realization" homomorphism of vertex algebras

$$\omega_{\kappa_c} : V_{\kappa_c}(\mathfrak{g}) \longrightarrow M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$$

where

- $M_{\mathfrak{g}} = \mathbb{C}[a_{\alpha,n}, a^*_{\alpha,m}]_{n<0,\, m\leq 0}$ is the Fock representation of the Weyl algebra A.
- $V_0(\mathfrak{h})$ = commutative vertex algebra associated to $L\mathfrak{h}$

This is the Vertex algebra version of the affine analogue of the map

$$\tilde{\rho} : U(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathfrak{h}^*] \underset{\mathbb{C}}{\otimes} D(N_+).$$

Recall in the fin. dim. case, this $\tilde{\rho}$ can be used to describe the center, by showing $\tilde{\rho}(Z(\mathfrak{g}))$ lands in the first factor, and also in its W-invariants. (See Exercise 2.11(1) in Daishi's notes.)

Now back to the affine case. Frenkel sets up in §7.1 the following plan, which Daniil reviewed for us:

✓ Done by Daniil

STEP 1: Show ω_{κ_c} is injective.

✓ STEP 2: Show $\mathfrak{z}(\hat{\mathfrak{g}}) \subset V_{\kappa_c}(\mathfrak{g})$ maps to $V_0(\mathfrak{h}) \subset M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$

(related to the operator $\tilde{S}_R$ constructed in Daniil's talk)

Our focus will be Steps 3 & 4

STEP 3: We'll construct the screening operators $\overline{S}_i$, $i=1,\dots,\ell$ from $W_{0,\kappa_c} = M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$ to some other modules, which commute with the action of $\hat{\mathfrak{g}}_{\kappa_c}$.

STEP 4 We'll show $\omega_{\kappa_c}(V_{\kappa_c}(\mathfrak{g}))$ is contained in

$$\bigcap_{i=1}^{\ell} \ker(\overline{S}_i)_{\mathfrak{g}} \Longrightarrow \boxed{\omega_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}})) \subset \bigcap_{i=1}^{\ell} \ker(\underset{\overline{S}_i|_{\pi_0}}{\overline{V}_i[1]})} \quad (1)$$

We will also work toward Step 5, to be completed in Vasya's talk.

STEP 5 Show above inclusion (1) is equality.

STEP 6 use Miura opers to identify RHS of (1) with $\mathrm{Fun}\,\mathrm{Op}_{^L G}(D)$.

Important ingredients from previous talks:

Daniil constructed a "screening operator of the first kind"

$$S_R : W_{0,\kappa} \longrightarrow W_{-2,\kappa}$$

along with other "screening operators of the second kind" $\tilde{S}_R$.
We will extend this sl_2 statement to arbitrary $\mathfrak{g}$ with Steps 3 & 4 above as our goal.

Recall also that Kenta's talk gives the following:
For $\mathfrak{p} \subset \mathfrak{g}$ parabolic with Levi $\mathfrak{m}$, there is an exact functor between smooth modules for $\hat{\mathfrak{m}}$ and $\hat{\mathfrak{g}}$ sending

$$W_{\lambda, \kappa|_{\mathfrak{m}} + \kappa_c(\mathfrak{m})} \longmapsto W_{\lambda, \kappa + \kappa_c(\mathfrak{g})}.$$

(it comes from the parabolic free field realization, as in Thm 1.25 of Kenta's notes)

This will allow us to produce homomorphisms on the RHS from those on the LHS. $(*)$

7.3.1 (Goal here: use screening operators for $\widehat{sl}_2$ to build ones for $\hat{\mathfrak{g}}_\kappa$!)

For $i \in \{1, \dots, \ell\}$, let

$$sl_2^{(i)} = \langle e_i, h_i, f_i \rangle \subset \mathfrak{g}$$

$$p^{(i)} = \langle b_-, e_i \rangle \subset \mathfrak{g}$$

$$m^{(i)} = sl_2^{(i)} \oplus \mathfrak{h}_i^\perp = \text{Levi subalg of } p^{(i)}.$$

orthogonal complement of h_i in $\mathfrak{h}$

Recall semi-infinite parabolic induction:

$$\mathrm{Wak}^{\mathfrak{g}}_{p^{(i)}} : \left\{ \begin{array}{l} \text{smooth reps of } \widehat{sl}_{2,k} \oplus \hat{\mathfrak{h}}^\perp_{i,\kappa_0} \\ \text{w/ } k \text{ and } \kappa_0 \text{ satisfying} \\ \text{some conditions} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{smooth} \\ \hat{\mathfrak{g}}_{\kappa+\kappa_c} \text{ modules.} \end{array} \right\}$$

special case of (*)

The conditions on k & κ_0: $(\kappa - \kappa_c)(h_i, h_i) = 2(k+2)$

$$\kappa_0 = \kappa|_{\mathfrak{h}_i^\perp}.$$

For R a smooth $\widehat{sl}_2$-module of level k,
L a smooth $\hat{\mathfrak{h}}^\perp_\kappa$-module,

$M_{\mathfrak{g}, p^{(i)}} \otimes R \otimes L$ is a smooth $\hat{\mathfrak{g}}_{\kappa+\kappa_c}$-module.

Letting R be the Wakimoto module $W_{\lambda, k}$ over sl_2,
L the Fock rep. $\pi^\kappa_{\lambda_0}$,
the corresponding $\hat{\mathfrak{g}}_\kappa$-module is isom. to $\boxed{W_{(\lambda,\lambda_0),\kappa+\kappa_c}}$

weight of $\mathfrak{g}$ built from λ & λ_0.

