§8. BRAID GROUP ACTIONS & A PBW-TYPE BASIS

MOTIVATION: ROOT VECTORS VIA A BRAID GROUP ACTION

Recall that in Leonardo's first talk, we saw that if $\mathcal{U}^+ :=$ subalgebra of $\mathcal{U}$ generated by all E_α, $\alpha \in \Pi$
$\mathcal{U}^- :=$ " " " " " " F_α, $\alpha \in \Pi$,
$\mathcal{U}^0 :=$ " " " " " " K_μ, $\mu \in \mathbb{Z}\Phi$,
then we had

THEOREM 4.21 i) $\mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \longrightarrow \mathcal{U}$, $u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$ is an isomorphism of vector spaces.
ii) K_μ for $\mu \in \mathbb{Z}\Phi$ are a basis for $\mathcal{U}^0$.

The subalgebras $\mathcal{U}^+$ and $\mathcal{U}^-$ are quantum analogues of $\mathcal{U}(\mathfrak{n}^+)$, $\mathcal{U}(\mathfrak{n}^-) \subset \mathcal{U}(\mathfrak{g})$. We have PBW theorems for $\mathcal{U}(\mathfrak{n}^+)$ & $\mathcal{U}(\mathfrak{n}^-)$. Explicitly, $\mathcal{U}(\mathfrak{n}^+)$ has a basis consisting of ordered monomials in the root vectors e_α, $\alpha \in \Phi^+$. In the quantum setting, we have E_α for $\alpha \in \Pi$, but we don't yet know how to make sense of E_α for $\alpha \in \Phi^+$.

THEOREM For all $\alpha \in \Phi^+$, there is an element $E_\alpha \in \mathcal{U}^+_\alpha$ such that $\mathcal{U}^+$ has a basis consisting of ordered monomials in these elements.

This statement (even the existence of E_α part) is nontrivial, as there is no underlying Lie algebra for $\mathcal{U}^+$.

Motivation for how we will construct these E_α again comes from the classical setting. For any $\beta \in \Phi^+$, there is $\alpha \in \Pi$ and $w \in W$ such that $w\alpha = \beta$. We write $s_{\alpha_1} \cdots s_{\alpha_r}(\alpha) = \beta$ for $w = s_{\alpha_1} \cdots s_{\alpha_r}$ a reduced expression. One can lift each s_{α_i} to an automorphism $\tilde{s}_{\alpha_i} : \mathfrak{g} \to \mathfrak{g}$ by

$$\tilde{s}_{\alpha_i}(X) = \exp(\operatorname{ad} e_{\alpha_i}) \exp(-\operatorname{ad} f_{\alpha_i}) \exp(\operatorname{ad} e_{\alpha_i}).$$

By the construction, $\tilde{s}_{\alpha_i} : \mathfrak{g}_\gamma \mapsto \mathfrak{g}_{s_{\alpha_i}\gamma}$ for $\gamma \in \Phi$. $\tilde{s}_{\alpha_i}$ also acts on any $\mathfrak{g}$-module.

The $\tilde{s}_{\alpha_i}$ do not quite form a Weyl group action (we don't always have $\tilde{s}_{\alpha_i}^2 = 1$) but they do form a braid group action:

<u>Definition:</u> The braid group associated to W is the group generated by simple reflections s_{α_i} ($\alpha_i \in \Pi$) but modulo only the braid relations: if $\alpha_i, \alpha_j \in \Pi$ and $s_{\alpha_i} s_{\alpha_j}$ has order m in W, we impose the relation

$$\underbrace{s_{\alpha_i} s_{\alpha_j} s_{\alpha_i} \cdots}_{m \text{ terms}} = \underbrace{s_{\alpha_j} s_{\alpha_i} s_{\alpha_j} \cdots}_{m \text{ terms}}.$$

[And we omit the relations $s_{\alpha_i}^2 = 1$ present in W.]

Any two reduced (minimal length) expressions for $w \in W$ are related by a sequence of braid relations. So if we accept that the $\tilde{s}_{\alpha_i}$ satisfy the braid relations, we can, for any $w \in W$, define $\tilde{w} := \tilde{s}_{\alpha_1} \cdots \tilde{s}_{\alpha_r}$ where $w = s_{\alpha_1} \cdots s_{\alpha_r}$ is reduced.

Then for any $\beta \in \Phi^+$, a way to define the root vector $e_\beta \in \mathcal{U}(\mathfrak{n}^+)$ is to set

$$e_\beta = \tilde{w}(e_\alpha) = \tilde{s}_{\alpha_1} \cdots \tilde{s}_{\alpha_r}(e_\alpha),$$

where $\beta = w\alpha$, $\alpha \in \Pi$, and $w = s_{\alpha_1} \cdots s_{\alpha_r}$ is reduced as before.

