FROM DAHA TO EHA

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1. Goals

The main purpose of this talk is two connect the two halves of our seminar. Specifically, we will follow the outline below:

- Consider the spherical double affine Hecke algebra (DAHA) of $\mathfrak{gl}_n$
- Define the limit $n \to \infty$
- Identify the generators and relations of the limit with those in the elliptic Hall algebra (EHA)

Note that the first bullet was introduced in José's talks, although we will recall it explicitly with focus on $\mathfrak{gl}_n$. Then we will use things from Chris' talks to work out the second bullet. Finally, the formulas we will work out in the third bullet will be compared with Mitya's talks next week. The reference is Schiffmann–Vasserot [1].

2. The spherical DAHA of $\mathfrak{gl}_n$

Recall (Definition 2.4.6 and Theorem 2.4.8 of José's notes) the DAHA:

$$\mathbb{H}_n = \mathbb{C}(q,v)\Big\langle T_1, ..., T_{n-1}, X_1^{\pm 1}, ..., X_n^{\pm 1}, Y_1^{\pm 1}, ..., Y_n^{\pm 1} \Big\rangle$$

subject to the relations that all X's commute, all Y's commute, and:

$$(T_i - v)(T_i + v^{-1}) = 0 \qquad T_i T_j = T_j T_i \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{1}$$

$$T_i X_j = X_j T_i \qquad T_i Y_j = Y_j T_i \qquad T_i X_i T_i = X_{i+1} \qquad T_i^{-1} Y_i T_i^{-1} = Y_{i+1} \tag{2}$$

$$Y_1 X_1 ... X_n = q X_1 ... X_n Y_1 \qquad Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2 \tag{3}$$

where i, j go over all possible indices such that $j \notin \{i-1, i, i+1\}$. Recall the action:

$$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}_n \tag{4}$$

in which the generators of $SL_2(\mathbb{Z})$ act as:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{cases} T_i \mapsto T_i \\ X_i \mapsto X_i \\ Y_i \mapsto Y_i X_i (T_1 ... T_{i-1})^{-1} (T_{i-1} ... T_1)^{-1} \end{cases} \tag{5}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : \begin{cases} T_i \mapsto T_i \\ X_i \mapsto X_i Y_i (T_{i-1} ... T_1)(T_1 ... T_{i-1}) \\ Y_i \mapsto Y_i \end{cases} \tag{6}$$

Finally, recall the idempotent:

$$e = \frac{1}{[n]!_v^+} \sum_{\sigma \in S_n} v^{l(\sigma)} T_\sigma$$

where $T_\sigma = T_{i_1} ... T_{i_r}$ corresponds to a reduced decomposition of σ as a product of transpositions. Recall that the v–factorial is defined by setting:

$$[i]_v^\pm = \frac{v^{\pm 2i} - 1}{v^{\pm 2} - 1} \qquad \Longrightarrow \qquad [n]!_v^\pm = [1]_v^\pm \cdot ... \cdot [n]_v^\pm \tag{7}$$

It is easy to show that:

$$e^2 = e \qquad \text{and} \qquad eT_i = T_i e = ve \tag{8}$$

As in Definition 3.3.3 of José's notes, let:

$$\mathbb{SH}_n = e\mathbb{H}_n e$$

denote the **spherical** subalgebra of $\mathbb{H}_n$, which is an algebra in its own right with unit e. As we will see in Proposition 3, the size of this subalgebra is "the same" as the size of $\mathbb{C}(q,v)[X_1^{\pm 1}, ..., X_n^{\pm 1}, Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^{S_n}$. Since the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}_n$ leaves e invariant, we conclude that $SL_2(\mathbb{Z})$ preserves $\mathbb{SH}_n$.

