38. BRAID GROUP ACTIONS & A PBW-TYPE BASIS MOTIVATION: ROOT VECTORS VIA A BRAID GROUP ACTION Recall that in Leonardo's first talk, we some that Ut := Subalgebra of U generated by all Ex, areTT íF then we had THEOREM 4.21 i) Urousout - U, u,ouzou3 Hu, uzu3 is an isomorphism of vector spaces. ii) Ku for MEZE are a basis for U°. The subalgebras Ut and Ut are quantum analogues of U(n+), U(n-) C U(op). We have PBW theorems for U(n+) & U(n-). Explicitly, U(n+) has a basis consisting of ordered monomials in the root vectors ex, are $\underline{\Phi}$. In the quantum setting, we have Ex for a ET, but We don't yet know how to make sense of Ex frace I! THEOREM For all $\alpha \in \overline{\Phi}^+$, there is an element Exe Ut such that Ut has a basis consisting of ordered monomials in these elements.

This statement (even the existence of Ex pourt) is nontrivial, as there is no underlying Lie algebra for Ut. Motivation for how we will construct these Ex again comes from the classical setting. For any BE Q⁺, there is are TT and we'W such that war= B. We write Soc, ... Socr (OS) = B for W= Soc, ... Socr a reduced expression. One can lift each Sai to an automorphism Sai of) of by $S_{\alpha_i}(X) = \exp(\operatorname{ad} e_{\alpha_i})\exp(-\operatorname{ad} f_{\alpha_i})\exp(\operatorname{ad} e_{\alpha_i}).$ By the construction, $\tilde{S}_{\alpha_i}: \mathcal{O}_{\mathcal{S}} \longrightarrow \mathcal{O}_{\mathcal{S}_{\alpha_i} \mathcal{X}}$ for $\mathcal{X} \in \overline{\Phi}$. \tilde{S}_{α_i} also acts on any $\mathcal{O}_{\mathcal{S}}$ -module. The Sa, do not quite form a Weyl group action (we don't always have $\tilde{S}_{x_i}^2 = 1$) but they do form a braid group action: Definition: The braid group associated to W is the group generated by Simple reflections Soci (aciETT) but modulo only the braid relations: if a, a; ETT and Sa; Sa; has order min W, we impose the relation Saci Saci Saci = $5a_i Sa_i Sacj \dots$ M terms M terms. [And we omit the relations $S_{a_i}^2 = 1$ Present in W.]

Any two reduced expressions for WEW are related by a sequence of braid relations. So if we accept that the Saci Satisfy the braid relations, we can for any wEW, define $\tilde{\omega} := \tilde{S}_{\alpha_1} \dots \tilde{S}_{\alpha_r}$ where w= Sa,... Sar is reduced. Then for any $B \in \mathbb{P}^+$, a way to define the root vector $e_B \in \mathcal{U}(n^+)$ is to set $e_{\beta} = \widetilde{\omega}(e_{\alpha}) = \widetilde{S}_{\alpha_{1}} \cdot \cdot \cdot \widetilde{S}_{\alpha_{r}}(e_{\alpha}),$ where B=was, areTTr and W= Sar, ... Sar is reduced as before. With this in mind, we'll toklow this outline in the quantum setting by constructing a braid group action on U and on all finite-dimensional U-modules, analogous to the one defined above for of. We will then Use this to define E_{β} for all $\beta \in \Phi^+$.

DEFINING THE ACTION (Sl2)

Let's start by restricting ourselves to the Case $01 = sl_2$. Write E, F, etc. instead of Ex, Fx, etc. and $q = q_x$. We continue to discuss only f.dim U-modules V of Type 1 So $V = \bigoplus V_m$, where $V_m = \{v \in V \mid Kv = q^m v\}$. Define $E^{(r)} = \frac{E^r}{[r]!}$, $F^{(r)} = \frac{F^r}{[r]!}$ for all $r \ge 0$. Recall the automorphism w of U with $\omega(E) = F, \quad \omega(F) = E, \quad \omega(K) = K^{-1}.$ Now we define four linear operators T, T', "T, "T', such that for all MEZ, VEVM, $T(v) = \sum_{a,b,c\geq 0}^{i} (-1)^{b} q^{b-ac} E^{(a)} F^{(b)} E^{(c)} v$ $a_{b,c}$ $T'(v) = \sum_{a,b,c^{20}}^{i} (-1)^{b} q^{ac-b} E^{(a)} F^{(b)} E^{(c)} V$ a,b,c:0- a+b-c=m $\omega T(v) = \sum_{\substack{a,b,c>0\\a-b+c=m}}^{(} (-1)^{b} q^{b-ac} F(a) E(b) F(c) v$ $WT'(v) = \sum_{a,b} (-1)^{b} q^{b-ac} F^{(a)} E^{(b)} F^{(c)} V$ a,b,c?0 $\alpha - 6 + c = m$

The latter two are obtained from the former two by twisting by ω . We will see see $T' = \omega T^{-1}$, $\omega T' = T^{-1}$. Recall the irreducible V = L(n, +) of highest Weight n & Type 1 for Ug(sl2). Let's determine the action of these operators on V. Let Vo be the highest weight vector of V, and vecall $v_i := F^{(i)} V_0$ form a basis for V with $F^{(r)}V_{i} = \begin{bmatrix} v+i \\ r \end{bmatrix} V_{i+r} \qquad E^{(r)}V_{i} = \begin{bmatrix} n+r-i \\ r \end{bmatrix} V_{i-r}$ $if V_j = 0''$ for j < 0, j > n. Now we'll describe the action of our operators using the following property of Gaussian binomial coefficients. Lemma 8.1 IF a,b, mEZ, b, m = 0, then $\sum_{i=0}^{m} \frac{ai-b(m-i)}{i} \begin{bmatrix} a \\ m-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} = \begin{bmatrix} a+b \\ m \end{bmatrix}$ Proof: (Note, this is a familiar combinatorial identity if q=1.) Exercise with binomial coeffs.