With this in mind, we'll follow this outline in the quantum setting by constructing a braid group action on $\mathcal{U}$ and on all finite-dimensional $\mathcal{U}$-modules, analogous to the one defined above for $\mathfrak{g}$. We will then use this to define E_β for all $\beta \in \Phi^+$.

DEFINING THE ACTION ($\mathfrak{sl}_2$)

Let's start by restricting ourselves to the case $\mathfrak{g} = \mathfrak{sl}_2$. Write E, F, etc. instead of E_α, F_α, etc. and $q = q_\alpha$. We continue to discuss only f.dim $\mathcal{U}$-modules V of Type 1.

So $V = \bigoplus_{m \in \mathbb{Z}} V_m$, where $V_m = \{ v \in V \mid Kv = q^m v \}$.

Define $E^{(r)} = \dfrac{E^r}{[r]!}$, $F^{(r)} = \dfrac{F^r}{[r]!}$ for all $r \geq 0$.

Recall the automorphism ω of $\mathcal{U}$ with $\omega(E) = F$, $\omega(F) = E$, $\omega(K) = K^{-1}$. Now we define four linear operators T, T', ${}^\omega T$, ${}^\omega T'$ such that for all $m \in \mathbb{Z}$, $v \in V_m$,

$$T(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=m}} (-1)^b q^{b-ac} E^{(a)} F^{(b)} E^{(c)} v$$

$$T'(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=m}} (-1)^b q^{ac-b} E^{(a)} F^{(b)} E^{(c)} v$$

$${}^\omega T(v) = \sum_{\substack{a,b,c \geq 0 \\ a-b+c=m}} (-1)^b q^{b-ac} F^{(a)} E^{(b)} F^{(c)} v$$

$${}^\omega T'(v) = \sum_{\substack{a,b,c \geq 0 \\ a-b+c=m}} (-1)^b q^{b-ac} F^{(a)} E^{(b)} F^{(c)} v$$

The latter two are obtained from the former two by twisting by ω. We will see see $T' = {}^{\omega}T^{-1}$, ${}^{\omega}T' = T^{-1}$.

Recall the irreducible $V = L(n,+)$ of highest weight n & Type 1 for $U_q(sl_2)$. Let's determine the action of these operators on V. Let v_0 be the highest weight vector of V, and recall $v_i := F^{(i)} v_0$ form a basis for V with

$$F^{(r)} v_i = \begin{bmatrix} r+i \\ r \end{bmatrix} v_{i+r} \qquad E^{(r)} v_i = \begin{bmatrix} n+r-i \\ r \end{bmatrix} v_{i-r}$$

if "$v_j = 0$" for $j<0$, $j>n$.

Now we'll describe the action of our operators using the following property of Gaussian binomial coefficients.

<u>Lemma 8.1</u> If $a, b, m \in \mathbb{Z}$, $b, m \geq 0$, then

$$\sum_{i=0}^{m} q^{ai-b(m-i)} \begin{bmatrix} a \\ m-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} = \begin{bmatrix} a+b \\ m \end{bmatrix}$$

(Note, this is a familiar combinatorial identity if $q=1$.)

Proof: Exercise with binomial coeffs.

Lemma 8.3 For all i,

$$T(v_i) = (-1)^{n-i} q^{(n-i)(i+1)} v_{n-i}$$
$$T'(v_i) = (-1)^{n-i} q^{-(n-i)(i+1)} v_{n-i}$$
$${}^{\omega}T(v_i) = (-1)^{i} q^{i(n+1-i)} v_{n-i}$$
$${}^{\omega}T'(v_i) = (-1)^{i} q^{-i(n+1-i)} v_{n-i}$$

Lemma 8.4 The operators T, T', ${}^{\omega}T$, ${}^{\omega}T'$ are bijective on each fin-dim Type 1 $U_q(\mathfrak{sl}_2)$-module V, and

(1) $T^{-1} = {}^{\omega}T'$
$T'^{-1} = {}^{\omega}T$

(2) for all $m \in \mathbb{Z}$, $v \in V_m$,

$${}^{\omega}T(v) = (-q)^{-m} T(v) \quad \text{and} \quad {}^{\omega}T'(v) = (-q)^{m} T'(v).$$

Proof. By complete reducibility, it suffices to consider $V = L(n,+)$. Bijectivity is clear by Lemma 8.3. We can check (1) & (2) on the basis v_i; then these too follow from Lemma 8.3.