3. The generators

For any $k > 0$, Schiffmann–Vasserot in [1] consider the following elements of $\mathbb{SH}_n$:

$$P_{0,k}^{(n)} = e(Y_1^k + ... + Y_n^k)e \tag{9}$$

They further generalize these elements to arbitrary $(a,b) \in \mathbb{Z}^2 \backslash 0$, by letting $k = \gcd(a,b)$ and defining:

$$P_{a,b}^{(n)} = \begin{pmatrix} * & \frac{a}{g} \\ * & \frac{b}{g} \end{pmatrix} \cdot P_{0,g}^{(n)} \tag{10}$$

where $*$ denote arbitrary integers such that the matrix on the left has determinant 1. We claim that there is no ambiguity here, since the integers denoted $*$ are determined up to multiplying the matrix (10) on the right with powers of the matrix (6). Since the latter preserves both e and the Y's, it preserves the elements (9), and so (10) are uniquely defined for any a and b.

Proposition 1. *For any $a, b \in \mathbb{Z}$, we have:*

$$P_{a,1}^{(n)} = [n]_v^- \cdot eY_1 X_1^a e \tag{11}$$

$$P_{1,b}^{(n)} = q[n]_v^+ \cdot eX_1 Y_1^b e \tag{12}$$

$$P_{-1,b}^{(n)} = [n]_v^- \cdot eY_1^b X_1^{-1} e \tag{13}$$

$$P_{a,-1}^{(n)} = q[n]_v^+ \cdot eX_1^a Y_1^{-1} e \tag{14}$$

Proof. Since $Y_{i+1} = T_i^{-1} Y_i T_i^{-1}$, we can use (8) to infer $eY_{i+1}e = v^{-2} eY_i e$. Iterating this relation gives us:

$$P_{0,1}^{(n)} = (1 + v^{-2} + ... + v^{-2n+2}) eY_1 e = [n]_v^- \cdot eY_1 e$$

Let us now hit this element with various matrices $\in SL_2(\mathbb{Z})$ to obtain (11):

$$P^{(n)}_{a,1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^a \cdot P^{(n)}_{0,1} = [n]^-_v e \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^a \cdot Y_1 \right] e = [n]^-_v \cdot eY_1X_1^a e$$

Hitting the case $a = 1$ with the other generator of $SL_2(\mathbb{Z})$ gives us:

$$P^{(n)}_{1,b} = \begin{pmatrix} 1 & 0 \\ b-1 & 1 \end{pmatrix} \cdot P^{(n)}_{1,1} = [n]^-_v e \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b-1} \cdot Y_1X_1 \right] e = [n]^-_v \cdot eY_1X_1Y_1^{b-1} e$$

Using formula (2.10) of [1] together with (8), we have:

$$Y_1X_1 = q(T_1...T_{n-1})(T_{n-1}...T_1)X_1Y_1 \quad \Longrightarrow \tag{15}$$
$$\Longrightarrow \quad eY_1X_1Y_1^{b-1}e = qv^{2n-2}eX_1Y_1^b e$$

and so the above relation implies (12). To obtain (13) and (14), note that:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ sends } X_1 \mapsto X_1Y_1X_1^{-1}, Y_1 \mapsto X_1^{-1} \tag{16}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ sends } X_1 \mapsto Y_1^{-1},\ Y_1 \mapsto Y_1X_1Y_1^{-1} \tag{17}$$

Therefore, formulas (11) and (12) imply:

$$P^{(n)}_{-1,b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P^{(n)}_{b,1} = [n]^-_v \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} eY_1X_1^b e = [n]^-_v \cdot eY_1^b X_1^{-1} e$$

$$P^{(n)}_{a,-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^{(n)}_{1,a} = q[n]^+_v \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} eX_1Y_1^a e = q[n]^+_v \cdot eY_1^{a-1}X_1^aY_1^{-1} e$$

thus proving (13) and (14).