So we have:

Prop. Any intertwining operator $a: W_{\lambda_1, k} \longrightarrow W_{\lambda_2, k}$ over $\widehat{sl}_2$ gives an intertwining operator

$$\mathrm{Wak}^{\mathfrak{g}}_{p^{(i)}}(a) : W_{(\lambda_1,\lambda_0),\kappa+\kappa_c} \longrightarrow W_{(\lambda_2,\lambda_0),\kappa+\kappa_c} \text{ over } \hat{\mathfrak{g}}_{\kappa+\kappa_c}$$

for any weight λ_0 of $\mathfrak{h}_i^\perp$.

We will also need the following formula.

Recall from Sec 1.3 of Ivan's 1st CDO Note, the morphism

$$\iota : V(\mathfrak{n}_+) \to CDO(N_+) \quad \left(\begin{array}{c}\text{Corresp. to the right action} \\ \text{of } N_+ \text{ on itself}\end{array}\right)$$

For all i, let $e^R_{i,-1}$ be the image of $e_{i,-1}$ under this map.

Def Let $e^R_i(z) = Y(e^R_{i,-1}, z)$, a field for the vertex alg. $CDO(N_+)$.
(Alternatively, in Frenkel: $e^R_i(z) = w^R(e_i(z))$, where
$w^R : L\mathfrak{n}_+ \to \mathcal{A}^{\mathfrak{g}}_{\leq 1, loc}$ induced by right action of $\mathfrak{n}_+$ on N_+).

Exercise: Show that for a general coordinate system on N_+, we have

$$e^R_i(z) = a_{\alpha_i}(z) + \sum_{\beta \in \Delta_+} P^{R,i}_\beta (a^*_\alpha(z)) a_\beta(z)$$

(c.f. Thm 6.2.1 in Zeyu's notes!)

7.3.2 If $\mathfrak{h}$ is any abelian Lie alg. with nondegenerate inner prod. κ, we can identify $\mathfrak{h} \simeq \mathfrak{h}^*$ via κ. Then let $\hat{\mathfrak{h}}_\kappa$ be the Heisenberg Lie alg, π^κ_λ $\lambda \in \mathfrak{h}^* \simeq \mathfrak{h}$ its Fock reps. Then for any $\chi \in \mathfrak{h}^*$, we can define $V^\kappa_\chi(z) : \pi^\kappa_0 \to \pi^\kappa_\chi$ by

$$V^\kappa_\chi(z) = T_\chi \exp\left(-\sum_{n<0} \frac{\chi_n}{n} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{\chi_n}{n} z^{-n}\right).$$

Of course, we will use this definition for $\mathfrak{h}$ the Cartan as in previous sections.

Def For $\kappa \neq \kappa_c$, let

$$S_{i,\kappa} = e^R_i(z) V^{\kappa-\kappa_c}_{-\alpha_i}(z) : W_{0,\kappa} \to W_{-\alpha_i,\kappa}$$

$(= Y_{W_{0,\kappa}, W_{-\alpha_i,\kappa}}(e^R_{i,-1}|-\alpha_i\rangle, z)$ in the notation of Daniil's talk).

And let
$$S_{i,\kappa} = \int S_{i,\kappa}(z)\,dz : W_{0,\kappa} \longrightarrow W_{-\alpha_i,\kappa}.$$
"the i^{th} screening operator of the first kind"

By Proposition 2.4 in Daniil's talk, $S_{i,\kappa}$ is induced by the screening operator S_R for the ith $\widehat{sl}_2$ subalgebra, with R satisfying $(\kappa-\kappa_c)(h_i,h_i) = 2(R+2)$. It also implies:

Proposition $S_{i,\kappa}$ is an intertwining operator between $W_{0,\kappa}$ and $W_{-\alpha_i,\kappa}$ for each $i=1,\dots,l$.

We won't use this next result, but it's useful for intuition as a step toward our main result later:

Proposition For generic κ, $V_\kappa(\mathfrak{g})$ is equal to the intersection of the kernels of $S_{i,\kappa}$ $i=1,\dots,l$.

7.3.3 We now approach defining screening operators of the second kind for $\hat{\mathfrak{g}}$. To do so, we'll need to make sense of $(e^R_i(z))^\gamma$ for $\gamma\in\mathbb{C}$.

First, fixing i, we can choose coordinates in N_+ s.t. $e^R_i(z) = a_{\alpha_i}(z)$ (we can get this naturally if we define Wakimoto modules over $\hat{\mathfrak{g}}$ via semiinfinite parabolic induction from the ith $\widehat{sl}_2$). Concretely, we choose coords $\{y_\alpha\}_{\alpha\in\Delta_+}$ on N_+ such that $\rho^R(e_i) = \partial/\partial y_{\alpha_i}$ where $\rho^R : n_+ \to D_{\le 1}(N_+)$ corresp. to $N_+ \circlearrowleft n_+$ (right action).

Now, recall the Friedan-Martinec-Shenker bosonization of the Weyl algebra generated by $a_{\alpha_i,n}, a^*_{\alpha_i,n}$ $n\in\mathbb{Z}$.

Def Let

$$\widetilde{W}^{(i)}_{0,0,\kappa} = \mathrm{Wak}^{\mathfrak{g}}_{p^{(i)}}(\widetilde{W}_{0,\gamma,k})$$

which is a $\hat{\mathfrak{g}}_\kappa$-module containing $W_{\lambda,\kappa}$ if $\gamma = 0$, via homomorphism $V_\kappa(\mathfrak{g}) \xrightarrow{w_{\kappa_c}} M_{\mathfrak{g}} \otimes \Pi_0 \longrightarrow M^{(i)}_{\mathfrak{g}} \otimes \widetilde{W}_{0,0,k}$.