Lemma 8.5 For all V as above & $v \in V$,

$$T(Ev) = (-FK)T(v) \qquad ET(v) = T((-K^{-1}F)v)$$
$$T(Fv) = (-K^{-1}E)T(v) \qquad FT(v) = T((-EK)v)$$
$$T(Kv) = K^{-1}T(v) \qquad KT(v) = T(K^{-1}v).$$

Proof sketch. It's enough to take $V = L(n,+)$, $v = v_i$ for some i. Let's do

$$T(Ev) = (-FK)T(v)$$

as an example, with the rest following similarly.

$$T(Ev_i) = T([n+1-i]\, v_{i-1})$$
$$= (-1)^{n-i+1} q^{i(n-i+1)} [n+1-i]\, v_{n-i+1},$$

while $FT(v_i) = F((-1)^{n-i} q^{(i+1)(n-i)} v_{n-i})$
$$= (-1)^{n-i} q^{(i+1)(n-i)} [n-i+1]\, v_{n-i+1},$$

$$\Rightarrow T(Ev_i) = -q^{2i-n} FT(v)$$
$$= (-FK)T(v)$$

since v_{n-i} has weight $= 2i-n$

and so on, for the rest. $\square$

DEFINING THE ACTION (IN GENERAL)

Return now to $\mathcal{U} = \mathcal{U}_q(\mathfrak{g})$ where $\mathfrak{g}$ is arbitrary. Extend all our definitions in the natural way:

For $\alpha \in \Pi$: set $E_\alpha^{(r)} = \dfrac{E_\alpha^r}{[r]!}$ $F_\alpha^{(r)} = \dfrac{F_\alpha^r}{[r]!}$ for all $r \geq 0$.

Let V be a f.dim $\mathcal{U}_q(\mathfrak{g})$-mod. Define four operators such that for all $\lambda \in \Lambda$, $v \in V_\lambda$,

$$T_\alpha(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=m}} (-1)^b q_\alpha^{b-ac} E_\alpha^{(a)} F_\alpha^{(b)} E_\alpha^{(c)} v$$

$$T_\alpha'(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b-c=m}} (-1)^b q_\alpha^{ac-b} E_\alpha^{(a)} F_\alpha^{(b)} E_\alpha^{(c)} v$$

$${}^\omega T_\alpha(v) = \sum_{\substack{a,b,c \geq 0 \\ a-b+c=m}} (-1)^b q_\alpha^{b-ac} F_\alpha^{(a)} E_\alpha^{(b)} F_\alpha^{(c)} v$$

$${}^\omega T_\alpha'(v) = \sum_{\substack{a,b,c \geq 0 \\ a-b+c=m}} (-1)^b q_\alpha^{b-ac} F_\alpha^{(a)} E_\alpha^{(b)} F_\alpha^{(c)} v$$

where $m := \langle \lambda, \alpha^\vee \rangle = \dfrac{2(\lambda,\alpha)}{(\alpha,\alpha)}$. (Recall $q_\alpha = q^{(\alpha,\alpha)/2}$)

Recall the embedding $\mathcal{U}_{q_\alpha}(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{g})$ defined by $E \mapsto E_\alpha$, $F \mapsto F_\alpha$, $K \mapsto K_\alpha$. Considering V as a $\mathcal{U}_{q_\alpha}(\mathfrak{sl}_2)$-module via the embedding, we get the T we defined earlier.

This means the results we proved for T actually lift straight away to give the results for T_α. Here are the properties we get immediately:

- T_α and T_α' are bijective, with inverses ${}^\omega T_\alpha'$ and ${}^\omega T_\alpha$.
- ${}^\omega T_\alpha(v) = (-q_\alpha)^{-\langle\lambda, \alpha^\vee\rangle} T_\alpha(v)$
  ${}^\omega T_\alpha'(v) = (-q_\alpha)^{\langle\lambda, \alpha^\vee\rangle} T_\alpha'(v)$
- $T_\alpha : V_\lambda \xrightarrow{\sim} V_{s_\alpha\lambda}$ (and same for ${}^\omega T_\alpha, {}^\omega T_\alpha', T_\alpha'$)
  We leave this as an exercise.
- Lemma 8.5 implies

$$T_\alpha(E_\alpha v) = (-F_\alpha K_\alpha) T_\alpha(v) \quad E_\alpha T_\alpha(v) = T_\alpha((-K_\alpha^{-1} F_\alpha) v)$$
$$T_\alpha(F_\alpha v) = (-K_\alpha^{-1} E_\alpha) T_\alpha(v) \quad F_\alpha T_\alpha(v) = T_\alpha((-E_\alpha K_\alpha) v).$$

- If $\beta \in \Pi$ with $(\alpha, \beta) = 0$, then E_β commutes with E_α and F_α $\Rightarrow$ with T_α. So

$$E_\beta T_\alpha(v) = T_\alpha(E_\beta v) \quad \text{and}$$
$$F_\beta T_\alpha(v) = T_\alpha(F_\beta v) \quad \text{when } (\alpha, \beta) = 0.$$

RELATIONS FOR THE T_α

Going back to the classical case for a brief moment, recall we had automorphisms $\tilde{s}_\alpha$ of f.dim $\mathfrak{g}$-modules and of $\mathfrak{g}$ itself. These satisfy

$$\tilde{s}_\alpha(Xv) = \tilde{s}_\alpha(X)\tilde{s}_\alpha(v).$$

We'd like similar behaviour from our T_α, and for this we'll need a bunch of formulas.