□

Proposition 2. *For any $k \in \mathbb{N}$, we have:*

$$P^{(n)}_{0,k} = e\sum_{i=1}^n Y_i^k e \tag{18}$$

$$P^{(n)}_{-k,0} = e\sum_{i=1}^n X_i^{-k} e \tag{19}$$

$$P^{(n)}_{0,-k} = q^k \cdot e\sum_{i=1}^n Y_i^{-k} e \tag{20}$$

$$P^{(n)}_{k,0} = q^k \cdot e\sum_{i=1}^n X_i^k e \tag{21}$$

Proof. The following element of $SL_2(\mathbb{Z})$:

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{takes} \qquad Y_1 \mapsto Y_1X_1^{-1} \mapsto X_1^{-1}$$

and $T_i \mapsto T_i$. Then one can iterate (2) to show that this matrix takes any $Y_i \mapsto X_i^{-1}$, and so it takes relation (18) to (19) and (20) to (21). However, note that (18) is simply the definition (9), so it remains to prove (20). To this end, let us consider the following element of $SL_2(\mathbb{Z})$:

$$\Gamma := \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

takes $\Gamma(Y_1) = X_1 Y_1^{-1} X_1^{-1}$. Using (15) and (2), we may rewrite this as:

$$\Gamma(Y_1) = qY_1^{-1} T_1 ... T_{n-1} T_{n-1} ... T_1 = qT_1^{-1} ... T_{n-1}^{-1} Y_n^{-1} T_{n-1} ... T_1$$

Because the product of T's on the left is the inverse of the product on the right, we may raise this relation to the k–th power:

$$\Gamma(Y_1^k) = q^k T_1^{-1} ... T_{n-1}^{-1} Y_n^{-k} T_{n-1} ... T_1$$

Because the idempotent e satisfies $eT_i = T_i e = ve$, see (8), we conclude that:

$$\Gamma(eY_1^k e) = q^k e Y_n^{-k} e \tag{22}$$

As [1] claims, there is a unique polynomial P with coefficients in $\mathbb{C}(q,v)$ such that:

$$P(eY_1 e, ..., eY_1^k e) = e \sum_{i=1}^n Y_i^k e \tag{23}$$

(in fact, this is true if one replaces $\sum_i Y_i^n$ with any other degree k symmetric polynomial in the Y variables), and that the same polynomial satisfies:

$$P(eY_n^{-1} e, ..., eY_n^{-k} e) = e \sum_{i=1}^n Y_i^{-k} e \tag{24}$$

The reason why these two equalities hold for the same polynomial P follows from the automorphism of the single affine Hecke algebra that sends $T_i \mapsto T_{n-i}$ and $Y_i \mapsto Y_{n+1-i}^{-1}$ (this automorphism can be checked either by hand, note that it is closely related to Theorem 3.3.3 of Seth's notes). Note that the degree of P in its first variable, plus twice its degree in the second variable, ... plus k times its degree in the last variable, equals k. Combining (22), (23), (24) yields:

$$\Gamma\left(e \sum_{i=1}^n Y_i^k e\right) = q^k \left(e \sum_{i=1}^n Y_i^{-k} e\right)$$

which is precisely what we needed to prove.

□

Proposition 3. *The elements $P_{a,b}^{(n)}$ generate $\mathbb{SH}_n$ as an algebra.*

Proof. Since all the structure constants in $\mathbb{SH}_n$ are Laurent polynomials in q and v, and $\mathbb{SH}_n$ is free over the ring $\mathbb{C}[q^{\pm 1}, v^{\pm 1}]$ (this was proved by Cherednik) it is enough to prove the proposition in the specialization $q = v = 1$. Note that:

$$\mathbb{SH}_n|_{q=v=1} \cong \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}, y_1^{\pm 1}, ..., y_n^{\pm 1}]^{S_n} \tag{25}$$

by sending $eP(X,Y)e$ to the symmetrization (that is, the average over all $n!$ permutations of the variables) of the Laurent polynomial $P(x,y)$. Since the symmetrizations of the polynomials in the right hand sides of (11)–(14) and (18)–(21) generate the right hand side of (25) (this is an exercise), the Proposition follows.

