LECTURE 3 (PART 1): MACDONALD POLYNOMIALS

CHRISTOPHER RYBA

ABSTRACT. These are notes for a seminar talk given at the MIT-Northeastern Double Affine Hecke Algebras and Elliptic Hall Algebras (DAHAEHA) seminar (Spring 2017).

CONTENTS

1. GOALS

The purpose of this document is to introduce Macdonald polynomials, and prove the Macdonald conjecture which concerns the value of certain bilinear forms when evaluated on these polynomials. The existence of these polynomials is not trivial; we will see two different approaches. Then we will explain how Affine Hecke Algebras and Double Affine Hecke Algebras can be used to prove the Macdonald conjecture.

2. REVIEW OF NOTATION

2.1. **Root System and Weyl Groups.** We write R for an irreducible finite root system in a vector space V, equipped with inner product $(-,-)$. We write R^a for the associated affine root system. We employ the following notation:

- The set of positive roots of R is denoted R_+, and the set of negative roots is denoted R_-. Similarly we write R^a_+ and R^a_- for the positive and negative roots of R^a, respectively.
- We write $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ for a choice of simple roots in R. The coroot associated to α_i is $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$.
- We write α_0 for $\delta - \theta$ where θ is the longest root in R. In this way $\{\alpha_0, \alpha_1, \cdots, \alpha_n$ form a set of simple roots for R^a.
- The root lattice is $Q = \mathbb{Z}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The coroot lattice is $Q^\vee = \mathbb{Z}\{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_n^\vee\}$.
- The weight lattice is $P = \mathbb{Z}\{\omega_1, \omega_2, \cdots, \omega_n\}$, where ω_i is the i-th fundamental weight ($P = \{\lambda \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in R\}$). The dominant weights are $P_+ = \mathbb{N}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ ($P_+ = \{\lambda \mid (\lambda, \alpha^\vee) \in \mathbb{N} \forall \alpha \in R\}$). Similarly we have the coweight lattice is $P^\vee = \{\lambda \mid (\lambda, \alpha) \in \mathbb{Z} \forall \alpha \in R\}$, and the dominant coweights are $P^\vee = \{\lambda \mid (\lambda, \alpha) \in \mathbb{N} \forall \alpha \in R\}$.
- The half-sum of positive roots is $\rho = \sum_{\alpha \in R_+} \alpha$, and it is well known that $\rho = \sum_{i=1}^n \omega_i$.

Date: March 13, 2017.

- We write $W = \langle s_1, s_2, \cdots, s_n \rangle$ for the (finite) Weyl group associated to R, generated by the simple reflections s_i. We also write $W^a = \langle s_0, s_1, \cdots, s_n \rangle$ for the affine Weyl group. We have the isomorphism
$$W^a = W \ltimes t(Q^\vee)$$
Here, as in previous lectures, t indicates translation in $Q^\vee$, so this is a subgroup of the group of invertible affine maps on $Q^\vee$.
- The extended affine Weyl group is $W^{ae} = W \ltimes t(P^\vee)$ (a subgroup of the group of invertible affine linear maps on $P^\vee$).
- We write $\Omega \subset W^{ae}$ for the set of all length zero elements of W^{ae}. It is a subgroup which acts faithfully on the set of simple roots of R^a. Furthermore, Ω is isomorphic to $P^\vee/Q^\vee$ and is in bijection with minuscule weights (to be discussed later).
- We actually have $W^{ae} = \Omega \ltimes W^a$.
- We write λ_+ for the unique dominant weight in the W-orbit of λ. Similarly we write λ^- for the unique anttdominant weight in the orbit of λ
- We will work over the field $\mathbb{C}(q,t)$, which we write $\mathbb{C}_{q,t}$.
- Shortly after the beginning, we will specialise to the case where $t = q^k$ (where $k \in \mathbb{Z}_{\geq 0}$). The notation t appears in parts of the theory directly related to Hecke algebras.