PROPOSITION 8.13 For all $u \in \mathcal{U}$ there is a unique $u' \in \mathcal{U}$ such that

$$T_\alpha(uv) = u' T_\alpha(v) \qquad \text{(for all } v \in V \text{ for all f.d. Type 1 } \mathcal{U}\text{-mods } V)$$

The map $u \mapsto u'$ is an automorphism of $\mathcal{U}$.

To prove this, it will be useful to have formulas for $E_\beta T_\alpha(v)$, $T_\alpha(E_\beta v)$ and similarly for E_β replaced by F_β.

PROPOSITION 8.10 "KEY COMPUTATION"

Let $\alpha, \beta \in \Pi$, $r := -\langle \beta, \alpha^\vee \rangle$. For all f.dim $\mathcal{U}$-modules V and all $v \in V$,

$$(*) \quad T_\alpha(E_\beta v) = \left(\mathrm{ad}\left(E_\alpha^{(r)}\right) E_\beta\right) T_\alpha(v).$$

Setting $q=1$, with correct signs this reduces to a familiar formula for $\tilde{S}_\alpha$.

Proof sketch This is a big computation, but here are the key steps:

- The formula $(m \geq 0)$:

$$\operatorname{ad}(E_\alpha^{(m)})E_\beta = \sum_{i=0}^{m} (-1)^i q_\alpha^{i(m-1-r)} E_\alpha^{(m-i)} E_\beta E_\alpha^{(i)}.$$

This comes from the formula for $\operatorname{ad}(E_\alpha^m)$ we saw in Leonardo's lecture using divided powers. The "r" comes into play here since $K_\alpha E_\beta K_\alpha^{-1} = q_\alpha^{\langle \beta, \alpha^\vee \rangle} E_\beta = q_\alpha^{-r} E_\beta$.

- We then use the above to prove that for integers $m, i \geq 0$,

$$(\operatorname{ad}(E_\alpha^{(m)})E_\beta)E_\alpha^{(i)}$$

(1) $$= \sum_{j=0}^{i} (-1)^j \begin{bmatrix} m+j \\ j \end{bmatrix}_\alpha q_\alpha^{i(r-2m)-j(i-1)} E^{(i-j)} (\operatorname{ad}(E_\alpha^{(m+j)})E_\beta)$$

and

$$(\operatorname{ad}(E_\alpha^{(m)})E_\beta)F_\alpha^{(i)}$$

(2) $$= \sum_{j=0}^{i} (-1)^j \begin{bmatrix} r-m+j \\ j \end{bmatrix}_\alpha q_\alpha^{j(i-1)} F_\alpha^{(i-j)} (\operatorname{ad}(E_\alpha^{(m-j)})E_\beta) K_\alpha^{-j}.$$

This is done by induction on i using the previous formula. Let $a(m) = \operatorname{ad}(E_\alpha^{(m)})E_\beta$.

- We use (1) and (2) to get a formula for $a(r)T_\alpha(v)$ (i.e. RHS of our goal). Indeed, $T_\alpha(v)$ is a linear combination of terms of the form $E_\alpha^{(a)}F_\alpha^{(b)}E_\alpha^{(c)}v$, and so applying (1) then (2) then (1) allows us to pass

$a(r)$ pass each of the three factors in such a term. We end up with a linear combination of terms $E_\alpha^{(a)}F_\alpha^{(b)}E_\alpha^{(c)}a(r-h)v$, where if $s=\langle\lambda,\alpha^\vee\rangle$ then $h=s+a-b+c$. Simplifying the coefficients on such terms, we see that they are zero unless $h=r$, and we get

$$a(r)T_\alpha(v) = \sum_{\substack{a,b,c\geq 0 \\ -a+b-c=s-r}} (-1)^b q_\alpha^{b-ac} E_\alpha^{(a)}F_\alpha^{(b)}E_\alpha^{(c)}a(0)v.$$

Since $a(0)=E_\beta$, $a(0)v\in V_{\lambda-\beta}$ with $\langle\lambda-\beta,\alpha^\vee\rangle=s-r$, this is $T_\alpha(E_\beta v)$, as desired.

Now note that we can get a similar result for $T_\alpha(F_\beta v)$ using the following:

<u>Lemma 8.7</u> If $u,u'\in\mathcal{U}$ s.t. $T_\alpha(uv)=u'T_\alpha(v)$ for all v in a finite-dimensional $\mathcal{U}$-module V, then
$${}^\omega T_\alpha(\omega(u)v)=\omega(u')\,{}^\omega T_\alpha(v) \text{ for all such } v.$$

If $u \in \mathcal{U}_\mu$ for $\mu \in \mathbb{Z}\Phi$ (recall $\mathcal{U}_\mu = \mu$-graded piece of $\mathcal{U}$).
then $T_\alpha(\omega(u)v)$
$= (-q_\alpha)^{-\langle \mu, \alpha^\vee \rangle} \omega(u') T_\alpha(v)$ for all such v.