□

4. The relations

In preparation for the stable limit, let us rescale our generators to:

$$u_{a,b} = \frac{v^k q^{-k} - v^{-k}}{k} \cdot P^{(n)}_{a,b} \tag{26}$$

where $k = \gcd(a,b)$. For all coprime a,b, define:

$$1 + \sum_{k=1}^{\infty} \frac{\theta_{ak,bk}}{x^k} = \exp\left(\sum_{k=1}^{\infty} (v^{-k} - v^k) \frac{u_{ak,bk}}{x^k}\right) \tag{27}$$

Proposition 4. *The elements $u_{a,b} \in \mathbb{SH}_n$ satisfy the commutation relations:*

$$[u_{a,b}, u_{a',b'}] = 0 \tag{28}$$

if $ab' = a'b$, and:

$$[u_{a,b}, u_{a',b'}] = \pm\theta_{a+a',b+b'} \cdot \frac{(q^l - 1)(v^l q^{-l} - v^{-l})}{l(v - v^{-1})} \tag{29}$$

if one of the following situations occurs (let $l = \gcd(a,b)$ above):

- *$ab' = a'b \pm k$, $\gcd(a,b) = k$, $\gcd(a',b') = 1$, $\gcd(a+a', b+b') = 1$*
- *$ab' = a'b \pm k$, $\gcd(a,b) = 1$, $\gcd(a',b') = 1$, $\gcd(a+a', b+b') = k$*

Remark 1. *By Pick's theorem, the lattice points (a,b), (a',b') that appear in (29) are those such that the triangle with vertices $(0,0), (a,b), (a+a', b+b')$ has no lattice points inside and on the edges, with the possible exception of the edge $(0,0), (a,b)$ in the case of the first bullet or the edge $(0,0), (a+a', b+b')$ in the case of the second bullet.*

Proof. Since the action of $SL_2(\mathbb{Z})$ on lattice points is transitive, it is enough to check (28) when $a = 0$. In this case, relations (18) and (20) tell us that:

$$u_{0,b} = \text{const} \cdot e \sum_i Y_i^b e \qquad \text{and} \qquad u_{0,b'} = \text{const} \cdot e \sum_i Y_i^{b'} e \tag{30}$$

Because e commutes with symmetric polynomials in the Y_i, (30) commute because the Y_i's commute with each other. By a similar logic, we can use the $SL_2(\mathbb{Z})$ action to make $(a,b), (a',b')$ equal to $(0, \pm k), (1,0)$ in the case of the first bullet, and $(k,-1), (0,1)$ in the case of the second bullet. Moreover, using one more rotation, we may assume $k > 0$. Therefore, it remains to prove:

$$[u_{0,\pm k}, u_{1,0}] = \pm u_{1,\pm k} \cdot \frac{(q^k - 1)(v^k q^{-k} - v^{-k})}{k} \tag{31}$$

$$[u_{k,-1}, u_{0,1}] = \theta_{k,0} \cdot \frac{(q-1)(vq^{-1} - v^{-1})}{v - v^{-1}} \tag{32}$$

(we used the fact that $\theta_{a,b} = u_{a,b}(v^{-1} - v)$ if $\gcd(a,b) = 1$). Let us prove the first. In the notation of the previous Subsection, it amounts to:

$$\left[P^{(n)}_{0,\pm k}, P^{(n)}_{1,0}\right] = \pm P^{(n)}_{1,\pm k} \cdot (q^k - 1)$$

When the sign is $\pm = +$, relations (12) and (18) make this relation is equivalent to:

$$\left[e\sum_{i=1}^n Y_i^k e, eX_1 e\right] = e\left[\sum_{i=1}^n Y_i^k, X_1\right] e = (q^k-1)eX_1Y_1^k e \tag{33}$$

The first equality holds on general grounds (because $\sum_{i=1} Y_i^k$ is symmetric), while the second equality is proved in a 6–page computation in Appendix A of [1]. When the sign is $\pm = -$, one must apply the following automorphism to (33):

$$T_i \mapsto T_1^{-1}, \qquad X_1 \mapsto Y_1X_1Y_1^{-1}, \qquad Y_i \mapsto Y_i^{-1}, \qquad v \mapsto v^{-1}, \qquad q \mapsto q^{-1}$$