2.2. **Double Affine Hecke Algebras.**

- We write H for the finite Hecke algebra attached to the root system R. Similarly H^a is the affine Hecke algebra, and H^{ae} is the extended affine Hecke algebra.
- $H^{ae} = \Omega \ltimes H^a$
- H^a is generated by $T_0, T_1, \cdots, T_n$ subject to the braid relations and the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$.
- Thinking of H^{ae} as a quotient of the affine extended braid group, one has the elements Y^λ coming from the lattice associated to integral coweights.
- $H^{ae} = H \otimes \mathbb{C}(t)[Y]$ as a vector space, where the Y^λ.
- The centre of H^{ae} is precisely $\mathbb{C}(t)[Y]^W$.
- We have Cherednik's basic representation of H^{ae} on $\mathbb{C}_{q,t}[X] = \mathbb{C}_{q,t}[P]$, where T_i acts as $t_i s_i + (t_i - t_i^{-1})\frac{s_i - 1}{X^{-\alpha_i} - 1}$.
- The extended affine Weyl group action on $\mathbb{C}_{q,t}[X]$ satisfies $t_\lambda)(X^\mu) = q^{2(\lambda,\mu)} X^\mu$, where $t_\lambda)$ is translation by λ.

3. Macdonald Polynomials

3.1. **Definition and Proof of Existence.** Recall that the Weyl group acts on the set of weights, P. We may therefore extend the action of W to the group algebra $\mathbb{C}_{q,t}[P]$. We will be concerned with elements of $\mathbb{C}_{q,t}[P]^W$, namely elements of the group algebra which are fixed by the Weyl group action. Note that $\mathbb{C}_{q,t}[P]^W$ is a linear subspace of $\mathbb{C}_{q,t}[P]$. Macdonald polynomials will form a basis of $\mathbb{C}_{q,t}[P]^W$. Note that an obvious basis of $\mathbb{C}_{q,t}[P]^W$ is given by the orbit sums $m_\lambda = \sum_{\mu \in W\lambda} e^\mu$ for $\lambda \in P_+$. By a standard theorem in Lie theory, there is a unique dominant weight in each Weyl group orbit on P, which shows that $\{m_\lambda \mid \lambda \in P_+\}$ is indeed a basis for $\mathbb{C}_{q,t}[P]^W$.

At this point, one might protest that it is unclear how these are polynomials. To answer this, recall that $P = \mathbb{Z}\{\omega_1, \omega_2, \cdots, \omega_n\}$ and therefore $\mathbb{C}_{q,t}[P]$ can be thought of as the algebra of Laurent polynomials in the variables $\omega_1, \omega_2, \cdots, \omega_n$ (with complex coefficients). To conform with standard notation, given $\lambda \in P$, we write e^λ instead of λ for the associated element in $\mathbb{C}_{q,t}[P]$ (this avoids ambiguity between additive and multiplicative notation).

Next, we introduce a bilinear form on $\mathbb{C}_{q,t}[P]$.

Definition 3.1. *If $f \in \mathbb{C}_{q,t}[P]$, write $[f]_0$ for the coefficient of e^0 in f, when expressed in the e^λ basis. Suppose that $f \mapsto \bar{f}$ is the involution of $\mathbb{C}_{q,t}[P]$ defined by $e^\lambda \mapsto e^{-\lambda}$. Let $\Delta_{q,t} = \prod_{\alpha \in R} \prod_{i=0}^{\infty} \frac{1 - q^{2i} e^\alpha}{1 - t^2 q^{2i} e^\alpha}$*

(consider this as a Laurent series in the variables q, t, having coefficients in $\mathbb{C}_{q,t}[P]$). Then, we define the bilinear form $\langle -, - \rangle_{q,t}$ on $\mathbb{C}_{q,t}[P]$ as follows:

$$\langle f, g\rangle_{q,t} = \frac{1}{|W|}[f\Delta_{q,t}\bar{g}]_0$$

We will just be interested in the restriction of the bilinear form to $\mathbb{C}_{q,t}[P]^W$. Motivation for this construction will come later.

Note that if we define $\Delta^+_{q,t} = \prod_{\alpha\in R_+}\prod_{i=0}^{\infty}\frac{1-q^{2i}e^{\alpha}}{1-t^2q^{2i}e^{\alpha}}$, then $\Delta_{q,t} = \Delta^+_{q,t}\overline{\Delta^+_{q,t}}$. This will be convenient in what follows. We are now able to give a definition of Macdonald polynomials, although it will not be immediately clear that they exist.