<u>Proof</u> Exercise (the first claim is easy, and the second follows from the first and the formulas relating ${}^\omega T_\alpha(v)$ and $T_\alpha(v)$ for $v \in V_\lambda$).

This lemma applied to $(*)$ gives us

$$T_\alpha(F_\beta) = \left(\sum_{i=0}^{r} (-1)^i q_\alpha^i F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)} \right) T_\alpha(v).$$

Now we want to deduce from this a formula for $E_\beta T_\alpha(v)$. We can do this by twisting the adjoint action. Recall the antiautomorphism τ defined by

$$\tau(E_\alpha) = E_\alpha, \quad \tau(F_\alpha) = F_\alpha, \quad \tau(K_\alpha) = K_\alpha^{-1}.$$

For all $x \in \mathcal{U}$, define ${}^\tau\mathrm{ad}(x) : \mathcal{U} \to \mathcal{U}$ by
${}^\tau\mathrm{ad}(x) = \tau \circ \mathrm{ad} \circ \tau$; τ is an antiautomorphism but the twist is still an action.

<u>Claim 8.12</u> With r as before,
$$T_\alpha\left[({}^\tau\mathrm{ad}(E_\alpha^{(m)}) E_\beta) v \right] = (\mathrm{ad}(E_\alpha^{(r-m)}) E_\beta) T_\alpha(v)$$

(in particular, setting $m=r$, we get

$$E_\beta T_\alpha(v) = T_\alpha\left[\left({}^\tau\mathrm{ad}\left(E_\alpha^{(r)}\right)E_\beta\right)v\right]$$

<u>Proof sketch</u>:

- First, we can easily see
$$\begin{aligned} {}^\tau\mathrm{ad}(E_\alpha)u &= uE_\alpha - E_\alpha K_\alpha u K_\alpha^{-1} \\ {}^\tau\mathrm{ad}(F_\alpha)u &= K_\alpha^{-1}(uF_\alpha - F_\alpha u) \\ {}^\tau\mathrm{ad}(K_\alpha)u &= K_\alpha u K_\alpha^{-1}. \end{aligned}$$

- Next, we claim that if $T_\alpha(uv) = u' T_\alpha(v)$ for all v in all finite-dimensional $\mathcal{U}$-modules V, then
$$T_\alpha\left[\left({}^\tau\mathrm{ad}(E_\alpha)u\right)v\right] = \left(\mathrm{ad}(F_\alpha)u'\right)T_\alpha(v)$$
and $T_\alpha\left[\left({}^\tau\mathrm{ad}(F_\alpha)u\right)v\right] = \left(\mathrm{ad}(E_\alpha)u'\right)T_\alpha(v)$ for v as above.
We'll do the first one:
$$\begin{aligned} T_\alpha\left[{}^\tau(\mathrm{ad}(E_\alpha)u)v\right] &= T_\alpha\left(uE_\alpha v - E_\alpha K_\alpha u K_\alpha^{-1} v\right) \\ &= u'(-F_\alpha K_\alpha)T_\alpha(v) - (-F_\alpha K_\alpha)K_\alpha^{-1}u' K_\alpha T_\alpha(v) \\ &= (F_\alpha u' - u' F_\alpha)K_\alpha T_\alpha(v) \\ &= (\mathrm{ad}(F_\alpha)u')T_\alpha(v). \end{aligned}$$

- Combining this observation with $(*)$, we get
$$\begin{aligned} T_\alpha\left[\left({}^\tau\mathrm{ad}\left(E_\alpha^{(m)}\right)E_\beta\right)v\right] &= \left[\mathrm{ad}\left(F_\alpha^{(m)}\right)\mathrm{ad}\left(E_\alpha^{(r)}\right)E_\beta\right]T_\alpha(v) \\ &= \left[\mathrm{ad}\left(E_\alpha^{(r-m)}\right)E_\beta\right]T_\alpha(v). \end{aligned}$$

Now let's move on and use these relations to show the computibility of the action of T on modules we hinted at earlier.

Proposition 8.13 For all $u \in \mathcal{U}$ there is a unique $u' \in \mathcal{U}$ such that

$$T_\alpha(uv) = u' T_\alpha(v)$$

(for all $v \in V$ for all f.d. Type 1 $\mathcal{U}$-mods V)

The map $u \mapsto u'$ is an automorphism of $\mathcal{U}$.