The above is the composition of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ of (17) and the automorphism:

$$T_i \mapsto T_1^{-1}, \qquad X_i \mapsto Y_i, \qquad Y_i \mapsto X_i, \qquad v \mapsto v^{-1}, \qquad q \mapsto q^{-1}$$

that was introduced in Theorem 2.2.7 of José's notes. As for relation (32), it reads:

$$\left[P_{k,-1}^{(n)}, P_{0,1}^{(n)}\right] = \frac{\theta_{k,0}^{(n)} \cdot (q-1)}{(v-v^{-1})(vq^{-1}-v)}$$

where $\theta_{k,0}^{(n)}$ is defined via the generating series:

$$1 + \sum_{k=1}^\infty \frac{\theta_{k,0}^{(n)}}{x^k} = \exp\left(\sum_{k=1}^\infty (v^{-k}-v^k)(v^kq^{-k}-v^{-k})q^k \cdot \frac{e\sum_{i=1}^n X_i^k e}{kx^k}\right) \in \mathbb{SH}_n[[x^{-1}]]$$

Then we may use (14), (18) and (21) to write the required relation as:

$$\frac{(1-qv^{-2})(v^{2n}-1)}{q-1} \cdot e\left[X_1^kY_1^{-1}, \sum_{i=1}^n Y_i\right] e = \theta_{k,0}^{(n)} \tag{34}$$

which is proved in a 5–page computation in Appendix B of [1].

□

5. The stable limit

Let us define the $\mathbb{C}(q,v)$–algebra:

$$\mathcal{A} = \langle u_{a,b}\rangle_{(a,b)\in\mathbb{Z}} \Big/ \text{relations (28) and (29)}$$

Mitya will discuss this algebra in more depth next week, and then Alexey and Tudor's talks will identify it with the elliptic Hall algebra. Meanwhile, note that Proposition 3 and 4 imply that there exist surjective ring homomorphisms:

$$\mathcal{A} \stackrel{\phi_n}{\twoheadrightarrow} \mathbb{SH}_n, \qquad u_{a,b} \mapsto \text{the RHS of (26)} \tag{35}$$

for any $n \in \mathbb{N}$. Our goal is to make the above into an isomorphism by letting $n \to \infty$. Unfortunately, this will only be possible when we restrict to the positive halves of the algebras in question:

$$\mathcal{A} \supset \mathcal{A}^+ = \mathbb{C}(q,v)\left\langle u_{a,b}\right\rangle_{(a,b)\in\mathbb{Z}^{2,+}}$$

$$\mathbb{SH}_n \supset \mathbb{SH}_n^+ = \mathbb{C}(q,v)\left\langle P_{a,b}^{(n)}\right\rangle_{(a,b)\in\mathbb{Z}^{2,+}}$$

where $\mathbb{Z}^2 \supset \mathbb{Z}^{2,+} = \{(a,b), a > 0 \text{ or } a = 0, b > 0\}$ denotes half of the lattice plane. Then the goal of the remainder of this talk is to prove the following Propositions:

Proposition 5. *There exists a morphism* $\mathbb{SH}_n^+ \to \mathbb{SH}_{n-1}^+$ *given by* $P_{a,b}^{(n)} \mapsto P_{a,b}^{(n-1)}$.

Clearly, the maps ϕ_n of (35) are compatible with the morphisms in Proposition 5.