More generally, by replacing the Fock representation Π_0 with Π_γ $\gamma \in \mathbb{C}$ and π_0^κ with π_λ^κ, we get a modified $\hat{\mathfrak{g}}$-action giving rise to a module $\widetilde{W}^{(i)}_{\lambda,\gamma,\kappa}$ for all λ, γ.

Def Let $\beta = \frac{1}{2}(\kappa - \kappa_c)(h_i, h_i)$ and define

$$\widetilde{S}_{i,\kappa}(z) = (e_i^R(z))^{-\beta} V_{\check{\alpha}_i}(z) : \widetilde{W}_{0,0,\kappa} \longrightarrow \widetilde{W}_{-\beta,\beta\check{\alpha}_i,\kappa}.$$

(well-defined as a map $\Pi_0 \to \Pi_{-\beta}$ by def of F-M-S bosonization)

$\check{\alpha}_i = h_i \in \mathfrak{h}$ ith coroot of $\mathfrak{g}$

Let $\widetilde{S}_{i,\kappa} = \int \widetilde{S}_{i,\kappa}(z)\, dz$.

As for $S_{i,\kappa}$, $\widetilde{S}_{i,\kappa}$ is induced by $\widetilde{S}_k$ for the ith $\widehat{\mathfrak{sl}}_2$, where $k = (\kappa - \kappa_c)(h_i, h_i) = 2(k+2)$

Prop. The operator $\widetilde{S}_{i,\kappa}$ is an intertwining operator of $\hat{\mathfrak{g}}_\kappa$-modules

$$\widetilde{W}^{(i)}_{0,0,k} \longrightarrow \widetilde{W}^{(i)}_{-\beta,\beta\check{\alpha}_i,\kappa}$$

Prop. For generic κ, $V_\kappa(\mathfrak{g})$ is the intersection of the kernels of $\widetilde{S}_{i,\kappa} : W_{0,\kappa} \longrightarrow \widetilde{W}^{(i)}_{-\beta, \beta\check{\alpha}_i, \kappa}$ $i = 1, \dots, l$.

Again, we won't use or prove this.

Exercise. Show that this intersection is a vertex subalgebra of $W_{0,\kappa}$.

7.3.4 We now want to define the limit of $\widetilde{S}_{i,\kappa}$ as $\kappa \longrightarrow \kappa_c$. We will start in the case $\mathfrak{g} = \mathfrak{sl}_2$, defining $\lim_{\kappa \to -2} \widetilde{S}_\kappa$.

We'll turn $\kappa+2$ into an indeterminate variable β and make $W_{0,\kappa}$ and $\widetilde{W}_{-(\kappa+2), 2(\kappa+2), \kappa}$ free modules over $\mathbb{C}[\beta]$, then quotient by (β).

Def. Let $\pi_0[\beta]$ (resp. $\pi_{2\beta}[\beta]$) be the free $\mathbb{C}[\beta]$-modules spanned by monomials in b_n $n<0$ applied to $|0\rangle$ (resp. $|2\beta\rangle$).

$\pi_0[\beta]$ is a vertex algebra (by Zeyu's talk) and $\pi_{2\beta}[\beta]$ a module over $\pi_0[\beta]$ (as in Daniil's talk).

We have $[b_n, b_m] = 2\beta n \delta_{n,-m}$. The quotient of $\pi_0[\beta]$ and $\pi_{2\beta}[\beta]$ by $(\beta - \kappa)$ $(\kappa \in \mathbb{C})$ are the π_0^κ and $\pi_{2\kappa}^\kappa$ introduced in Zeyu's talk.

let $W_0[\beta] = M \otimes_{\mathbb{C}} \pi_0[\beta]$, $\widetilde{W}_{0,0}[\beta] = \Pi_0 \otimes_{\mathbb{C}} \pi_0[\beta]$,
(vertex algebras & free $\mathbb{C}[\beta]$-modules, w/ quotients $W_{0,\kappa}$ and $\widetilde{W}_{0,0,\kappa}$ after $/(\beta-\kappa)$ $\kappa \in \mathbb{C}$.

Now let $\Pi_{-\beta+n,-\beta+n}$ be the free $\mathbb{C}[\beta]$-module spanned by

p_n, q_n, $n<0$ applied to $|-\beta+n,-\beta+n\rangle$,

with $\Pi_{-\beta} = \bigoplus_{n\in\mathbb{Z}} \Pi_{-\beta+n,-\beta+n}$, $\widetilde{W}_{-\beta,2\beta} = \Pi_{-\beta} \otimes_{\mathbb{C}[\beta]} \pi_{2\beta}[\beta]$.