Theorem 3.2. *For each $\lambda \in P_+$, there exists a unique $P_\lambda \in \mathbb{C}_{q,t}[P]^W$ such that:*

(1) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu}m_\mu$
(2) $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$ *whenever* $\lambda \neq \mu$

Here, $\mu < \lambda$ means that $\lambda - \mu \in Q_+$.

We will prove this theorem (at least, in some special cases), but first we discuss it.

Remark 3.3. *Gram-Schmidt orthogonalisation cannot be applied here because $<$ is not a total order on P_+. However, it does imply uniqueness.*

To see why this is could be an interesting construction, we consider some examples.

Example 3.4. *Suppose that $t = 1$, so that $\Delta_{q,t} = 1$. Then $\langle f, g\rangle_{q,t} = [f\bar{g}]_0$, and it is easy to see that $P_\lambda = m_\lambda$ satisfy the statement of the theorem (and this does not depend on q).*

Now suppose that $t = q$. Then $\Delta^+_{q,t} = \prod_{\alpha\in R_+}(1 - e^\alpha)$ because the product telescopes. Let χ_λ be given by the Weyl Character Formula for $\lambda \in P_+$:

$$\chi_\lambda = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}$$

We calculate $\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t}$.

$$\begin{aligned}\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t} &= \frac{1}{|W|}[\chi_{\lambda_1}\Delta_{q,t}\overline{\chi_{\lambda_2}}]_0\\ &= \frac{1}{|W|}[\left(\chi_{\lambda_1}\overline{\Delta^+_{q,t}}\right)(\Delta^+_{q,t}\overline{\chi_{\lambda_2}})]_0\\ &= \frac{1}{|W|}[\sum_{w_1\in W}\varepsilon(w_1)e^{w_1(\lambda_1+\rho)-\rho}\sum_{w_2\in W}\varepsilon(w_2)e^{-w_2(\lambda_2+\rho)+\rho}]_0\\ &= \frac{1}{|W|}\sum_{w_1\in W}\sum_{w_2\in W}\varepsilon(w_1w_2)[e^{w_1(\lambda_1+\rho)-w_2(\lambda_2+\rho)}]_0\end{aligned}$$

Note that the nonzero terms are precisely those for which $w_1(\lambda_1 + \rho) = w_2(\lambda_2 + \rho)$. This is equivalent to $\lambda_1 + \rho = w_1^{-1}w_2(\lambda_2 + \rho)$. Using the fact that each weight has a unique dominant weight in its orbit in the Weyl group, we see that this equation can only hold if $\lambda_1 + \rho = \lambda_2 + \rho$ (the latter is the unique dominant weight in its orbit). So we get zero unless $\lambda_1 = \lambda_2$. Furthermore, the terms which contribute 1 are those for which $w_1^{-1}w_2$ fixes $\lambda_2 + \rho$. Recall the length of a Weyl group element is equal to the number of positive roots that it maps to negative roots, so that

$$w_1^{-1}w_2(\rho) = \frac{1}{2}\sum_{\alpha\in R_+}w_1^{-1}w_2(\alpha) \leq \frac{1}{2}\sum_{\alpha\in R_+}\alpha = \rho$$

with equality if and only if $w_1^{-1}w_2$ is the identity element of W. Since $w_1^{-1}w_2(\lambda) \leq \lambda$, we may add these two inequalities to find that the identity is the only element of W that fixes $\lambda_2 + \rho$. The number of solutions

(w_1, w_2) to $w_1^{-1}w_2 = \mathrm{Id}_W$ is clearly $|W|$, so we obtain

$$\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t} = \frac{|W|}{|W|} = 1$$

Finally, we write $\chi_\lambda = \sum_\mu a_{\lambda,\mu} m_\mu$, where the $a_{\lambda,\mu}$ correspond to the dimensions of weight spaces in the irreducible representation of the relevant simple Lie algebra of highest weight λ. Recall that the irreducible representation is generated by the action of the lower triangular part of the Lie algebra (usually written $\mathfrak{n}_-$) on a highest weight vector, which itself is unique up to scalar multiplication. This implies that only $\mu \leq \lambda$ appear in the sum, and that $a_{\lambda,\lambda} = 1$. This proves that χ_λ satisfy the conditions of the theorem.