Proposition 6. *The induced map:*

$$\mathcal{A}^+ \xrightarrow{\theta^+} \lim_{\leftarrow} \mathbb{SH}_n^+$$

given by $u_{a,b} \mapsto (..., P_{a,b}^{(n)}, ...)$, *is an isomorphism.*

Proof. of Proposition 5: Recall Cherednik's **basic representation**:

$$\mathbb{H}_n \hookrightarrow \text{Diff}(\mathbb{A}^{*n}) \rtimes S_n \tag{36}$$

where $\text{Diff}(\mathbb{A}^{*n}) = \mathbb{C}(q,v)[x_1^{\pm 1}, ..., x_n^{\pm 1}, D_1^{\pm 1}, ..., D_n^{\pm n}]$ is the ring of q–difference operators on punctured n–dimensional space, whose generators satisfy the relations:

$$[x_i, x_j] = [D_i, D_j] = 0 \qquad\qquad D_i x_j = q^{\delta_i^j} x_j D_i$$

Specifically, the map (36) is given by:

$$X_i \quad \mapsto \quad \text{multiplication by } x_i \tag{37}$$

$$T_i \quad \mapsto \quad v s_i + \frac{x_{i+1}(v - v^{-1})(s_i - 1)}{x_i - x_{i+1}} \tag{38}$$

$$T_{m-1}^{-1} ... T_i^{-1} Y_i T_{i-1} ... T_1 \quad \mapsto \quad s_{m-1} ... s_1 D_1 \tag{39}$$

where $s_i \in S_n$ denotes the transposition of i and $i+1$. Note that the basic representation was discussed in both Seth's and José's notes (Theorem 2.4.5 of the latter, together with the first formula after Definition 2.4.1). Because of the denominators $x_i - x_{i+1}$, the target of the map (36) is more precisely a certain localization of the ring $\text{Diff}(\mathbb{A}^{*n})$, but there's no need to burden the notation with detail. When we restrict this embedding to the spherical Hall algebra, we obtain the composition:

$$\mathbb{SH}_n \hookrightarrow \text{Diff}(\mathbb{A}^{*n})^{S_n} \rtimes S_n \twoheadrightarrow \text{Diff}(\mathbb{A}^{*n})^{S_n} \tag{40}$$

which is also an embedding. The map on the right is the projection $D \rtimes \sigma \mapsto D$ for all $D \in \text{Diff}(\mathbb{A}^{*n})$ and $\sigma \in S_n$. Let us consider the smaller subalgebras:

$$\mathbb{SH}_n^+ \supset \mathbb{SH}_n^{++} = \mathbb{C}(q,v) \left\langle P_{a,b}^{(n)} \right\rangle_{a,b,\geq 0}$$

$$\text{Diff}(\mathbb{A}^{*n}) \supset \text{Diff}^{++}(\mathbb{A}^{*n}) = \mathbb{C}(q,v)[x_1, ..., x_n, D_1, ..., D_n]$$

We claim that the maps (40) restrict to:

$$\psi_n : \mathbb{SH}_n^{++} \hookrightarrow \text{Diff}^{++}(\mathbb{A}^{*n})^{S_n} \tag{41}$$

Indeed, Propositions 3 and 4 imply that the domain is generated by $P_{0,k}^{(n)}$ and $P_{k,0}^{(n)}$ (this kind of generation statement will be discussed in more detail in Mitya's talk) so it is enough to show that these elements land in $\text{Diff}^{++}(\mathbb{A}^{*n}) \subset \text{Diff}(\mathbb{A}^{*n})$. Comparing formulas (18), (21) with (37), (39), this statement is clear since no negative powers of x_i and D_i come up in the latter formulas.

Lemma 1. *The maps ψ_n of (41) can be completed to a commuting square:*

$$\begin{tikzcd} \mathbb{SH}^{++}_n \arrow[r, "\psi_n"] \arrow[d, dotted] & \mathrm{Diff}^{++}(\mathbb{A}^{*n})^{S_n} \arrow[d, "\gamma"] \\ \mathbb{SH}^{++}_{n-1} \arrow[r, "\psi_{n-1}"] & \mathrm{Diff}^{++}(\mathbb{A}^{*n-1})^{S_{n-1}} \end{tikzcd} \tag{42}$$

where the dotted map on the left takes $P^{(n)}_{a,b} \mapsto P^{(n-1)}_{a,b}$, and the map γ is given by:

$$\gamma(x_i) = x_i \qquad \gamma(x_n) = 0 \tag{43}$$

$$\gamma(D_i) = \frac{D_i}{v} \qquad \gamma(D_n) = 0 \tag{44}$$

for all $i \in \{1, ..., n-1\}$. Note that γ is a homomorphism.