Proof Suppose $u_1, u_2 \in \mathcal{U}$ and we've already found u_1', u_2'. Then we can take $(u_1 u_2)' = u_1' u_2'$ and $(au_1 + bu_2)' = au_1' + bu_2'$ so it suffices to show existence for generators of $\mathcal{U}$. But this is exactly what we've done:

basic observations from earlier.

$$\begin{cases} T_\alpha(K_\mu v) = K_{s_\alpha \mu} T_\alpha(v) \\ T_\alpha(E_\alpha v) = -(F_\alpha K_\alpha) T_\alpha(v) \\ T_\alpha(F_\alpha v) = (-K_\alpha^{-1} E_\alpha) T_\alpha(v) \end{cases}$$

key computation(s) above

$$\begin{cases} T_\alpha(E_\beta v) = \left(\mathrm{ad}(E_\alpha^{(r)}) E_\beta\right) T_\alpha(v) & (*) \\ T_\alpha(F_\beta v) = \left(\sum_{i=0}^{r} (-1)^i q_\alpha^i F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)}\right) T_\alpha(v). \end{cases}$$

Now for uniqueness. If u', u'' both satisfy the condition for a given u, then

$(u' - u'') T_\alpha(v) = 0 \quad \forall\, V,\ v \in V$. Since T_α is bijective, this means $u' - u''$ annihilates every fin. dim $\mathcal{U}$-module. So $u' - u'' = 0$ by Leonardo's lecture.

By definition it's an algebra endomorphism. But note we've already shown surjectivity of this map $u \mapsto u'$:

Since we have formulas for

$$K_\mu T_\alpha(v),\ E_\alpha T_\alpha(v)\ \&\ F_\alpha T_\alpha(v),$$
$$E_\beta T_\alpha(v)\ \&\ F_\beta T_\alpha(v),$$

this means all the generators appear in the image of our map.

Injectivity: If u is in the kernel, then $T_\alpha(uv) = 0$ for all $v \in V$, all V. Since T_α is bijective, the result from Leonardo's lecture again says $u = 0$. So $u \mapsto u'$ is an automorphism.

By abuse of notation, we also denote this autom. $u \mapsto u'$ by $T_\alpha : \mathcal{U} \to \mathcal{U}$. So by definition,

$$T_\alpha(uv) = T_\alpha(u) T_\alpha(v) \quad \text{for all } v \in \text{all } V \ \left(\begin{smallmatrix}\text{Type 1}\\ \text{fin. dim.}\end{smallmatrix}\right)$$

Our formulas from before tell us:

$$T_\alpha(K_\mu) = K_{s_\alpha \mu} = T_\alpha^{-1}(K_\mu) \quad \forall\ \mu \in \mathbb{Z}\Phi.$$
$$T_\alpha(E_\alpha) = -F_\alpha K_\alpha \qquad T_\alpha^{-1}(E_\alpha) = -K_\alpha^{-1} F_\alpha$$
$$T_\alpha(F_\alpha) = -K_\alpha^{-1} E_\alpha \qquad T_\alpha^{-1}(F_\alpha) = -E_\alpha K_\alpha$$

as well

as

$$T_\alpha(E_\beta) = \sum_{i=0}^{r} (-1)^i q_\alpha^{-i} E_\alpha^{(r-i)} E_\beta E_\alpha^{(i)}$$

$$T_\alpha^{-1}(E_\beta) = \sum_{i=0}^{r} (-1)^i q_\alpha^{-i} E_\alpha^{(i)} E_\beta E_\alpha^{(r-i)}$$

$$T_\alpha(F_\beta) = \sum_{i=0}^{r} (-1)^i q_\alpha^{i} F_\alpha^{(i)} F_\beta F_\alpha^{(r-i)}$$

$$T_\alpha^{-1}(F_\beta) = \sum_{i=0}^{r} (-1)^i q_\alpha^{i} F_\alpha^{(r-i)} F_\beta F_\alpha^{(i)}.$$

<u>Note</u> These imply $T_\alpha^{-1} = \tau \circ T_\alpha \circ \tau$.

BRAID RELATIONS

Lusztig showed that the T_α (on modules and on $\mathcal{U}$) satisfy the braid relations, i.e. for $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$, if $s_\alpha s_\beta \in W$ has order m, then

$$\underbrace{T_\alpha T_\beta \cdots}_{m} = \underbrace{T_\beta T_\alpha \cdots}_{m}$$

where both sides have m factors. This generalizes the behaviour of the $\tilde{s}_\alpha$ from the classical case.

The proof, in general, requires long calculations. We'll do it for $m=3$ to give an idea of the proof. The case $m=2$ is an easy exercise using commutativity of E_α & E_β when $(\alpha, \beta) = 0$.