Lemma 8.3 For all i, $T(v_{i}) = (-1)^{n-i} q^{(n-i)(i+1)} V_{n-i}$ $T'(v_{i}) = (-1)^{n-i} q^{-(n-i)(i+1)} V_{n-i}$ $\omega_{T}(v_{i}) = (-1)^{i} q^{i(n+i-i)} V_{n-i}$ $\omega_{T}(v_{i}) = (-1)^{i} q^{i(n+i-i)} V_{n-i}$ $(\omega T^{1}(v_{i}) = (-1)^{i} q^{-i(m+1-2)} V_{n-i}$ Lemma 8.4 The operators T, T', WT, WT, are bijective on each fin-dim Type 1 Ug(slz)module V, and $(1) \quad \begin{array}{c} T^{-i} = \omega T \\ T^{i-i} = \omega T \end{array}$ (2) for all mEZ, VEVm, $wT(v) = (-q)^{-m}T(v)$ and $wT'(v) = (-q)^{m}T'(v)$. Proof. By complete reducibility, it suffices to consider V = L(n, +). Bijectivity is clear by Lemma 8.3. We can check (1) & (2) on the pasis Vi; then these too fillow from Lenna 8.3. Lemma 8.5 For all V as above & VEV, T(Ev) = (-Fk)T(v)ET(v) = T((-k'F)v)FT(v) = T((-EK)v)T(Fv) = (-K'E)T(v) $T(K_v) = K^{-1}T(v)$ $\mathsf{KT}(\mathsf{u}) = \mathsf{T}(\mathsf{k}^{-1}\mathsf{u}) \quad .$

Proof sketch. It's enough to take V=L(n,+), V=V. for some i. Let's do $T(E_v) = (-FK)T(v)$ as an example, with the rest following similarly. $T(\Xi_{v_i}) = T([n+1-i]v_{i-1})$ $= (-1)^{n-i+1} \frac{i(n-i+1)}{[n+1-i]} \frac{V_{n-i+1}}{V_{n-i}},$ while $FT(v_i) = F((-1)^{n-i} \frac{q^{(i+1)}(n-i)}{V_{n-i}} \frac{V_{n-i}}{V_{n-i}})$ $= (-1)^{n-2} q^{(i+1)} (n-i) [n-i+i] V_{n-i+1} g$ \Rightarrow T(Ev_i) = $-q^{2i-n}$ FT(v) Since Vn-i has weight = 2i-n = (-FK) T(v) and so on, for the vest.

DEFINING THE ACTION (IN GENERAL) Peturn now to U= Ug(oj) where of is arbitrary. Extend all our definitions in the natural way: For all T: set $E_{\alpha}^{(r)} = \frac{E_{\alpha}^{r}}{[r]!}$ $F_{\alpha}^{(r)} = \frac{F_{\alpha}^{r}}{[r]!}$ for all $r \ge 0$. Let V be a f.dim $U_q(o_q)$ -mod. Define four operators such that for all $\lambda \in \Lambda$, $v \in V_{\lambda}$, $T_{cs}(v) = \sum_{a,b,c\geq 0}^{l} (-1)^{b} q_{cs}^{b-ac} E_{cs}^{(a)} F_{cs}^{(b)} E_{cs}^{(c)} V$ - a, b, c \ge 0 - a, b, c \ge 0 - a, b, c \ge 0 $T_{ax}^{9}(v) = \sum_{a,b,c^{2}0}^{I} (-1)^{b} q_{ac-b} = (a) + (b) + (c)^{c} + (c)$ $\omega T_{(v)} = \sum_{\alpha,b,c^{20}} (-1)^{b} q_{c\alpha}^{b-ac} F_{c\alpha}^{(a)} F_{(c)}^{(b)} F_{(c)}^{(c)}$ a-b+c=m $\omega T_{a}^{9}(v) = \sum_{a,b,c}^{1} (-1)^{b} q_{b}^{b-ac} F^{(a)}(-1) F^{(b)}(c) = \sum_{a,b,c}^{1} (-1)^{b} q_{a}^{b-ac} F^{(a)}(c) = \sum_{a,b}^{1} (-1)^{b} q_{a}^{b-ac} F^{$ where $m := \langle \lambda, \alpha v \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \quad (\text{Recell } q_{\alpha} = q^{(\alpha, \alpha)} r_2)$ Recall the enbedding Uq. (st2) -> Ug(og) defined by ENEX, FNFx, KNKx. Considering V as a Uqa(slz)-module via the enbedding, me get the T we defined earlier.

This means the results we proved for Tachally lift straight away to give the results for Tx. Here are the properties we get immediately: • Tax and Ta' are bijective, with inverses "Ta' and "Tar. • $WT_{\alpha}(v) = (-q_{\alpha})^{-\langle \lambda, \alpha v \rangle} T_{\alpha}(v)$ · Ta: V, ~ Vsax (and some for WTa, WTa, Ta) We leave this as an exercise. · Lemma 8.5 implies $T_{\alpha}(E_{\alpha}v) = (-F_{\alpha}k_{\alpha})T_{\alpha}(v) = T_{\alpha}((-k_{\alpha}^{-}F_{\alpha})v)$ $T_{\alpha}(F_{\alpha}v) = (-K_{\alpha}^{\prime}E_{\alpha})T_{\alpha}(v) \quad F_{\alpha}T_{\alpha}(v) = T_{\alpha}((-E_{\alpha}K_{\alpha})v).$ • If BETT with (a,B)=0, then EB commutes with Es and Fr => with Tr. So EBTa(V) = Ta(EBV) and $F_{\beta}T_{\alpha}(v) = T_{\alpha}(F_{\beta}v)$ when $(\alpha, \beta) = 0$.