In light of the preceding example, it might be reasonable to view Macdonald polynomials as a deformation of characters of representations of simple Lie algebras.

Example 3.5. *Let $V = \mathbb{R}^n$ with standard basis e_i, and let $R = \{e_i - e_j \mid i \neq j\}$ so that R is a root system of type A_{n-1} and we may take $\alpha_i = e_i - e_{i+1}$. The associated simple Lie algebra is $\mathfrak{sl}_n$, and the Weyl group is $W = S_n$, which acts on V by permutation of coordinates. Since $\alpha = \alpha^\vee$ for all $\alpha \in R$, the weights and the coweights of R are the same. Let $\mathbb{C}_{q,t}[P]$ be presented by letting $x_i = \exp(e_i)$, so that $e^{\alpha_i} = \frac{x_i}{x_{i+1}}$. Then $\mathbb{C}_{q,t}[P]$ is realised as the space of Laurent polynomials in $x_1, x_2, \cdots, x_n$ of total degree zero. Then it is easily seen that the positive roots correspond to $\frac{x_i}{x_j}$ with $i < j$. It is also easy to see that if $\lambda \in P$ corresponds to $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$ (where necessarily the λ_i sum to zero), then the value of the fundamental weight ω_r applied to λ is $\omega_r(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. It is also easy to see that being dominant is equivalent to having the λ_i forming a weakly decreasing sequence, and λ is integral if and only if the λ_i are integers.*

We now prove theorem 3.2 in the case where $t = q^k$, for $k \in \mathbb{Z}_{\geq 0}$, and when there are minuscule weights associated to the root system R (so R cannot be G_2, F_4, E_8). Although these restrictions are not required, they mitigate technical difficulties. The specialisation of the parameter t is the case relevant to the Macdonald conjecture, so not much will be lost to us. The reader who is interested in greater generality is directed to [Mac00].

Proof. Firstly, note that the product in the definition of $\Delta^+_{q,t}$ telescopes:

$$\begin{aligned}\Delta^+_{q,t} &= \prod_{\alpha\in R_+}\prod_{i=0}^{\infty}\frac{1-q^{2i}e^\alpha}{1-t^2q^{2i}e^\alpha}\\ &= \prod_{\alpha\in R_+}\prod_{i=0}^{\infty}\frac{1-q^{2i}e^\alpha}{1-q^{2k}q^{2i}e^\alpha}\\ &= \prod_{\alpha\in R_+}\prod_{i=0}^{k-1}(1-q^{2i}e^\alpha)\end{aligned}$$

In particular, we obtain a finite expression. Next, if $\pi \in P^\vee$ we define $T_\pi(e^\lambda) = q^{2(\pi,\lambda)e^\lambda}$ (and extend linearly), where we may have to include fractional powers of q in our ring. Now let us write

$$D_\pi(f) = \sum_{w\in W} w\left(\frac{T_\pi(\Delta^+_{q,t}(f))}{\Delta^+_{q,t}}\right)$$

In the case where π is a minuscule coweight (i.e. $0 \leq (\lambda, \alpha) \leq 1$ for all positive roots $\alpha \in R_+$), this simplifies as follows.

$$D_\pi(f) = \left(\sum_{w\in W} w\right)\left(\prod_{\alpha\in R_+,(\pi,\alpha)=1}\frac{1-q^{2k}e^\alpha}{1-e^\alpha}\right)T_\pi(f) \tag{1}$$

This is clearly the symmetrisation of some rational function, whose denominator is a product of distinct terms of the form $(1 - e^\alpha)$. It is certainly W-invariant. In particular, let $\delta = \prod_{\alpha\in R_+}\frac{1}{e^{\alpha/2}-e^{-\alpha/2}}$ be the Weyl denominator, which is antisymmetric (implicitly we are now working in $e^\rho\mathbb{C}_{q,t}[P]$). To see this, recall that the action of $s_i \in W$ permutes the positive roots except for α_i which it maps to $-\alpha_i$. Thus s_i acts by multiplication by -1 on $\frac{1}{e^{\alpha_i/2}-e^{-\alpha_i/2}}$, and permutes the other factors of δ. Thus $\delta T_\pi(f)$ is antisymmetric with respect to the W-action, and is also a polynomial (we have removed the denominators). In particular, for any s_i, the coefficient of e^λ must be minus the coefficient of $e^{s_i(\lambda)}$. This means that no e^λ fixed by s_i can occur in $\delta T_\pi(f)$, so $\delta T_\pi(f)$ is a linear combination of $e^\lambda - e^{s_i(\lambda)}$. Now observe that