Recall

$$V_{2\beta}(z) : \pi_0[\beta] \to \pi_{2\beta}[\beta] \otimes_{\mathbb{C}} \mathbb{C}[\beta]$$

$$\tilde{a}(z)^{-\beta} : \Pi_0 \otimes_{\mathbb{C}} \mathbb{C}[\beta] \longrightarrow \Pi_{-\beta},$$

(This depends polynomially on β by the formula (**) below, which follows from the expansion of $\tilde{a}(z)^{-\beta} = e^{-\beta(u+v)}$ as in 7.2.3)

whose Fourier coeffs are well-def'd linear operators

$$\widetilde{W}_0[\beta] \longrightarrow \widetilde{W}_{-\beta,2\beta}.$$

Write $V_{2\beta}(z) = \sum_{n\in\mathbb{Z}} V_{2\beta}[n] z^{-n}$, and define $\overline{V}[n]$ $(n\le 0)$ by

$$\sum_{n\le 0} \overline{V}[n] z^{-n} = \exp\left(\sum_{m>0} \frac{b_{-m}}{m} z^m\right).$$

From Formula (7.2-11) in Frenkel (see eq'n (6) in Daniil's talk), we get

$$V_{2\beta}[n] = \begin{cases} \overline{V}[n] + \beta(\ldots) & n\le 0 \\ \beta\overline{V}[n] + \beta^2(\ldots) & n>0, \end{cases}$$

$$\overline{V}[n] = -2 \sum_{m \leq 0} \overline{V}[m] \frac{\partial}{\partial b_{m-n}}, \quad n > 0.$$

Similarly, one can show

$$\tilde{a}(z)^{-\beta}_{[n]} = \begin{cases} 1 + \beta(\cdots) & n = 0 \\ \beta \frac{p_n + q_n}{n} + \beta^2(\cdots), & n \neq 0. \end{cases} \qquad (**)$$

These imply

$$\int \tilde{a}(z)^{-\beta} V_{2\beta}(z)\, dz = \beta \left(\overline{V}[1] + \sum_{n>0} \frac{1}{n} \overline{V}[-n+1](p_n + q_n) \right) + \beta^2(\cdots).$$

So we define the limit of $\tilde{S}_{\beta-2}$ at $\beta = 0$ $(k = -2)$

as ...

Def $\quad \overline{S} = \overline{V}[1] + \sum_{n>0} \frac{1}{n} \overline{V}[-n+1](p_n + q_n),$

a map from $\underset{\underset{M_{sl_2} \otimes V_0(\mathfrak{h})}{\shortparallel}}{W_{0,-2}} \longrightarrow \underset{\underset{\Pi_0 \otimes V_0(\mathfrak{h})}{\shortparallel}}{\tilde{W}_{0,0,-2}}$ intertwining the $\hat{sl}_2$ action.

More generally, drawing from the $\hat{sl}_2$ case, we define

Def For any i, let

$$\overline{S}_i = \overline{V}_i[1] + \sum_{n>0} \frac{1}{n} \overline{V}_i[-n+1](p_{i,n} + q_{i,n}).$$

In this definition, $\overline{V}_i[n] : V_0(\mathfrak{h}) \to V_0(\mathfrak{h})$ are given by

$$\sum_{n\le 0}' V_i[n] z^{-n} = \exp\left(\sum_{m>0}' \frac{b_{i,-m}}{m} z^m \right)$$

$$\overline{V}_i[1] = -\sum_{m\le 0}' \overline{V}_i[m] D_{b_{i,m-1}}, \qquad \text{(Formula A)}$$

where $\quad D_{b_{i,m}} \cdot b_{j,n} = a_{ji} \delta_{n,m}$. $\quad \left((a_{k\ell}) = \text{Cartan matrix for } \mathfrak{g} \right)$

(derivative in the direction of $b_{i,m}$.)

Prop. The image of $V_{\kappa_c}(\mathfrak{g})$ under w_{κ_c} is contained in the intersection of the kernels of the operators

$$\overline{S}_i : W_{0,\kappa_c} \longrightarrow \widetilde{W}^{(i)}_{0,0,\kappa_c} \qquad i = 1, \dots, \ell.$$

Pf. The $\overline{S}_i$ commute w/ $\hat{\mathfrak{g}}_{\kappa_c}$ by construction, and they each annihilate the highest-weight vector of W_{0,κ_c}. So they annihilate all of $V_{\kappa_c}(\mathfrak{g})$!

Prop. The center $\mathfrak{z}(\hat{\mathfrak{g}})$ of $V_{\kappa_c}(\mathfrak{g})$ is contained in the intersection of the kernels of $\overline{V}_i[1]$ $\quad i = 1, \dots, \ell$ (in $V_0(\mathfrak{h})$).

Pf We saw in Lemma 1.2 of Daniil's notes that $w_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}}))$ lies in $\pi_0 \subset W_{0,\kappa_c}$, and so this reduces to the fact that $\overline{S}_i|_{V_0(\mathfrak{h})} = \overline{V}_i[1] : V_0(\mathfrak{h}) \to V_0(\mathfrak{h})$.

So, Step 4 from our outline is now complete!

END of CH. 7.

CHAPTER 8 Now our focus will be on completing STEP 5 from the intro, i.e. showing the inclusion in the preceding proposition is equality.

8.1.1: Computing the character of $\mathfrak{z}(\hat{\mathfrak{g}})$.

Recall that $V_{\kappa_c}(\mathfrak{g})$ inherits a "PBW filtration", which then gives a filtration on $\mathfrak{z}(\hat{\mathfrak{g}})$. We can then consider its associated graded $\mathrm{gr}\,\mathfrak{z}(\hat{\mathfrak{g}})$.

(i.e., coming from the actual PBW filt. on $\mathcal{U}(\hat{\mathfrak{g}}_{\kappa_c})$).

Recall:

Prop. 3.10 from Hamilton's Talk. There is an injective map of graded $\mathbb{C}$-algebras

$\mathbb{C}[\mathfrak{g}_0^*]^{G(0)}$ in Hamilton's notes.

$$\mathrm{gr}\,\mathfrak{z}(\hat{\mathfrak{g}}) \hookrightarrow \mathbb{C}[J\mathfrak{g}^*]^{JG} \qquad (***)$$

where $JG \curvearrowright J\mathfrak{g}^*$ is induced by the adjoint action.