RELATIONS FOR THE To

Going back to the classical case for a brief moment, recall we had automorphisms Sos of fidim σ_{J} -modules and of σ_{J} itself. These satisfy $\widetilde{S}_{\alpha}(X_{\nu}) = \widetilde{S}_{\alpha}(X)\widetilde{S}_{\alpha}(\nu)$. We'd like similar behaviour from our Tax, and for this we'll need a bunch of formulas. PROPOSITION 8.13 For all nell there is a unique u'ell such that $T_{\alpha}(uv) = u'T_{\alpha}(v) \qquad (for all veV for all f.l. Type 1 U-modes V)$ The map U+>U? is an automorphism of U. To prove this, it will be useful to have formulas for EBTa(U), Ta(EBV) and Similarly for EB replaced by FB. PROPOSITION 8.10 "KEY COMPUTATION" let a, BETT, r:=- < B, avy. For all f.dom U-modules V and all VEV, (*) $T_{\alpha}(E_{\beta}v) = (od(E_{\alpha}'')E_{\beta})T_{\alpha}(v)$.

Setting 9=1, with correct signs this reduces to a familiar formula for Sa. Proof sketch This is a big computation, but here are the key steps: • The formula (m20): $ad(E_{\alpha}^{(m)})E_{\beta} = \sum_{i=0}^{m} (-1)^{i} q_{\alpha}^{i(m-1-r)} E_{\alpha}^{(m-i)}E_{\beta}E_{\alpha}^{(i)}$ This comes from the formula for ad(Ex) we E Saw in Leonardo's lecture using divided powers. The "r" comes into play here since KaEpka'= qs, ar p = q & EB. · We then use the above to prove that for integers M, i 20, $(ad(E_{\alpha}^{(m)})E_{\beta})E_{\alpha}^{(i)}$ $(1) = \sum_{j=0}^{i} (-1)^{j} \begin{bmatrix} m+j \\ j \end{bmatrix}_{\alpha} q_{\alpha}^{i(r-2m)-j(i-i)} E^{(i-j)} \left(\alpha d \left(E_{\alpha}^{(m+j)} \right) E_{\beta} \right)$ and $(2) \left(ad \left(E_{\alpha}^{(m)} \right) E_{\beta} \right) F_{\alpha}^{(i)}$ $= \sum_{j=0}^{n} (-1)j \begin{bmatrix} r-m+j \\ j \end{bmatrix} q_{\alpha}^{j(i-1)} F_{\alpha}^{(i-j)} \left(ad \left(E_{\alpha}^{(m-j)} \right) E_{\beta} \right) K_{\alpha}^{-j}.$ This is done by induction on i using the previous formula. Let a(m) = ad (E a) EB.

(1) and (2) to get a formula for · We use a(r)T_a(v) (i.e. RHS of our goal). Indeed, Ta(v) is a linear combination of terms of the form Ex Fa Ex, and so applying (1) then (2) then (1) allows us to pass a(r) pass each of the three factors in such a term. We end up with a linear combination of terms $E_{\alpha}^{(a)}F_{\alpha}^{(b)}E_{\alpha}^{(c)}a(r-h)v$, where if $c=<1, \alpha >$ then h = s ta-b+c. Simplifying the coefficients on such terms, we see that they are zero unless h=r, and we get $\alpha(r) T_{\alpha}(v) = \sum_{\substack{a,b,c \ge 0 \\ -a+b-c=s-v}} (-1)^{b} q_{\alpha}^{b-\alpha c} E_{\alpha}^{(c)} F_{\alpha}^{(b)} E_{\alpha}^{(c)} \alpha(0) v$ Since $a(0) = E_{\beta}$, $a(0)v \in V_{\lambda-\beta}$ with $\langle \lambda - \beta, \sigma^{\nu} \rangle = s - v$, Hhis is $T_{\alpha}(E_{\beta}v)$, as desired. Now note that we can get a similar result for Tac(Fpv) using the following: Lemma 8.7 If u, u' EU s.t. Ta(uv) = u' Ta(v) for all V in a finite-dimensional U-module V, Hen $\omega T_{\alpha}(\omega(u)v) = \omega(u')^{\omega} T_{\alpha}(v)$ for all such v.

IF $U \in U_{\mu}$ for $\mu \in \mathbb{Z}$ (recall $U_{\mu} = \mu$ -graded) then $T_{\alpha}(w(u)v)$ (piece of U). $= (-q_{\alpha})^{-\varsigma_{\mu,\alpha\nu}} \omega(u^{\prime}) T_{\alpha}(\nu) \quad \text{for all such } \nu.$ Proof Exercise (the first claim is easy, and the second tollows from the forst and the formulas relating "Ta(v) and Ta(v) For vEVA). This lemma applied to (*) gives us $T_{\alpha}(F_{\beta}) = \left(\sum_{i=0}^{r} (-i)^{i} q_{\alpha}^{i} F_{\alpha}^{(i)} F_{\beta} F_{\alpha}^{(r-i)}\right) T_{\alpha}(v).$ Now we want to deduce from this a formula For EBTa(v). We can do this by twisting the adjoint action. Recall the articutomorphism T defined by $T(E_{\alpha}) = E_{\alpha}, T(F_{\alpha}) = F_{\alpha}, T(K_{\alpha}) = K_{\alpha}^{-1}.$ For all XEU, define Tad(x): U->U by Tad(x) = ToadoT; T is an outiontomorphism but the twist is still on action. Chaim 8.12 With r as before, $T_{\alpha}\left[\left(T_{\alpha}d\left(E_{\alpha}^{(m)}\right)E_{\beta}\right)v\right] = \left(ad\left(E_{\alpha}^{(r-m)}\right)E_{\beta}\right)T_{\alpha}(v)$

