§8. BRAID GROUP ACTIONS & A PBW-TYPE BASIS

MOTIVATION: ROOT VECTORS VIA A BRAID GROUP ACTION

Recall that in Leonardo's first talk, we saw that
if $\mathcal{U}^+ :=$ subalgebra of $\mathcal{U}$ generated by all E_α, $\alpha \in \Pi$
$\mathcal{U}^- :=$ " " " " " " F_α, $\alpha \in \Pi$,
$\mathcal{U}^0 :=$ " " " " " " K_μ, $\mu \in \mathbb{Z}\Phi$,
then we had

THEOREM 4.21 i) $\mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \longrightarrow \mathcal{U}$, $u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$
is an isomorphism of vector spaces.
ii) K_μ for $\mu \in \mathbb{Z}\Phi$ are a basis for $\mathcal{U}^0$.

The subalgebras $\mathcal{U}^+$ and $\mathcal{U}^-$ are quantum analogues of $\mathcal{U}(\mathfrak{n}^+)$, $\mathcal{U}(\mathfrak{n}^-) \subset \mathcal{U}(\mathfrak{g})$. We have PBW theorems for $\mathcal{U}(\mathfrak{n}^+)$ & $\mathcal{U}(\mathfrak{n}^-)$. Explicitly, $\mathcal{U}(\mathfrak{n}^+)$ has a basis consisting of ordered monomials in the root vectors e_α, $\alpha \in \Phi^+$.
In the quantum setting, we have E_α for $\alpha \in \Pi$, but we don't yet know how to make sense of E_α for $\alpha \in \Phi^+$.

THEOREM For all $\alpha \in \Phi^+$, there is an element $E_\alpha \in \mathcal{U}^+_\alpha$ such that $\mathcal{U}^+$ has a basis consisting of ordered monomials in these elements.

This statement (even the existence of E_α part) is nontrivial, as there is no underlying Lie algebra for $\mathcal{U}^+$.

Motivation for how we will construct these E_α again comes from the classical setting. For any $\beta \in \Phi^+$, there is $\alpha \in \Pi$ and $w \in W$ such that $w\alpha = \beta$. We write $s_{\alpha_1} \cdots s_{\alpha_r}(\alpha) = \beta$ for $w = s_{\alpha_1} \cdots s_{\alpha_r}$ a reduced expression. One can lift each s_{α_i} to an automorphism $\tilde{s}_{\alpha_i} : \mathfrak{g} \to \mathfrak{g}$ by

$$\tilde{s}_{\alpha_i}(X) = \exp(\operatorname{ad} e_{\alpha_i}) \exp(-\operatorname{ad} f_{\alpha_i}) \exp(\operatorname{ad} e_{\alpha_i}).$$

By the construction, $\tilde{s}_{\alpha_i} : \mathfrak{g}_\gamma \mapsto \mathfrak{g}_{s_{\alpha_i}\gamma}$ for $\gamma \in \Phi$. $\tilde{s}_{\alpha_i}$ also acts on any $\mathfrak{g}$-module.

The $\tilde{s}_{\alpha_i}$ do not quite form a Weyl group action (we don't always have $\tilde{s}_{\alpha_i}^2 = 1$) but they do form a braid group action:

<u>Definition:</u> The braid group associated to W is the group generated by simple reflections s_{α_i} ($\alpha_i \in \Pi$) but modulo only the braid relations: if $\alpha_i, \alpha_j \in \Pi$ and $s_{\alpha_i} s_{\alpha_j}$ has order m in W, we impose the relation

$$\underbrace{s_{\alpha_i} s_{\alpha_j} s_{\alpha_i} \cdots}_{m \text{ terms}} = \underbrace{s_{\alpha_j} s_{\alpha_i} s_{\alpha_j} \cdots}_{m \text{ terms}}.$$

[And we omit the relations $s_{\alpha_i}^2 = 1$ present in W.]