In this part, our main result will be that $(***)$ is an isomorphism. An important ingredient in the proof will be:

Thm 2.3 from Kenta's talk. The Wakimoto module W_{0,κ_c}^+ is isomorphic to the Verma module $\mathbb{M}_{0,\kappa_c}$.

Recall that $\mathbb{C}[J\mathfrak{g}^*]^{JG}$ is identified in Thm 1.3.1 of Ivan K.'s notes, with $\mathbb{C}[J(\mathfrak{h}/W)]$, i.e. it's freely generated by the polynomials $\overline{P}_{i,n}$ (affine version of Harish-Chandra isom.).

$(i=1,\dots,\ell,\ n<0)$

Now observe that $L_0 = -t\partial_t$ acts on $\mathbb{C}[J\mathfrak{g}^*]$. This defines a $\mathbb{Z}$-grading on $\mathbb{C}[\mathfrak{g}_0^*]$ such that $\deg \overline{J}_n^a = -n$.

Then

$\deg \overline{P}_{i,n} = d_i - n$. So we get

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$\hookrightarrow$ $\mathrm{ch}\, \mathbb{C}[J\mathfrak{g}^*]^{JG} = \prod_{i=1}^{\ell} \prod_{n_i \geq d_i+1} (1-q^{n_i})^{-1}$.

Now let $\widetilde{\mathfrak{b}}_+ = (\mathfrak{b}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]]) \subset \mathfrak{g}[[t]]$, an Iwahori subalgebra.

The natural surjection $\mathbb{M}_{0,\kappa_c} \twoheadrightarrow V_{\kappa_c}(\mathfrak{g})$ gives rise to:

$$\phi : (\mathbb{M}_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} \longrightarrow V_{\kappa_c}(\mathfrak{g})^{\widetilde{\mathfrak{b}}_+}.$$

The PBW filt. on $\mathcal{U}_{\kappa_c}(\hat{\mathfrak{g}})$ equips each of $\mathbb{M}_{0,\kappa_c}$ and $V_{\kappa_c}(\mathfrak{g})$ with natural filtrations such that the epimorphism $\mathbb{M}_{0,\kappa_c} \twoheadrightarrow V_{\kappa_c}(\mathfrak{g})$ is filtration-preserving.

$$\phi_{cl} : (\mathrm{gr}\, \mathbb{M}_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} \longrightarrow (\mathrm{gr}\, V_{\kappa_c}(\mathfrak{g}))^{\widetilde{\mathfrak{b}}_+}.$$

They are also preserved by the $\widetilde{\mathfrak{b}}_+$-action, and the filtration on the target is also preserved under the $\mathfrak{g}[[t]]$-action.

Since $V_{\kappa_c}(\mathfrak{g})$ is a direct sum of fin. dim. reps of $\mathfrak{g} \otimes 1 \subset \mathfrak{g}[[t]]$, any $\widetilde{\mathfrak{b}}_+$-invariant in $V_{\kappa_c}(\mathfrak{g})$ or $\mathrm{gr}\, V_{\kappa_c}(\mathfrak{g})$ is automatically $\mathfrak{g}[[t]]$-invariant.

So, $V_{\kappa_c}(\mathfrak{g})^{\widetilde{\mathfrak{b}}_+} = V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}$

$(\mathrm{gr}\, V_{\kappa_c}(\mathfrak{g}))^{\widetilde{\mathfrak{b}}_+} = (\mathrm{gr}\, V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]} = \mathbb{C}[\overline{P}_{i,m}]_{i=1,\dots,m,\, n<0}$.

We want a description of the source of ϕ_{cl} similar to the description we had for $\mathbb{C}[\mathfrak{g}_0^*]^{G(\mathcal{O})}$.

We have $\mathrm{gr}\, \mathbb{M}_{0,\kappa_c} = \mathrm{Sym}\, \mathfrak{g}((t))/\widetilde{\mathfrak{b}}_+ \simeq \mathbb{C}[\mathfrak{g}^*[[t]]_{(-1)}]$

where

$$\mathfrak{g}^*[[t]]_{(-1)} = ((\mathfrak{n}_-)^* \otimes t^{-1}) \oplus \mathfrak{g}^*[[t]] \simeq (\mathfrak{g}((t))/\widetilde{\mathfrak{b}}_+)^*.$$

So ϕ_{cl} can be identified with the map $\mathbb{C}[\mathfrak{g}^*[[t]]_{(-1)}]^{\widetilde{\mathfrak{b}}_+} \to \mathbb{C}[\mathfrak{g}^*[[t]]]^{\widetilde{\mathfrak{b}}_+}$ induced by $\mathfrak{g}^*[[t]] \hookrightarrow \mathfrak{g}^*[[t]]_{(-1)}$.

Suppose we chose the basis $\{J^a\}$ of $\mathfrak{g}$ as a union of a basis for $\mathfrak{b}_+$ and one for $\mathfrak{n}_-$. If we let $\bar{J}^a_n$ be the polynomial on $\mathfrak{g}^*[[t]]$ defined by

$$\bar{J}^a_n(A(t)) = \mathrm{Res}_{t=0}\langle A(t), J^a\rangle t^n\, dt$$

Then $\mathbb{C}[\mathfrak{g}^*[[t]]_{(-1)}]$ is generated as an alg by $\bar{J}^a_n$ $n<0$ and $\bar{J}^a_0$ for $J^a\in\mathfrak{n}_-$.