When $s_\alpha s_\beta$ has order 3 is when $\langle \beta, \alpha^\vee \rangle = -1 = \langle \alpha, \beta^\vee \rangle$, and $(\alpha, \alpha) = (\beta, \beta)$, so $q_\alpha = q_\beta$. Our formulas from earlier give that (writing $q = q_\alpha$ for convenience) in this case,

$$\left.\begin{aligned}
T_\alpha(E_\beta) &= E_\alpha E_\beta - q^{-1} E_\beta E_\alpha = T_\beta^{-1}(E_\alpha) \\
T_\alpha^{-1}(E_\beta) &= E_\beta E_\alpha - q^{-1} E_\alpha E_\beta = T_\beta(E_\alpha) \\
T_\alpha(F_\beta) &= F_\beta F_\alpha - q F_\alpha F_\beta = T_\beta^{-1}(F_\alpha) \\
T_\alpha^{-1}(F_\beta) &= F_\alpha F_\beta - q F_\beta F_\alpha = T_\beta(F_\alpha).
\end{aligned}\right\} (3)$$

<u>Lemma 8.16</u> Let $\alpha, \beta \in \Pi$ s.t. $s_\alpha s_\beta$ has order 3. Then

$$T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta \quad \text{as automorphisms of } \mathcal{U}.$$

<u>Proof</u> It's enough to show the two sides coincide on generators of $\mathcal{U}$, and it's an easy exercise to show on $\mathcal{U}^0$, so it remains to look at E_γ, F_γ, $\gamma \in \Pi$. For $\gamma = \alpha$ or β, we just check both sides:

e.g.
$$\begin{aligned}
T_\alpha T_\beta T_\alpha(E_\beta) &= T_\alpha T_\beta T_\beta^{-1}(E_\alpha) \\
&= T_\alpha(E_\alpha) \\
&= -F_\alpha K_\alpha \\
T_\beta T_\alpha T_\beta(E_\beta) &= T_\beta T_\alpha(-F_\beta K_\beta) \\
&= -F_\alpha K_{s_\beta s_\alpha \beta} \\
&= -F_\alpha K_\alpha,
\end{aligned}$$

and similarly for F_β, (plus symmetry in α and β).

Now assume $\gamma \neq \alpha, \beta$. Since the Dynkin diagram contains no loops, $\langle \gamma, \alpha^\vee \rangle = 0$ or $\langle \gamma, \beta^\vee \rangle = 0$. WLOG $\langle \gamma, \beta^\vee \rangle = 0$, so

$$[E_\gamma, E_\beta] = 0 = [E_\gamma, F_\beta]$$
$$\Rightarrow T_\beta(E_\gamma) = E_\gamma.$$

Applying $T_\beta T_\alpha$ to [the relations above], it's easy to see from (3) that

$$[T_\beta T_\alpha(E_\gamma), E_\alpha] = 0 = [T_\beta T_\alpha(E_\gamma), F_\alpha],$$

which implies

$$T_\alpha(T_\beta T_\alpha(E_\gamma)) = T_\beta T_\alpha(E_\gamma),$$

so $T_\alpha T_\beta T_\alpha(E_\gamma) = T_\beta T_\alpha(E_\gamma) = T_\beta T_\alpha T_\beta(E_\gamma)$, similarly for F_γ. $\square$

Now recall: W is generated by the s_α, $\alpha \in \Pi$. Reduced expressions for an element $w \in W$ are related by braid relations, and remain reduced after applying braid relations.

So, since the T_α satisfy braid relations, we can define for $w \in W$:

$$T_w := T_{\alpha_1} \cdots T_{\alpha_r} \quad \text{for } w = \underbrace{s_{\alpha_1} \cdots s_{\alpha_t}}^{\text{reduced}}$$

and this is well-defined.

If $l(ww') = l(w) + l(w')$, get

$$T_{ww'} = T_w T_{w'} .$$

It is clear that

$$T_w(K_\mu) = K_{w\mu} \text{ for all } w \in W,\ \mu \in \mathbb{Z}\Phi.$$

$$T_w(\mathcal{U}_\mu) = \mathcal{U}_{w\mu} \text{ for all } w \in W,\ \mu \in \mathbb{Z}\Phi.$$

And our <u>Note</u> earlier gives $T_w^{-1} = \tau \circ T_{w^{-1}} \circ \tau$.
(using $\tau^2 = id$).

ROOT VECTORS & PBW BASIS

Now we can start to approach our goal by using the T_w to construct "root vectors" to ultimately get a PBW-type theorem.

From now on, let's restrict to the simply-laced case (i.e. "m" = 2 or 3) for the following, just for simplicity.

Proposition 8.20 Let $w \in W$ and $\alpha \in \Pi$. If $w\alpha > 0$, then $T_w(E_\alpha) \in \mathcal{U}^+$; if $w\alpha \in \Pi$, then $T_w(E_\alpha) = E_{w\alpha}$.

Proof sketch We go by induction on $\ell(w)$.