(in particular, setting m=r, we get $E_{\beta}T_{\alpha}(v) = T_{\alpha}\left[\left(T_{\alpha}d\left(E_{\alpha}^{(r)}\right)E_{\beta}\right)v\right]$ Proof sketch: $Tad(E_{\alpha})u = u E_{\alpha} - E_{\alpha} K_{\alpha} u K_{\alpha}^{-1}$ · First, we can easily see $T_{ad}(F_{a})u = V_{a}(uF_{a}-F_{a}u)$ Tad (Ka) u= Kan Ka. · Next, we claim that if Ta(uv) = u'Ta(v) for all V in all Finite-dimensional U-modules V, then $T_{\alpha}\left[\left(T_{\alpha}d\left(E_{\alpha}\right)u\right)V\right] = \left(\alpha d\left(F_{\alpha}\right)u^{2}\right)T_{\alpha}(V)$ and Tar [(Iad (Fa) u)v] = (ad (Ea) u') Tar(v) for v as above. We'll do the first one: $T_{\alpha}[T(\alpha d(E_{\alpha})u)v] = [\alpha(uE_{\alpha}v - E_{\alpha}K_{\alpha}uK_{\alpha}^{-1}v)$ = u' (-Faka) Tas(v) - (-Faka) Karu' KarTar(v) $= (F_{\alpha}u' - u'F_{\alpha}) K_{\alpha} T_{\alpha}(v)$ $= (ad(F_{a})u')T_{a}(v).$ · Combining this observation with (X), we get $T_{\alpha}\left[\left(T_{\alpha d}\left(E_{\alpha}^{(m)}\right)E_{\beta}\right)v\right] = \left[ad\left(F_{\alpha}^{(m)}\right)ad\left(E_{\alpha}^{(r)}\right)E_{\beta}\right]T_{\alpha}(v)$ = $\left[ad(E_{x}^{(v-m)})E_{3}\right]T_{a}(v)$ Now let's move on and use these relations to show the computibility of the action of T on modules me hinkel at earlier.

PROPOSITION 8.13 For all nell there is a unique u'ell such that $T_{\alpha}(uv) = u'T_{\alpha}(v) \qquad (for all veV for all f.d. Type 1 U-modes V)$ The map U+)U' is an automorphism of U. PROOF Suppose u, uzell and we've already found u', u'z. Then we can take (u, uz) = u', u'z and (au, + buz) = au? + buz so it suffices to show existence for generators of U. But this is exactly what we've done: basic observations $\begin{cases} T\alpha(K_{\mu}v) = K_{Sam} T\alpha(v) \\ T\alpha(Eav) = -(F\alpha Ka) T\alpha(v) \\ from earlier (T\alpha(Fav) = (-Ka^{-1}Ea) T\alpha(v)) \end{cases}$ $\begin{array}{l} \text{Key computation(s)} \\ \text{above} \\ \\ \text{Tax}(E_{\beta V}) = (\text{ad}(E_{\alpha}^{(r)})E_{\beta}) T_{\alpha}(v) \quad (\star) \\ \\ \text{Tax}(F_{\beta V}) = (\sum_{i=0}^{r} (-i)^{i} q_{i}^{i} F_{\alpha}^{(i)} F_{\beta} F_{\alpha}^{(r-i)}) T_{\alpha}(v) . \end{array}$ Now for unsquences. If u', u' both satisfy the Condition for a given u, then $(u'-u'')T_{\alpha}(u) = 0 \quad \forall V, v \in V.$ Since T_{α} is bijective, this means w-w" annihilates every fin. dim U-module. So u'-u''= O by Leonardo's lecture.

By definition it's an algebra endomorphism. But note verve already shown surjectivity of this map ut yu?; Since lue have formuleus for Ky Ta(V), EaTa(V) & FaTa(V), EBTa(V) & FBTa(V), this means all the generators appoor in the image of our map. Injectivity: If u is in the kernel, then $T_{\alpha}(uv) = 0$ for all $v \in V$, all V. Since T_{α} is bijective, the result from Leonardo's lecture again says u=0. So $u \mapsto u'$ is an automorphism. By abuse of notation, we also denote this cuitom. $\mathcal{U} \rightarrow \mathcal{U}$ by $T_{\alpha}: \mathcal{U} \rightarrow \mathcal{U}$. So by definition, $T_{\alpha}(\mathcal{U}\mathcal{V}) = T_{\alpha}(\mathcal{U}) T_{\alpha}(\mathcal{V})$ for all $\mathcal{V} \in all \mathcal{V}$ (Type 1) for all $\mathcal{V} \in all \mathcal{V}$ (Fin. dim.) Our formulas from boone tell us: $T_{\alpha}(k_{\mu}) = k_{sa\mu} = T_{\alpha}(k_{\mu}) \forall \mu \in \mathbb{Z}\Phi.$ $T_{\alpha}(E_{\alpha}) = -F_{\alpha}K_{\alpha} \quad T_{\alpha}(E_{\alpha}) = -K_{\alpha}^{-1}F_{\alpha}$ $T_{\alpha}(F_{\alpha}) = -K_{\alpha}^{-1}E_{\alpha} \quad T_{\alpha}^{-1}(F_{\alpha}) = -E_{\alpha}K_{\alpha}$



BRAID RELATIONS

Lusztig showed that the Ta (on modules and on U) Satisfy the braid relations, i.e. for a, BETT with a #B, if SaspEW has order m, then Tate = TBtan m where both sides have in factors. This generalizes the behaviour of the Sa from the classical case. classical case. The proof, in general, requires long calculations. We'll do it for m=3 to give on idea of the proof. The case m=2 is on easy exercise using commutativity of Eas & Eps when $(\alpha, \beta) = 0$. When Sasp has order 3 is when SB, as >= $-1 = \langle \alpha, \beta^{\vee} \rangle$, and $(\alpha, \alpha) = (\beta, \beta)/50$ Ja= JB. Our formulas from earlier give that (writing q= ga for conversionce) in this case,