Now consider

$$\bar{P}_i(\bar{J}^a(z)) = \sum_{m\in\mathbb{Z}} \bar{P}_{i,m} z^{-m-1}$$

for $\bar{J}^a(z) = \sum_n \bar{J}^a_n z^{-n-1}$ (summing over $n<0$ if $J^a\in\mathfrak{b}_+$, over $n\le 0$ if $J^a\in\mathfrak{n}_-$),

which is a construction of $\tilde{\mathfrak{b}}_+$-inv't functions on $\mathfrak{g}^*[[t]]_{(-1)}$.

similar to the one used in §3.4 of Hamilton's talk.

Since $\bar{J}^a(z)$ has nonzero z^{-1} coeff. if $J^a\in\mathfrak{n}_-$, for $J^a\in\mathfrak{n}_-$.

the coeffs $\bar{P}_{i,m}$ are zero unless $m<d_i$. (since $\bar{J}^a(z) = \bar{J}^a_{-1}z^{-1} + \sum_{m\ge 0}\bar{J}^a_{-m-1}z^m$ and each $\bar{P}_i$ has degree d_i.)

This gives a natural homomorphism

$$\mathbb{C}[P_{i,m_i}]_{i=1,\dots,\ell;\ m_i<d_i} \longrightarrow \mathbb{C}[\mathfrak{g}^*[[t]]_{(-1)}]^{\tilde{\mathfrak{b}}_+}$$

Lemma This homomorphism is an isomorphism.

Pf Let $\mathfrak{g}^*[[t]]_{(0)} = ((\mathfrak{n}_-)^*\otimes 1)\times(\mathfrak{g}^*\otimes t\mathbb{C}[[t]])$

$$= t\,\mathfrak{g}^*[[t]]_{(-1)}$$

Multiplication by t gives rise to a $\tilde{\mathfrak{b}}_+$-equivariant isomorphism $\mathfrak{g}^*[[t]]_{(-1)} \xrightarrow{\sim} \mathfrak{g}^*[[t]]_{(0)}$, hence an isomorphism

$$\mathbb{C}[\mathfrak{g}^*[[t]]_{(0)}]^{\tilde{\mathfrak{b}}_+} \longrightarrow \mathbb{C}[\mathfrak{g}^*[[t]]_{(-1)}]^{\tilde{\mathfrak{b}}_+}.$$

Let $\mathfrak{g}^*[[t]]^{reg}_{(0)}$ be the intersection of $\mathfrak{g}^*[[t]]_{(0)}$ with

$$J\mathfrak{g}^*_{reg} = \mathfrak{g}^*_{reg} \times (\mathfrak{g}^* \otimes t\mathbb{C}[[t]]).$$

(Recall $x \in \mathfrak{g}^*$ is in $\mathfrak{g}^*_{reg}$ iff $\dim Z_{\mathfrak{g}}(x) = \operatorname{rk} \mathfrak{g}$).

So $\mathfrak{g}^*[[t]]^{reg}_{(0)} = ((\mathfrak{n}_-)^{*,reg} \otimes 1) \oplus (\mathfrak{g}^* \otimes t\mathbb{C}[[t]])$

open & dense in $(\mathfrak{n}_-)^*$, so $\hat{\mathfrak{n}}_+^{reg}$ is open & dense in $\hat{\mathfrak{n}}_+$.

Recall $Jp : J\mathfrak{g}^*_{reg} \longrightarrow \operatorname{Spec} \mathbb{C}[\bar{P}_{i,n}]_{i=1,\dots,\ell,\, n<0}$ (as in Ivan K's talk) the jet homomorphism corresponding to $p : \mathfrak{g}^*_{reg} \twoheadrightarrow \mathfrak{g}/\!/G \simeq \operatorname{Spec}[\bar{P}_i]_{i=1,\dots,\ell}$.

The group $G[[t]]$ acts transitively along the fibers of Jp.

One can also show that $\tilde{B}_+$, the subgroup of $G[[t]]$ corresponding to $\tilde{\mathfrak{b}}_+ \subset \mathfrak{g}[[t]]$, acts transitively on the fibers of $Jp|_{\mathfrak{g}^*[[t]]^{reg}_{(0)}}$:

The group $JG = G[[t]]$ acts transitively on the fibers of Jp. For any $x \in \mathfrak{g}^*[[t]]^{reg}_{(0)}$, $\tilde{B}_+$ is the subgroup of all elements $g \in G[[t]]$ such that $g \cdot x \in \mathfrak{g}^*[[t]]^{reg}_{(0)}$, so it acts transitively as stated.

This implies that the ring of $\tilde{B}_+$-inv't polynomials on $\mathfrak{g}^*[[t]]^{reg}_{(0)}$ is functions on the image $Jp(\mathfrak{g}^*[[t]]^{reg}_{(0)})$.

This image is the subspace determined by

$\bar{P}_{i,m} = 0 \quad i = 1, \dots, \ell, \quad m = -1,$ since $m=-1$ corresponds to the "degree -1 in the variable t" part, which is excluded from $\mathfrak{g}^*[[t]]_{(0)} = ((\mathfrak{n}_-^*) \otimes 1) \oplus (\mathfrak{g}^* \otimes t\mathbb{C}[[t]])$

So the ring of $\tilde{\mathfrak{b}}_+$-inv't polynomials on $\mathfrak{g}^*[[t]]^{reg}_{(0)}$ is =

$\mathbb{C}[\overline{P}_{i,m_i}]_{i=1,\dots,\ell,\ m_i<-1}$.