Suppose $\ell(w) > 0$. We can then pick $\beta \in \Pi$ $(\neq \alpha)$ s.t. $w(\beta) < 0$. There exists w', w'' s.t. $w = w'w''$ with $w'' \in \langle s_\alpha, s_\beta \rangle$ s.t. $w'\beta > 0$, $w'\alpha > 0$,

$$\ell(w) = \ell(w') + \ell(w'') \quad \& \quad w''\alpha > 0, \; w''\beta < 0.$$

So $w'' \neq 1 \Rightarrow \ell(w') < \ell(w)$, so by induction, get (e.g.) $T_{w'}(E_\alpha) \in \mathcal{U}^+$, $T_{w'}(E_\beta) \in \mathcal{U}^+$. (4)

We deal with $T_{w''}$ as follows:

<u>Lemma 8.19</u> Under above assumptions on w'', $T_{w''}(E_\alpha)$ is contained in the subalg generated by E_α & E_β, & if $w''\in\Pi$, $T_{w''}(E_\alpha)=E_{w\alpha}$.

<u>Pf</u> Since we're assuming $m=2$ or $m=3$ it's easy:

- If $m=2$, $w''=s_\beta$, $T_w(E_\alpha)=E_\alpha$.
- If $m=3$, $w''\in\{s_\beta, s_\alpha s_\beta\}$, so
  $T_w(E_\alpha)\in\{E_\beta E_\alpha - q_\alpha^{-1}E_\alpha E_\beta, E_\beta\}$. □

So $T_{w''}(E_\alpha)\in U^+$, so we're done since $T_w=T_{w'}T_{w''}$, and so we can apply (4) to conclude.

For the second claim, assume $w\alpha\in\Pi$. If we show $w''\alpha\in\Pi$, we're done by induction applied to w' and $w''\alpha$.

Exercise (about root systems): show that $w''\alpha\in\Pi$ under the hypotheses above. □.

Now we're ready to define root vectors. The naïve approach would be: for any $\gamma\in\Phi^+$, pick $\beta\in\Pi$ and $w\in W$ s.t. $w\beta=\gamma$, and then define $E_\gamma=T_w(E_\beta)$.

But this doesn't actually provide a consistent choice:

E.G. If Φ is of Type A_2, $\Pi = \{\alpha, \beta\}$, $\gamma = \alpha + \beta$, then $\gamma = s_\alpha \beta = s_\beta \alpha$, but

$$T_\alpha(E_\beta) = E_\alpha E_\beta - q^{-1} E_\beta E_\alpha$$

$$T_\beta(E_\alpha) = E_\beta E_\alpha - q^{-1} E_\alpha E_\beta$$

two linearly independent vectors!

Luckily, there is still a way to make a consistent choice of root vectors, but only by first making a choice for a reduced expression of $w_0 \in W$, the longest element.

If we choose $w_0 = s_{\alpha_1} \cdots s_{\alpha_t}$ a reduced expression, then

$$\alpha_1,\ s_{\alpha_1}\alpha_2,\ s_{\alpha_1}s_{\alpha_2}\alpha_3, \ldots,\ s_{\alpha_{t-1}}\alpha_t$$

is a list of all positive roots. So for $\gamma \in \Phi^+$, let i be s.t. $s_{\alpha_1} \cdots s_{\alpha_{i-1}} \alpha_i = \gamma$, and define

$$X_\gamma := T_{\alpha_1} \cdots T_{\alpha_{i-1}}(E_{\alpha_i}).$$

("the root vector associated to γ").

We would now like to prove the PBW-type theorem we sought.

<u>THEOREM 8.24</u> In the above setup, (recalling again our assumption that q is not a root of unity),

$$T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_{t-1}}(E_{\alpha_t}^{a_t}) \cdots T_{\alpha_1} T_{\alpha_2}(E_{\alpha_3}^{a_3}) T_{\alpha_1}(E_{\alpha_2}^{a_2}) E_{\alpha_1}^{a_1}$$

(a_i nonneg. integers) are a basis for $\mathcal{U}^+$.

To prove it, define for any $w \in W$ the subspace $\mathcal{U}^+[w] \subset \mathcal{U}^+$ spanned by expressions

$$T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_{r-1}} (E_{\alpha_r}^{a_r}) \cdots T_{\alpha_1}(E_{\alpha_2}^{a_2}) E_{\alpha_1}^{a_1} \qquad (5)$$

where $s_{\alpha_1} \cdots s_{\alpha_r}$ is a reduced expression for w.

<u>PROPOSITION 8.22</u>

(a) This is well-defined ($\mathcal{U}^+[w]$ does not depend on the choice of reduced expression).

(b) Let $\alpha \neq \beta$ be simple roots. If w is the longest element in $\langle s_\alpha, s_\beta \rangle$, then $\mathcal{U}^+[w]$ spans the subalg of $\mathcal{U}$ gen by E_α & E_β.