 $T_{\alpha}(E_{\beta}) = E_{\alpha}E_{\beta} - q^{-1}E_{\beta}E_{\alpha}T_{\beta}^{-1}(E_{\alpha})'$ $T_{\alpha}(E_{\beta}) = E_{\beta} E_{\alpha} - q^{-1} E_{\alpha} E_{\beta} = T_{\beta}(E_{\alpha}) \qquad (3)$ $T_{\alpha}(F_{\beta}) = F_{\beta} F_{\alpha} - q F_{\alpha} F_{\beta} = T_{\beta}(F_{\alpha}) \qquad (3)$ $T_{\alpha}(F_{\beta}) = F_{\alpha} F_{\beta} - q F_{\beta} F_{\alpha} = T_{\beta}(F_{\alpha}) \qquad (3)$ $T_{a}(F_{\beta}) = F_{a}F_{\beta} - qF_{\beta}F_{a} = T_{\beta}(F_{\alpha}).$ let a, BETT S.t. Sasp has emma 8.16 order 3. Then as automorphisms Tatpta=Tptatp of U. Proof It's erough to show the two sides coincide on generators of M, and it's on easy exercise to show on U°, so it remains to look at Ex, Fx, XETT. For X= a or B, we just check both sides! eg. $T_{\alpha}T_{\beta}T_{\alpha}(E_{\beta}) = T_{\alpha}T_{\beta}T_{\beta}(E_{\alpha})$ $=T_{\alpha}(E_{\alpha})$ = - Fac Ka $T_{\beta}T_{\alpha}T_{\beta}(E_{\beta}) = T_{\beta}T_{\alpha}(-F_{\beta}K_{\beta})$ =-Fakspsaß =-Faka, and similarly for FB, (plus symmetry in a and B).

Now assume & Fa, B. Since the Dynkin diagram contains no loops, $\langle X, x' \rangle = 0$ or $\langle X, B' \rangle = 0$. WLOG $\langle X, B' \rangle = 0$, So LEX, EBJ=O=LEX, FBJ Applying TBTato, it's easy to see from (3) that $[T_{\beta}T_{\alpha}(E_{\delta}), E_{\alpha}] = 0 = [T_{\beta}T_{\alpha}(E_{\delta}), F_{\alpha}],$ which implies which imploses $T_{\alpha}(T_{\beta}T_{\alpha}(E_{\delta})) = T_{\beta}T_{\alpha}(E_{\delta}),$ So $T_{\alpha}T_{\beta}T_{\alpha}(E_{\delta}) = T_{\beta}T_{\alpha}(E_{\delta}) = T_{\beta}T_{\alpha}T_{\beta}(E_{\delta}),$ similarly for F_{δ} . Now recall: W is generated by the Soc, at T. Reduced expressions for an element well are related by braid relations, and remain reduced after applying braid relations. So, since the Ta Satisfy braid relations, We can define for wEW: reduced Twi= Tar, Tar for W= Soc, ... Sock and this is well-defined.

If l(ww') = l(w) + l(w'), get Twwi = TwTwi. It is clear that $T_{W}(K_{\mu}) = K_{w\mu}$ for all well, $\mu \in \overline{R} \Phi$. $T_{W}(\mathcal{U}_{\mu}) = \mathcal{U}_{w\mu}$ for all well, $\mu \in \overline{R} \Phi$. And our Note earlier gives $T_w^{-L} = \overline{c} \circ T_{w^{-1}} \circ \overline{C}$. (using $\overline{T}^2 = id$).

ROOT VECTORS & PBW BASIS

Now we can start to approach our goal by using the Tw to construct "root vectors" to ultimately get a PBW-type theorem. From now on, let's restrict to the simply-laced case (i.e. "m" = 2 or 3) for the following, just for simplicity. PROPOSITION 8.20 Let WEW and AETT. IF war>o, fhen Tw(Ea) eUt; if water, then Tw (Ear) = Ewas. Proof sketch me go by induction on R(w). (\$\$) Suppose l(w)>0. We can then pick BETT s.t. W(B)<0. There exists w', w" s.t. w= w'w" with $w'' \in \langle s_{\alpha}, s_{\beta} \rangle \ s.t. \ w'_{\beta} > 0, \ w'_{\alpha} > 0, \ l(w) = l(w') + l(w'') \ \& \ w''_{\alpha} > 0, \ w''_{\beta} < 0.$ So $w'' \neq 1 \Rightarrow l(w') < l(w)$, So by induction, get (e.g) $T_{w'}(E_{\alpha}) \in \mathcal{U}^{\dagger}, T_{w'}(E_{\beta}) \in \mathcal{U}^{\dagger}.$ (4) We deal with Two as follows:

Lemma 8.19 Under above assumptions on W", Twill (Ed) is cothined in the Subalg generated by Ea & EB, & if wett, $T_{\omega''}(E_{\alpha}) = E_{\omega\alpha}$. Pf since we're assuming m=2 or m=3 it's easy: • If m=2, $w=S_{\beta}$, $T_w(E_{\alpha})=E_{\alpha}$. • IF m= 3, W'E { 5 B, Sa 5 B }, 50 Tw (Ear) EZEBEar - Ja EaEB, EBS. II So Two (Ex) eUt, so me're dore since Tw=TwTw"; and so we can apply (4) to conclude. For the Second claim, assume water. If he Show wate IT, we're done by induction applied to w and w"ac. Exercise (about root systems): Show that w" or ETT under the hypotheses above. Щ· Now we've ready to define root vectors. The naïve approach would be: for any $Y \in \mathbb{Q}^+$, pick BETT and WEW s.t. WB = 8, and then define $E_{\chi} = T_{w}(E_{\beta}).$ But this doesn't actually provide a consistent choice:

E.G. If Φ is of Type A₂, $TT = \{\alpha, \beta\}$ $\forall = \alpha + \beta$, then $\forall = S \propto \beta = S_{\beta} \propto \beta$, but $I_{\alpha}(E_{\beta}) = E_{\alpha}E_{\beta} - q^{-1}E_{\beta}E_{\alpha}$ two linearly independent vectors! TB(Ea)=EBEa-q"EaEB " Luckily, there is still a way to make a consistent choice of root vectors, but only by first making a choice for a reduced expression of WOEW, the largest elevert. If we choose $W_0 = S_{\infty_1} \cdots S_{\infty_k}$ a reduced expression, then α_1 , $S_{\alpha_1}\alpha_2$, $S_{\alpha_1}S_{\alpha_2}\alpha_3$, ..., $S_{\alpha_{t-1}}\alpha_t$ is a list of all positive roots. So for XE It let i be s.t. Sau - Sau = V, and define $\chi_{\gamma} := T_{\alpha_1} \cdots T_{\alpha_{i-1}}(E_{\alpha_i}) \cdot \begin{pmatrix} u + he voot \\ ve e + or associated \\ t \gamma \gamma'' \end{pmatrix}$ We would now like to prove the PIBW-type theorem we sought. THEOREM 8.24 In the above setup, (recalling again our assumption that q is not a root of unity), $T\alpha_{t}T\alpha_{2}\cdots T\alpha_{t-1}(E\alpha_{t})\cdots T\alpha_{t}T\alpha_{t}(E\alpha_{t})T\alpha_{t}(E\alpha_{t})E\alpha_{t}$ (ai nonneg. integers) are a basis for Ut.

TO prove it define for any well the subspace Ut[w] < Ut spanned by expressions $T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{r-1}}(E_{\alpha_r})\cdots T_{\alpha_r}(E_{\alpha_1})E_{\alpha_r} (5)$ where Sal Sar is a reduced expression for w. PROPOSITION 8.22 (a) This is well-defined (Uttw) does not depend on the choice of reduced expression). (b) let a 7 B be simple voots. If w is the longest element in < Sa, SB>, then U+[W] Spans the subalg of U ger by Ea & Ep. PROOF SKETCH: First we'll show (b) => (a), by induction on l(w). Suppose llw)>1 & the claim for all w' s.t. l(w') < l(w). let W= Sp,... Spr (= Soc,... Socr). We can assume the two expressions differ by a single braid more. If or,=B,, then take w'= Sa, W, and note that $T_{\alpha}\left(\mathcal{U}^{\dagger}\left[w^{\prime}\right]\right)\cdot\left(\sum_{a\geq0}^{\prime}kE_{\alpha}^{a}\right)\quad agnees with both possible \\ definitions of \mathcal{U}^{\dagger}\left[w^{\prime}\right] \vee.$ If a, 7 B, let w" be the longest elevent in < sa, sp.7. Since the expressions for wave related by one braid move ie. both reduced expressions of w begin with the two reduced expressions for w" and agree offerward; write W=w"w'.

Then by (b), both definitions give Twill (21+Ew']) U+Ew"]. It remains to prove (b). This is casework based on m= the order of Sass. For m=2, it's easy Since EarEB = EBEa. For m=3, we get expressions (6) $T_{\alpha}T_{\beta}(E_{\alpha})T_{\alpha}(E_{\beta})E_{\alpha}^{\circ} = E_{\beta}^{\circ}T_{\alpha}(E_{\beta}^{b})E_{\alpha}^{\circ} \quad (w=s_{\alpha}s_{\beta}s_{\alpha})$ = $E_{\beta}^{\circ}(E_{\alpha}E_{\beta}-q^{-i}E_{\beta}E_{\alpha})^{b}E_{\alpha}^{\circ}$ It's erough to show stability of the span wher left-mult. by Ep & Eco. Ep is obvious, Eco is a computation about which we give a brief idea: Using Ta(EB)Ea=q='EaTa(EB), we can derive a commutator formula by induction: $E_{\alpha}E_{\beta} = q_{\alpha}E_{\beta}E_{\alpha} + [n]_{\alpha}E_{\beta} = q_{\alpha}(E_{\beta})$. Using this, We can then simplify the expression $E_{\alpha}E_{\beta}^{c}T_{\alpha}(E_{\beta})^{b}E_{\alpha}^{c}$ = $q_{\alpha}^{b-c}E_{\beta}^{c}T_{\alpha}(E_{\beta})^{b}E_{\alpha}^{c+i} + [c]_{\alpha}E_{\beta}^{c-i}T_{\alpha}(E_{\beta})^{b+i}E_{\alpha}^{c}$, which is in the span of expressions of the form (6). The appendix in Jantzer does m=4. Now we've ready to prove the main theorem: Recall THEOREM 8.24 In the above setup, (recalling again our assumption that q is not a root of unity), (5) $T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{t-1}}(E_{\alpha_t}^{\alpha_t})\cdots T_{\alpha_1}T_{\alpha_1}(E_{\alpha_3}^{\alpha_3})T_{\alpha_1}(E_{\alpha_1}^{\alpha_2})E_{\alpha_1}^{\alpha_1}$ (ai nonneg. integers) are a basis for Ut.