$$e^\lambda - e^{s_i(\lambda)} = e^\lambda - e^{\lambda-(\lambda,\alpha_i^\vee)\alpha_i} = e^\lambda(1-e^{-(\lambda,\alpha_i^\vee)\alpha_i}) = (1-e^{-\alpha_i})(1+e^{-\alpha_i}+\cdots+e^{-((\lambda,\alpha_i^\vee)-1)\alpha_i})$$

This is therefore divisible by $e^{\alpha_i/2}-e^{-\alpha_i/2}$ for each simple root α_i. As a result, the same is true of $\delta D_\pi(f)$. Since each root is in the orbit of a simple root, by applying the action of a suitable element of W we find that $e^{\alpha/2}-e^{-\alpha/2}$ divides $\delta D_\pi(f)$ for all positive roots α. It is not difficult to check that these are coprime in the UFD $\mathbb{C}_{q,t}[P/2]$. Hence, $\delta D_\pi(f)$ is divisible by $\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2}) = \delta$ (in the sense of polynomial divisibility). We conclude that $D_\pi(f)$ is actually a polynomial (rather than a rational function), which preserves $\mathbb{C}_{q,t}[P]^W$.

If we can show that D_π is triangular with respect to the e^λ basis, and is self-adjoint with respect to $\langle -,-\rangle_{q,t}$ with distinct eigenvalues, then the theorem will follow. This is because triangularity allows us to restrict to the finite dimensional subspace spanned by m_μ for $\mu \le \lambda$, whence distinct eigenvalues guarantee diagonalisability. Finally, self adjointness (and distinctness of eigenvalues) implies the eigenvectors are orthogonal. The Macdonald polynomials will be the eigenvectors of this operator.

To see the self-adjoint property, recall that $\Delta_{q,t}$ is W-invariant, as is the m_λ basis:

$$\begin{aligned}
\frac{1}{|W|}[D_\pi(m_\lambda)\Delta_{q,t}\bar{e^\mu}]_0 &= \frac{1}{|W|}[\sum_{w\in W} w\left(\frac{T_\pi(\Delta_{q,t}^+ m_\lambda)}{\Delta_{q,t}^+}\right)\Delta_{q,t}\overline{m_\mu}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W}\left(\frac{T_\pi(\Delta_{q,t}^+ m_\lambda)}{\Delta_{q,t}^+}\right)\Delta_{q,t}\overline{m_{w^{-1}\mu}}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} T_\pi(\Delta_{q,t}^+ m_\lambda)\overline{\Delta_{q,t}^+}\overline{m_\mu}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} \Delta_{q,t}^+ m_\lambda T_{-\pi}(\overline{\Delta_{q,t}^+}\overline{m_\mu})]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} m_\lambda \Delta_{q,t}^+\overline{T_\pi((\Delta_{q,t}^+ m_\mu))}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\sum_{w\in W}\frac{\overline{T_\pi((\Delta_{q,t}^+ m_\mu))}}{\overline{\Delta_{q,t}^+}}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\sum_{w\in W} w^{-1}\overline{\frac{T_\pi(\Delta_{q,t}^+ m_\mu)}{\Delta_{q,t}^+}}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\overline{D_\pi(m_\mu)}]_0
\end{aligned}$$

We calculate the leading order term, and in doing so, observe triangularity. For this, we use a deformed version of the Weyl characters χ_λ. Throughout we consider everything as formal series of the form $c_\lambda e^\lambda + \sum_{\mu<\lambda} c_\mu e^\mu$ (the c_λ being constants), where we refer to $c_\lambda e^\lambda$ as the leading term.

$$\chi_\lambda^{(k)} = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+k\rho)}}{\prod_{\alpha\in R_+}(e^{k\alpha/2}-e^{-k\alpha/2})}$$

