Elaborations on Lec 17, 18, 20.

Elaborated parts are marked in the text w. C#

Lecture 17:

C1 – for Theorem on page 2 of the lecture – why the highest weight is dominant: this is because the set of weights of V is closed under the W-action. Each W-orbit contains the unique maximal element, which is the unique dominant element, see Step 1 of the proof of Proposition in Sec 2 of Lec 22.

C2 – Corollary on page 5 – why $L(\lambda)$ is the unique irreducible submodule of $M(\lambda)$. Step 5 of the proof of Theorem in that section show that every other irreducible submodule of $M(\lambda)$ must be different from $L(\lambda)$. Let's say $\mathrm{Hom}_G(L(\mu), M(\lambda)) \neq 0$. This Hom is $\mathrm{Hom}_B(L(\mu), \mathbb{F}_{w_0\lambda})$. In particular, due to the W-invariance of the set of weights of $L(\mu)$, if the latter Hom is nonzero, then $\mu \leq \lambda$. On the other hand, if $\mu > \lambda$, then $L(\mu)_\mu$ lies in the kernel of every homomorphism $L(\mu) \to M(\lambda)$. Since $L(\mu)$ is irreducible, this implies the every homomorphism is zero, which completes the proof of our claim.

Lecture 18:

C3 – 1) of Corollary on page 4: in 1) of this Corollary we claim that for the isomorphism $(\mathbb{C}[G], *) \xrightarrow{\sim} \mathbb{C}G$ of Example on page 3, denote it by ι, we have $\iota(f)m = f * m \ \forall f \in \mathbb{C}[G]$

$\mathfrak{m} = \mathbb{C}[G/H]$.

C4 – proof of 2) on page 6, why $P = BsB \sqcup B$: P is a subgroup containing B so is the disjoint union of $B\times B$-orbits. For $w\in W$, the $B\times B$-orbit BwB is contained in $P \Leftrightarrow w=1$ or s. The equality $P = BsB \sqcup B$.

Lecture 19:

C5 – Lemma on page 5: there are also h-h-relations: $[h_i, h_j]=0$ $\forall i\neq j$. They follow from (i) & (ii): $[h_i,h_j] = [h_i,[e_j,f_j]] = [[h_i,e_j],f_j]+[e_j,[h_i,f_j]]$ $= a_{ji}[e_j,f_j] - a_{ji}[e_j,f_j] = 0$.

Lecture 20, C6 – the group $W(\tilde{A}_n)$, example 2) on pages 4-5. Let $\underline{\mathfrak{h}} := \mathrm{Span}(h_1,\dots h_{n-1}) \subset \mathfrak{h} = \mathrm{Span}(h_0,\dots h_{n-1})$ so that $\mathfrak{h} = \underline{\mathfrak{h}} \oplus \mathbb{C}\delta$. We have an embedding $\underline{\mathfrak{h}} \hookrightarrow \mathbb{C}^n$ via $h_i \mapsto e_i - e_{i+1}$. In particular the real locus $\underline{\mathfrak{h}}_{\mathbb{R}}$ acquires a Euclidian structure restricted from $\mathbb{R}^n$. Then $\delta^{-1}(0) \xrightarrow{\sim} \underline{\mathfrak{h}}^*$ (via the restriction map $\mathfrak{h}^* \twoheadrightarrow \underline{\mathfrak{h}}^*$) in particular $\delta^{-1}(0)_{\mathbb{R}}$ is also a Euclidian space. We identify the affine space $\delta^{-1}(1)_{\mathbb{R}}$ w. $\delta^{-1}(0)_{\mathbb{R}}$ by choosing the unique point in $\delta^{-1}(1)_{\mathbb{R}}$ w. $h_1 = \dots = h_{n-1} = 0$ for the origin. So $\delta^{-1}(1)_{\mathbb{R}} = \delta^{-1}(0)_{\mathbb{R}}$ becomes the Euclidian space $\{(x_1,\dots x_n)\in\mathbb{R}^n \mid x_1+\dots+x_n=0\}$ w. scalar product restricted from $\mathbb{R}^n$. The hyperplane $h_i=0$ for $i=1,\dots,n-1$ is given by $x_i = x_{i+1}$ for $i>0$ and by $x_1 = x_n+1$ for $i=0$. The group $W(\tilde{A}_n)$ in its action of $\delta^{-1}(1)_{\mathbb{R}}$ is generated by the orthogonal reflections about these hyperplanes. Those are:

- s_i = permutation of coordinates i & $i+1$ for $i>0$. These reflections generate $W = S_n$.
- s_0 is recoved as follows. It's associated linear map swaps x_1 & x_n, so $s_0(x_1, x_2, \dots x_{n-1}, x_n) = (x_n, x_2, \dots x_{n-1}, x_1) + v$ for some fixed $v \in \delta^{-1}(0)_{\mathbb{R}}$. Since s_0 fixes the hyperplane $x_1 = x_n + 1$. So we find that $v = (1, 0, \dots, 0, -1)$, hence $s_0(x_1, x_2, \dots x_{n-1}, x_n) = (x_n + 1, x_2, \dots, x_{n-1}, x_1 - 1)$.

In particular consider the element $s_0' = (1, n) \in S_n$. The composition, $s_0 s_0'$ is the translation by $(1, 0, \dots, 0, -1)$. It follows that $W(\tilde{A}_n)$ contains all translations by the elements of the form $(0, \dots, 1, 0, \dots, -1, \dots 0)$ and hence the translations by all elements of the lattice $\Lambda_0 = \{(z_1, \dots, z_n) \in \mathbb{Z}^n \mid z_1 + \dots + z_n = 0\}$. Hence

$$W(\tilde{A}_n) \supset S_n \ltimes \Lambda_0.$$

To establish $W(\tilde{A}_n) = S_n \ltimes \Lambda_0$ we need to show that $s_0 \in W \ltimes \Lambda_0$. This is because $s_0' \in S_n$ & $s_0 s_0'$ is a translation by an element of Λ_0.