Proof of Theorem:

One can show that U+[w] is a subalg of "Ut for all w. But for now, all we need is that Ut[wo] Eac Ut[wo] for all aETT. This is true since we can pick a reduced expression for Wo starting with ac, and so we get Ut [wa] is spanned by Tai, Taz ... Take (Ear) ... Tai (Eaz) Eai, (we're using we're using Clearly Stuble under right unlt. by Ear. (of U+Ews) Since the Ear generate Ut and here! 1 e U+ [wo], we get that U+ [wo] = U+! It remains to show expressions above are linearly independent. A steetch of the proof is as follows: First, we note that if $\alpha \in TT$ and $u_i \in \mathcal{U}^+$ with $T_{\alpha}^{-1}(u_i) \in \mathcal{U}^+$, then m (a) $\sum_{i=0}^{m} E_{\alpha}^{i} u_{i} = 0$ or $\sum_{i=0}^{m} u_{i} E_{\alpha}^{i} = 0 \Rightarrow u_{i} = 0$ for all i. Indeed, $T_{\alpha}(E_{\alpha}) = -K_{\alpha}F_{\alpha}$, so $0 = T_{\alpha} \left(\sum_{i=0}^{c} u_i E_{\alpha}^i \right) = \sum_{i=0}^{c} T_{\alpha} \left(u_i \right) \left(-K_{\alpha}^i F_{\alpha} \right)^i, \text{ and}$ Since UtoU=0 ~ U is an iso of vect spaces, & (-K=iFa)i are lin indep in U=0, we must have

 $T_{\infty}(u_{i})=0$ for all i, so $u_{i}=0$. Now ve go by induction on t: for t>0, the products in (5) are of the form $T_{\alpha}(v_i) E_{\alpha}^i$ where $c_{x=c_{\alpha}}$, and v_i are products as in (5) but for Saz... Say = SaW. If $\sum_{i,j} \alpha_{ij} T_{\alpha}(v_j) E_{\alpha}^i = 0$ for scalars α_{ij} , then set $u_i = T_{\alpha}(Z_i a_{ij}v_j)$ and apply (a), so $Z_{\alpha_{ij}v_j} = 0$ for all *i*. By induction, the v_j are line malep, so all $a_{ij} = 0$. It's an analogous argument using what we've discussed, that the products (call the factors) (5-) $T_{\alpha_1}T_{\alpha_2} \dots T_{\alpha_{k-1}}(F_{\alpha_k}^{\alpha_k}) \dots T_{\alpha_n}(F_{\alpha_n}^{\alpha_n})F_{\alpha_n}^{\alpha_n}(Y_{X})$ are a basis for U. (This is not a formal consequence, though, as applying w to our basis does not give us the basis in (57).)

THE BILINEAR FORM

The remainder of this talk will be dedicated to showing that the bases in (5) and (5-) are related via a bilinear form (,). Now, recall the bilinear pairing $(,): \mathcal{U}^{\leq 0} \times \mathcal{U}^{\geq 0} \rightarrow \mathbb{R}$ From Chapter 6. Recall For $x, x^2 \in \mathcal{U}^{\frac{3}{2}}, y, y^2 \in \mathcal{U}^{\frac{1}{2}}, \mu, \nu \in \mathbb{Z} \Phi$, a, BETT: $(y_1 x x') = (\Delta(y), x' \otimes x) (yy'), x) = (y \otimes y', \Delta(x))$ $(K_{\mu}, K_{\nu}) = q^{-(\mu, \nu)} \quad (F_{\alpha}, E_{\beta}) = -S_{\alpha\beta}(q_{\alpha} - q_{\alpha}^{-})^{-1} \\ (K_{\mu}, E_{\alpha}) = 0 \qquad (F_{\alpha}, K_{\mu}) = 0 \\ We are going to use our "PBW basis" to$ Produce explicit dual bases of Up and Up. Recall if W = Sa, ... Sa, is reduced, Sa,... Sa:- ((a;)=8, then $X_{\chi} = T_{\alpha_i} - T_{\alpha_{i-1}}(E_{\alpha_i}), \quad Y_{\chi} = T_{\alpha_i} - T_{\alpha_{i-1}}(F_{\alpha_i}).$ THEOREM 8.29 Let W= Sa, ... Sar be reduced. Then $(T_{\alpha_1}, \dots, T_{\alpha_{t-1}}(F_{\alpha_r}), \dots, T_{\alpha_t}(F_{\alpha_2}), F_{\alpha_t}, g, T_{\alpha_1}, \dots, T_{\alpha_{t-1}}(E_{\alpha_r}), \dots, E_{\alpha_t})$ = 50 if there is an i with at 76; (TTill Fai Eai) otherwise.

We will first need the following lemma: PROPOSITION 8.28 Let $\propto \in TI$, $\chi \in \mathcal{U}^+[s_{\alpha}w_{\sigma}]$, y $\in \mathcal{U}^-[s_{\alpha}w_{\sigma}]$. Then $(T_{\alpha}(y), T_{\alpha}(\chi)) = (y, \chi)$. The proof is very long, and Chapter 8A of Jantzen's book is dedicated to it, so we leave this to those interested. Before we sketch a proof of the Theorem, note that if we fix a reduced expression Wo = Sa,... Sat, then X^{at}_{xt}... X^a, with Z'a; X; = M form a basis for Ut, and since by Chapter 6, $(F_{\alpha_{i}}, E_{\alpha_{i}}^{r}) = (-1)^{r} q_{\alpha_{i}}^{r(r-1)/2} \frac{[r]_{\alpha_{i}}^{l}}{(q_{\alpha_{i}} - q_{\alpha_{i}}^{-1})^{r}},$ So after scaling by appropriate constants, the Yat ... Yai are a dual basis for U-n with respect to our pairing. In particular: Corollory 8.30: The restarction of (1) to U-m×U+ with METHE is nondegenerate. This allows us to extend some previous theorems (where we assumed or proved nondeguency under more restrictive carditions, such as "q transcendental") to all q that are not a root of unity. It remains only to skeetch the proof of THM 8.29.