This is a formal series rather than a polynomial. The numerator has leading order term $e^{\lambda+k\rho}$, and the denominator has leading order term $e^{k\rho}$, so it is easy to see that the $\chi_\lambda^{(k)}$ (for $\lambda \ge 0$) have leading term e^λ and so are related to the e^λ by a triangular matrix. Thus, it will be enough to consider $D_\pi(\chi_\lambda^{(k)})$ to prove triangularity and calculate the eigenvalues. So, we write

$$D_\pi(\chi_\lambda^{(k)}) = \sum_{w' \in W} \left[w' \left(\prod_{\alpha \in R_+, (\pi,\alpha)=1} \frac{1 - q^{2k} e^\alpha}{1 - e^\alpha} \right) \right] \left[w'(T_\pi(\chi_\lambda^{(k)})) \right]$$

We calculate the leading order term of each set of square brackets separately. This means we will write it as a constant times e^μ plus terms indexed by weights lower than μ in the dominance order. We first consider the action of w' on each factor of

$$\left(\prod_{\alpha \in R_+, (\pi,\alpha)=1} \frac{1 - q^{2k} e^\alpha}{1 - e^\alpha} \right) = \left(\prod_{\alpha \in R_+} \frac{1 - q^{2k(\pi,\alpha)} e^\alpha}{1 - e^\alpha} \right)$$

The action of w' on

$$\frac{1 - q^{2k(\pi,\alpha)} e^\alpha}{1 - e^\alpha}$$

is

$$\frac{1 - q^{2k(\pi,\alpha)} e^{w'(\alpha)}}{1 - e^{w'(\alpha)}}$$

If $w'(\alpha)$ is negative root, the leading term is just 1, otherwise we may write this as

$$\frac{q^{2k(\pi,\alpha)} - e^{-w'(\alpha)}}{1 - e^{-w'(\alpha)}}$$

whence the leading term is clearly $q^{2k(\pi,\alpha)}$. Thus the total contribution to the leading order term is $q^{(\pi,2k\nu)}$, where ν is the sum of all positive roots α such that $w'(\alpha)$ is also positive.

Upon applying T_π to $\chi_\lambda^{(k)}$, we get

$$T_\pi(\chi_\lambda^{(k)}) = \frac{\sum_{w \in W} \varepsilon(w) q^{2(\pi, w(\lambda + k\rho))} e^{w(\lambda+k\rho)}}{\prod_{\alpha \in R_+} (q^{(\pi,k\alpha)} e^{k\alpha/2} - q^{(\pi,-k\alpha)} e^{-k\alpha/2})}$$

If the Weyl group element w' is applied to this expression, we get

$$w'(T_\pi(\chi_\lambda^{(k)})) = \frac{\sum_{w \in W} \varepsilon(w) q^{2(\pi, w(\lambda + k\rho))} e^{w'(w(\lambda+k\rho))}}{\prod_{\alpha \in R_+} w'(q^{(\pi,k\alpha)} e^{k\alpha/2} - q^{(\pi,-k\alpha)} e^{-k\alpha/2})}$$

We expand the denominator as a series of the form e^μ plus lower order terms. To extract the leading order term from $\prod_{\alpha \in R_+} w'(q^{(\pi,k\alpha)} e^{k\alpha/2} - q^{(\pi,-k\alpha)} e^{-k\alpha/2})$, we first note that we pick up a sign for each α mapped to a negative root, thus obtaining $\varepsilon(w')$. We pick the term corresponding to the positive root in each $w'(\alpha)/2, -w'(\alpha)/2$ pair, obtaining the following:

$$\varepsilon(w') e^{k\rho} q^{(\pi,k\nu')}$$

Here $\nu' = \sum_{\alpha \in R_+} \sigma(w'(\alpha))\alpha$, where $\sigma(\alpha)$ is the sign of a root. Clearly $2\nu - \nu' = 2\rho$. So our leading term so far is $q^{(\pi,2k\rho)}/\varepsilon(w')e^{k\rho}$. The leading term of the numerator arises when $w = (w')^{-1}$, when we get $\varepsilon((w')^{-1}) q^{2(\pi,(w')^{-1}(\lambda+k\rho))} e^{\lambda+k\rho}$. Taking the product of these (and noting that $\varepsilon(w') = \varepsilon((w')^{-1})$), we obtain

$$q^{2(\pi,(w')^{-1}(\lambda+k\rho))} q^{(\pi,k(2\rho))} e^\lambda$$

Finally, we sum over $w' \in W$ to get the coefficient

$$q^{2(\pi,k\rho)} \sum_{w \in W} q^{2(\pi,w(\lambda+k\rho))}$$

These are not necessarily distinct for distinct $\lambda \in P_+$. However, one can find a suitable coweight π in types A and B (and E_6 and E_7). In type D it is possible to find a linear combination of the operators D_π with this property. We demonstrate the case of type A below. $\square$