Note that the map γ would not have been defined if we had considered anything greater than the $++$ algebras, because the $P^{(n)}_{a,b}$ with $b < 0$ involve inverse powers of Y_i, and we could not have set $D_n \mapsto 0$ in (44). Let us show how this Lemma implies Proposition 5. Consider any relation:

$$0 = \sum \mathrm{const} \prod_i P^{(n)}_{a_i,b_i} \in \mathbb{SH}^+_n \tag{45}$$

for various choices of $a_i > 0$ or $a_i \geq 0, b_i > 0$. Because the sum is finite, we may choose some k large enough so that $b_i + ka_i \geq 0$ for all i that appear in (45). The $SL_2(\mathbb{Z})$ invariance of spherical DAHAs implies that we have a relation:

$$0 = \sum \mathrm{const} \prod_i P^{(n)}_{a_i,b_i+ka_i} \in \mathbb{SH}^{++}_n \tag{46}$$

By Lemma 1, relation (46) also holds with n replaced by $n-1$, and therefore $SL_2(\mathbb{Z})$ invariance implies that so does (45). Having showed that any relation between the generators of $\mathbb{SH}^+_n$ also holds in $\mathbb{SH}^+_{n-1}$, this concludes the proof of Proposition 5.

Proof. of Lemma 1: Relations (28) and (29) imply that for any a, b, there exists a finite polynomial $Q_{a,b}$ in the variables $\alpha_1, \alpha_2, ..., \beta_1, \beta_2, ...$ such that:

$$Q_{a,b}\left(P^{(n)}_{0,1}, P^{(n)}_{0,2}, ..., P^{(n)}_{1,0}, P^{(n)}_{2,0}, ...\right) = P^{(n)}_{a,b}$$

for all n. It's really important that the above relation holds in $\mathbb{SH}_n$ for all n, for a fixed polynomial $Q_{a,b}$ (the combinatorics which establishes this fact will be discussed in more detail by Mitya in the next talk, but it's not hard to believe). Since $\psi_n, \psi_{n-1}, \gamma$ in (42) are homomorphism, to establish the commutativity of the square, it is therefore enough to prove that:

$$\gamma \circ \psi_n \left(P^{(n)}_{0,a}\right) \quad \text{and} \quad \gamma \circ \psi_n \left(P^{(n)}_{a,0}\right) \quad \in \quad \psi_{n-1}\left(\mathbb{SH}^{++}_{n-1}\right)$$

This is obvious for the latter, namely $P^{(n)}_{a,0}$, because its image under ψ_n is $q^k \sum_{i=1}^n x_i^a$. As for the former, it is true that for any symmetric polynomial $f(Y_1, ..., Y_n)$ we have:

$$\gamma \circ \psi_n \left(ef(Y_1, ..., Y_{n-1}, Y_n)e\right) = \gamma \circ \psi_n \left(ef(Y_1, ..., Y_{n-1}, 0)e\right) \tag{47}$$

One way to see this is to chase through the definitions and observe that:

$$\psi_n\left(ef(Y_1, ..., Y_{n-1}, Y_n)e\right) = L_f$$

where the operator $L_f \in \mathrm{Diff}(\mathbb{A}^{*n})^{S_n}$ was introduced in Lemma 4.3.5 of José's talk, or Definition 3.13 of Chris' talk. Then equation (47) is merely the compatibility of L_f's for n and $n-1$ via the homomorphism γ. Alternatively, since both sides of (47) are additive and multiplicative in f, it is enough to check the equality when f is the k–th elementary symmetric function in $Y_1, ..., Y_n$. In this case, Macdonald shows (see Lemma 4.5 of [1]) that:

$$\psi_n\left(e \sum_{1\leq i_1<...<i_k\leq n} Y_{i_1}...Y_{i_k} e\right) = \sum_{I\subset\{1,...,n\}}^{|I|=k} \prod_{j\notin I}^{i\in I} \frac{x_i v - x_j v^{-1}}{x_i - x_j} \prod_{i\in I} D_i$$

and it is clear that setting $x_n, D_n \mapsto 0$ in the right hand side produces the corresponding expression when n is replaced by $n-1$ (up to a power of v, which is accounted for in (44)). $\square$