PROOF SKETCH : We go by induction on l(w). For l(w) = 1, it's clear. For t = l(w) > 1, set $\chi = T_{\alpha_2} \cdots T_{\alpha_{t-1}} (E_{\alpha_t}^{\alpha_t}) \cdots T_{\alpha_2} (E_{\alpha_3}^{\alpha_3}) E_{\alpha_2}^{\alpha_1}$ $y = T_{\alpha_2} \cdots T_{\alpha_{d+1}} (F_{\alpha_d}^{b_d}) \cdots T_{\alpha_2} (F_{\alpha_3}^{b_3}) F_{\alpha_2}^{b_2}$ and $\alpha := \alpha_i$. By def, $x \in U + [sawo], y \in U [sawo].$ So the term in our proposition is $(T_{\alpha}(y)F_{\alpha}^{b'}, T_{\alpha}(x)E_{\alpha}^{a'})$ (7) The next steps are: (i) Show (7) is equal to Say, b, (Fa, Ea) (Ta(y), Ta(x)) (ii) Use invariance to show () = Sand, (Fai, Ea)(y, x) (iii) Appeal to our induction hypothesis, which gives the value of (y, x). Since (ii) comes by Proposition 8.28, it remains only to justify (i). This follows by the following lemma: LEMMA 8.27 Let XEUITESawo], YEUTESawo]. Then $(T_{\alpha}(y)F_{\alpha}, T_{\alpha}(x)E_{\alpha}) = (T_{\alpha}(y), T_{\alpha}(x))(F_{\alpha}, E_{\alpha})$ for all it Z and if $i \neq j$ then $(T_{\alpha}(y)F_{\alpha}, T_{\alpha}(x)E_{\alpha}) = 0$.

Proof sleet d Induction on it jusing results from Chapter 6 covered in Arun's talk, namely $r_{\alpha}(xx') = \chi r_{\alpha}(x') + q^{(\alpha_{1}\mu')} r_{\alpha}(x)\chi'$ (6.14(4)) $\mathbf{r}_{\alpha}^{\prime}(\mathbf{x}\mathbf{x}^{\prime}) = q^{(\alpha, \mu)} \mathbf{x} \mathbf{r}_{\alpha}^{\prime}(\mathbf{x}^{\prime}) + \mathbf{r}_{\alpha}^{\prime}(\mathbf{x}) \mathbf{x}^{\prime}, \quad \mathbf{x} \in \mathcal{U}_{\mu}^{+}, \quad \mathbf{x}^{\prime} \in \mathcal{U}_{\mu}^{+}.$ where $V_{\infty}(x)$, $V_{\infty}'(x)$ were defined in Arun's talk Such that the component of $\Delta(x)$ in $U_{\mu}^{\pm} \vee K_{\nu} \otimes U_{\nu}^{\pm}$ is equal to Va(X) Ka®Eas U=at, and equals Eak-a @Va(X) for v=n-a, along with an inductive proof that $(+)V_{\alpha}(E_{\alpha}^{m}) = V_{\alpha}^{2}(E_{\alpha}^{m}) = \frac{q^{2m-1}}{q^{2}\alpha - 1} E_{\alpha}^{m-1}$ (m=1 was covered by Arnu's talk). How ra and r? Come into play in Jantzen's proof: Lemma Let $\alpha \in TT$, then $T_{\alpha}U^{\dagger}[s_{\alpha}w_{0}] = \{x \in U^{\dagger} | v_{\alpha}(x) = 0\}$ $\frac{\int de_{\alpha}: Write, for u_{i}, v_{j} \in \mathcal{U}^{+}[s_{\alpha}w_{o}]:}{v_{\alpha}(x) = (q_{\alpha}-q_{\alpha})\sum_{i>0}^{i} T_{\alpha}(u_{i}) E_{\alpha}^{i}}$ $r_{\alpha}(x) = (q_{\alpha} - q_{\overline{\alpha}}^{-1}) \sum_{j \ge 0} T_{\alpha}(u_{j}) E_{\alpha}^{j}$ (using (+).

(Second) Idea: Use this lemma to show $(F_{\alpha}U^{\dagger}, U^{\dagger}[s_{\alpha}w_{\sigma}]) = 0 = (U^{\dagger}F_{\alpha}, T_{\alpha}U^{\dagger}[s_{\alpha}w_{\sigma}])$ By using these results we end up with $(T_{\alpha}(y)F_{\alpha}^{i},T_{\alpha}(x)E_{\alpha}^{j})=\frac{q_{\alpha}^{2}-1}{q_{\alpha}^{2}-1}(F_{\alpha},E_{\alpha})(T_{\alpha}(y)F_{\alpha}^{i-1},T_{\alpha}(x)E_{\alpha}^{j-1})$ for j > 0... Then induction here yields the result. This proves Lemma 8.27, completing the proof of Proposition 8.29. Remark In previous lectures, me used nondegeneracy of (,) after assuming stricter Conditions on q (e.g. transcedetal).

We're now proved that these results extend to all q hot a voot of unity. In Chap 7/ Elijah's talk, the R-matrix was described in terms of a dual basis wrt (,). As Pavel commented in Elijch's 1st lecture, the quantum Weyl group action can be used to get a new description for the universal R-matrix: Define $\Theta^{[i]} = \sum_{r=0}^{i} (-1)^{r} q^{-r(r-1)/2} \frac{(q_{a_{i}} - q_{a_{i}})^{r}}{[r]_{a_{i}}^{i}} \frac{(q_{a_{i}} - q_{a_{i}})^{r}}{$ Then the Θ_{μ} used in Elijah's lecture is the $(\mathcal{U}_{-\mu} \otimes \mathcal{U}_{\mu})^{-}$ component of the product $\Theta^{[t]}\Theta^{[t-1]}\dots\Theta^{[t^2]}\Theta^{[t]}(=\Theta).$ See Leverdorskiik Spibelman or Kirillouk Reshetikhin for mone. In Conclusion, we have · Defined a braid action on U · Used it to define "rout vedors" to formulate a PBW-type theorem. · Proved this theorem, giving us nice bases for Ut & U-· Proved these bases are dual with respect to (,).