By density of $\mathfrak{g}^*[[t]]^{reg}_{(0)}$ in $\mathfrak{g}^*[[t]]_{(0)}$, we can erase "reg" and this statement is still true.

To pass back from $\mathfrak{g}^*[[t]]_{(0)}$ to $\mathfrak{g}^*[[t]]_{(-1)}$, we shift $\overline{J}^a_n \mapsto \overline{J}^a_{n+1}$, so get $\overline{P}_{i,m_i} \mapsto \overline{P}_{i,m_i+d_i+1}$. $\square$

Corollary. The map ϕ_{cl} is surjective.

Pf The map ϕ_{cl} corresponds to taking the quotient of $\mathbb{C}[P_{i,m_i}]$, $i=1,\dots,\ell$, $m_i<d_i$ by $(P_{i,m_i})_{i=1,\dots,\ell,\ 0\le m_i<d_i}$.

Theorem "The center $\mathfrak{z}(\mathfrak{g})$ is as 'large as possible'."

$$\operatorname{gr} \mathfrak{z}(\hat{\mathfrak{g}}) = \mathbb{C}[J\mathfrak{g}^*]^{JG},$$ so there exist central elements $S_i \in \mathfrak{z}(\hat{\mathfrak{g}}) \subset V_{\kappa_c}(\mathfrak{g})$ whose symbols are $\overline{P}_{i,-1}$ $i=1,\dots,\ell$ so that

$$\mathfrak{z}(\hat{\mathfrak{g}}) = \mathbb{C}[S_{i,(n)}]_{i=1,\dots,\ell;\ n<0}|0\rangle$$

where the $S_{i,(n)}$ are Fourier coeffs of $Y(S_i,z)$.

Pf We have $\deg \overline{P}_{i,m} = d_i - m$, so the Lemma gives

$$\operatorname{ch}(\operatorname{gr} \mathbb{M}_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+} = \prod_{m>0}(1-q^m)^{-\ell} \qquad (\star)$$

Now recall by Thm 2.3 of Kenta's talk that $\mathbb{M}_{0,\kappa_c} \simeq \mathbb{W}^+_{0,\kappa_c}$.

In his notes, we have $(\mathbb{W}^+_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+} = \Pi_0$, whose character is also as in $(\star)$.

To see this, note that $\mathbb{M}^{\tilde{\mathfrak{b}}_+}_{0,\kappa_c} \simeq \operatorname{End}(\mathbb{M}_{0,\kappa_c})$, and the following.

Exercise Π_0, thought of as a commutative algebra, acts faithfully by endomorphisms of $\mathbb{W}^+_{0,\kappa_c}$ when acting on the right.

This gives the bound $\mathrm{ch}(\pi_0) \leq \mathrm{ch}(W_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+}$. Now since we have the natural embedding $\mathrm{gr}(M_{0,\kappa_c}^{\tilde{\mathfrak{b}}_+}) \hookrightarrow (\mathrm{gr}\, M_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+}$, this gives the opposite bound and proves equality of the characters, so $(W^+_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+} \quad \pi_0$.

So the natural embedding $\mathrm{gr}(M_{0,\kappa_c}^{\tilde{\mathfrak{b}}_+}) \hookrightarrow (\mathrm{gr}\, M_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+}$ is an isomorphism.

In the diagram

$$\begin{tikzcd}
\mathrm{gr}(M_{0,\kappa_c}^{\tilde{\mathfrak{b}}_+}) \arrow[r] \arrow[d] & \mathrm{gr}(V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}) \arrow[d] \\
(\mathrm{gr}\, M_{0,\kappa_c})^{\tilde{\mathfrak{b}}_+} \arrow[r] & \mathrm{gr}(V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}
\end{tikzcd}$$

- the left arrow is an iso. by what we just said
- the bottom arrow is surj. by the Corollary.

$\Rightarrow$ the right vertical arrow must be surj. (but we already know it's injective).

$\Rightarrow$ it's an isomorphism.

$\Rightarrow$ the character of $\mathrm{gr}\, \mathfrak{z}(\mathfrak{g})$ is equal to that of $\mathrm{gr}(V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]})$, so

$$\mathrm{ch}\, \mathfrak{z}(\mathfrak{g}) = \mathrm{ch}\, \mathrm{gr}\, \mathfrak{z}(\mathfrak{g}) = \prod_{i=1}^{\ell} \prod_{n_i \geq d_i+1} (1-q^{n_i})^{-1}. \qquad \square.$$

This is a nontrivial result which tells us a lot about the center, but again, we want to understand the geometric meaning of the center, and in particular the action of $\mathrm{Aut}\, \mathcal{O}$ on $\mathfrak{z}(\hat{\mathfrak{g}})$. To do so, we need to complete Steps 5 & 6 of the plan at the start.

8.1.2: The center & the classical W-algebra.

Recall we showed $\mathfrak{z}(\hat{\mathfrak{g}})$ is contained in the intersection of the kernels of $\overline{V}_i[1]$ $i=1,\dots,\ell$ on Π_0. Our next goal is to compute the character of this intersection, to use later for a proof of equality.

Let $\hat{h}_\nu$ be a copy of the Heisenberg Lie algebra with generators $b_{i,n}$ $i=1,\dots,\ell$; $n\in\mathbb{Z}$, for ν an inv't inner product on $\mathfrak{g}$.