$\square$

Proof. of Proposition 6: Since the $P^{(n)}_{a,b}$ generate $\mathbb{SH}^+_n$ by Proposition 4, ι^+ is surjective. To prove it is also injective, it is enough to show that the analogous map:

$$\mathcal{A}^{++} \xrightarrow{\iota^{++}} \lim_{\leftarrow} \mathbb{SH}^{++}_n$$

is injective (we have already seen the reason for this: if there's a relation of the form (45) in the kernel of ι^+, then we could act with an element of $SL_2(\mathbb{Z})$ to turn it into a relation of the form (46) in the kernel of ι^{++}). We claim the following:

- The algebra $\mathcal{A}^{++}$ is graded by $\mathbb{N}_0 \times \mathbb{N}_0$, with $u_{a,b}$ in degree (a, b)
- The dimension of $\mathcal{A}^{++}_{a,b}$ is equal to the number of unordered collections:

$$(a_1, b_1), ..., (a_t, b_t) \quad \text{with} \quad \sum a_i = a \text{ and } \sum b_i = b \tag{48}$$

The first bullet is immediate, and the second bullet will be explained by Mitya in more detail. The intuition behind it is the following: elements of $\mathcal{A}^{++}$ are linear combinations of ordered products $u_{a_1,b_1}...u_{a_t,b_t}$ with $a_i, b_i \geq 0$, and the second bullet claims that we can always use relations (28) and (29) to rearrgange the terms in this product such that the lattice points $(a_1, b_1), ..., (a_t, b_t)$ form a convex path. The number of convex paths is equal to the number of unordered collections (48).

Therefore, to prove the injectivity of ι^{++}, it is enough to show that:

$$\dim\left(\lim_{\leftarrow} \mathbb{SH}^{++}_n \text{ in degree } (a, b)\right) = \text{the number in the second bullet} \tag{49}$$

To prove this, we will invoke the argument used in the proof of Proposition 3. Since the integral form of $\mathbb{SH}^{++}_n$ is a free module over the ring $\mathbb{C}[q^{\pm1}, v^{\pm1}]$, its rank can be computed in the specialization $q = v = 1$:

$$\dim\left(\mathbb{SH}^{++}_n \text{ in degree } (a, b)\right) = \dim_{\mathbb{C}}\left(\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]^{S_n} \text{ in degree } (a, b)\right)$$

where a refers to the degree in the x variables and b refers to the degree in the y variables. The polynomial rings in the right hand side have a well-known inverse limit, the ring of polynomials in infinitely many variables:

$$\dim \left(\lim_{\leftarrow} \mathbb{SH}_n^{++} \text{ in degree } (a,b) \right) = \dim_{\mathbb{C}} \left(\mathbb{C}[x_1, ..., y_1, ...]^{\text{sym}} \text{ in degree } (a,b) \right)$$

All that remains is to observe that a basis of the space in the RHS is given by:

$$\text{Sym } x_1^{a_1} x_2^{a_2} ... y_1^{b_1} y_2^{b_2} ...$$

with $\sum a_i = a$, $\sum b_i = b$. The number of such basis vectors is precisely the number in the second bullet, which appears in (49).

$\square$

References

[1] Schiffmann O., Vasserot E., *Cherednik algebras, W–algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbb{A}^2$, **Publ. Math. Inst. Hautes Etud. Sci.**, 118 (2013), Issue 1, 213-342