Recall the vertex operator $V^\nu_{-\alpha_i}(z):\Pi^\nu_0\to\Pi^\nu_{-\alpha_i}$ defined by

$$V^\nu_\chi = T_\chi \exp\Big(-\sum_{n<0}\frac{\chi_n}{n}z^{-n}\Big)\exp\Big(-\sum_{n>0}\frac{\chi_n}{n}z^{-n}\Big)$$

and let $V^\nu_{-\alpha_i}[1]=\int V^\nu_{-\alpha_i}(z)\,dz$. We call it a W-algebra screening operator. Since $V^\nu_{-\alpha_i}(z)=Y_{\Pi^\nu_0,\Pi^\nu_{-\alpha_i}}(|-\alpha_i\rangle,z)$, the intersection of the kernels of $V^\nu_{-\alpha_i}[1]$ $i=1,\dots,\ell$ is a vertex subalgebra of Π^ν_0, which we know by the commutation relation

$$\Big[\int Y_{V,M}(B,z)\,dz,\,Y(A,w)\Big]=Y_{V,M}\Big(\int Y_{V,M}(B,z)\,dz\cdot A,\,w\Big).$$

We take this intersection as the definition of the affine W-algebra $W_\nu(\mathfrak{g})$, although other definitions are also possible, (c.f. Frenkel & Ben-Zvi, 2004). This is a deformation of the algebra of functions on $\mathrm{Op}_G(D)$.

We now want to define the limit of $W_\nu(\mathfrak{g})$ as $\nu\to\infty$. To do so, we fix an inv't inner prod. ν_0 on $\mathfrak{g}$ and let $\varepsilon=\nu/\nu_0$. Then:

$$\alpha_i=\varepsilon\frac{2}{\nu_0(h_i,h_i)}h_i \qquad (\text{identifying } \mathfrak{h}^*\simeq\mathfrak{h} \text{ via } \nu).$$

Let $b'_{i,n}=\varepsilon\dfrac{2}{\nu_0(h_i,h_i)}b_{i,n}$. Consider the $\mathbb{C}[\varepsilon]$-lattice in $\pi_0^\vee \underset{\mathbb{C}}{\otimes} \mathbb{C}[\varepsilon]$ spanned by all monomials in $b'_{i,n}$ and its specialization at $\varepsilon = 0$. (The latter is a commutative vertex alg).

In the limit $\varepsilon \to 0$, we get the expansion

$$V^\nu_{-\alpha_i}[1] = \varepsilon \frac{2}{\nu_0(h_i, h_i)} V_i[1] + \underbrace{\cdots}_{\text{higher order in } \varepsilon \text{ terms}}$$

and the action of $V_i[1]$ on $\pi_0^\vee$ is given by

(Formula B) $$V_i[1] = \sum_{m \le 0} V_i[m] D_{b'_{i,m-1}}$$ where $D_{b'_{i,m}} \cdot b'_{j,n} = a_{ij} \delta_{n,m}$,

(a_{ij}) the Cartan of $\mathfrak{g}$ and

$$\sum_{n \le 0} V_i[n] z^{-n} = \exp\left(-\sum_{m>0} \frac{b'_{i,-m}}{m} z^m\right)$$

Def. Let $W(\mathfrak{g})$ (the classical W-algebra associated to $\mathfrak{g}$) be the commutative vertex subalgebra of $\pi_0^\vee$ which is the intersection of the kernels of the operators $V_i[1]$ $i = 1, \ldots, \ell$.

(It's independent of changing ν_0, since this only rescales $V_i[1]$).

8.1.3 The appearance of ${}^L\mathfrak{g}$.

Recall:

$$\overline{V}_i[1] = -\sum_{m \le 0} \overline{V}_i[m] D_{b_{i,m-1}}, \quad \text{(Formula A)}$$

By comparing FORMULA A and FORMULA B, we see that if we substitute $b_{i,n} \mapsto -b'_{i,n}$, the operators $\overline{V}_i[1]$ <u>almost</u> become $V_i[1]$, except $a_{ji} \leftrightarrow a_{ij}$ are flipped.

This is because $\overline{V}_i[1]$ was associated to a coroot, and $V_i[1]$ to a root.

Swapping roots & coroots, i.e. transposing the Cartan matrix, corresponds to swapping $\mathfrak{g}$ with ${}^L\mathfrak{g}$, the Langlands dual Lie algebra.

$$\pi_0 \text{ for } \mathfrak{g} \simeq \pi_0^\vee \text{ for } {}^L\mathfrak{g} \text{ by } b_{i,n} \longmapsto -b^\vee_{i,n}, \text{ and}$$

$$\bar{V}_i[1] \text{ for } \mathfrak{g} \longmapsto V_i[1] \text{ for } {}^L\mathfrak{g}.$$

So, by this isomorphism, $\mathfrak{z}(\hat{\mathfrak{g}})$ is actually embedded into the intersection of the kernels of $V_i[1]$ $i=1,\dots,l$ on $\pi_0^\vee$ for ${}^L\mathfrak{g}$, i.e. into the classical W-alg. $W({}^L\mathfrak{g})$.

Lemma The character of $W({}^L\mathfrak{g})$ is equal to that of $\mathfrak{z}(\hat{\mathfrak{g}})$.
(To be proved in Vasya's talk).

$\Rightarrow$ **Theorem** There is an isomorphism
$\mathfrak{z}(\hat{\mathfrak{g}}) \simeq W({}^L\mathfrak{g})$ of graded commutative vertex algebras.

This completes Step 5 of the plan above (once this lemma is proven!)