Example 3.6. *Suppose that R is the root system A_{n-1} as before. Then the positive roots are $e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \le i < j \le n$. All coefficients are zero or one, so it is clear that the fundamental weights ω_i are minuscule coweights (note that $\alpha = \alpha^\vee$ for all $\alpha \in R$). Additionally $\rho = (n-1, n-3, \cdots, 1-n)$.*

We are given (a constant independent of λ multiplied by) $\sum_{w \in W} q^{2(\pi, w(\lambda+k\rho))}$, which is equivalent to knowing the multiset of values $(w^{-1}(\pi), \lambda + k\rho)$ as w ranges across W. In general, this does not determine λ, but for certain π, it does. For example, in type A_3, we may choose $\lambda_1 = (4, 0, -2, -2)$ and $\lambda_2 = (2, 2, 0, -4)$, so that $\lambda_1 + k\rho = (3k + 4, k, -k - 2, -3k - 2)$ and $\lambda_2 + k\rho = (3k + 2, k + 2, -k, -3k - 4)$. Then we see that $\pi_2 = (1/2, 1/2, -1/2, -1/2)$ is a fundamental (co)weight. But, one can check that in both cases, the multiset of values of $(w^{-1}(\pi_2), \lambda_i + k\rho)$ is $\{4k + 4, 2k + 2, 2, -2, -2k - 2, -4k - 4\}$. However, if π_1 is used instead, it is easy to restrict λ from the multiset $(w^{-1}(\pi_1), \lambda + k\rho)$. For example, we may add any multiple of $(1, 1, \cdots, 1)$ to π_1 without affecting the inner product, allowing us to assume $\pi_1 = (1, 0, 0, \cdots, 0)$. Thus, the inner product gives the coordinates of the vector $\lambda + k\rho$. Since these are strictly decreasing, they determine $\lambda + k\rho$, and hence λ.

Proposition 3.7. *The D_π operators commute.*

Proof. Let D be the operator with distinct eigenvalues that was used to construct the Macdonald Polynomials. For π a minuscule coweight, consider $D_\pi + cD$, where $c \in \mathbb{C}_{q,t}$. Since D has distinct eigenvalues, this linear combination has district eigenvalues for generic c. This means that this linear combination of operators is diagonalisable, and as before, its eigenvectors are the Macdonald polynomials. Since the Macodonald polynomials are unique, this means that D and $D_\pi + cD$ are diagonalisable with the same eigenbasis. We conclude that D_π is diagonalisable, with Macdonald polynomials as eigenvectors. This means that in the basis of Macdonald polynomials, the D_π are diagonal operators, and hence commute. □

Example 3.8. *Continuing with R being of type A_{n-1} as in example 3.5, we recall that the positive roots correspond to $\frac{x_i}{x_j}$ with $i < j$. In this setting, T_{ω_r} can be taken to send x_i to $q^2 x_i$ if $i \le r$ and to x_i otherwise. In this way $\frac{x_i}{x_{i+1}}$ is unchanged unless $i = r$, in which case it is multiplied by q^2, which is the correct action. In fact, this makes it easy to write down explicit formulae for D_{ω_r} (acting on $\mathbb{C}_{q,t}[P]^W$) in terms of the "shift operators" T_i. If $I \subset \{1, 2, \cdots, n\}$, we write $T_I = \prod_{i \in I} T_i$ (the order of composition is unimportant since these operators clearly commute). Using equation 1, we have*

$$\begin{aligned}
D_{\omega_r} &= \sum_{w \in S_n} w \left(\prod_{1 \le i \le r < j \le n} \frac{1 - q^{2k} \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} T_{\{1,2,\cdots,r\}} \right) \\
&= \sum_{w \in S_n} w \left(\prod_{1 \le i \le r < j \le n} \frac{x_j - q^{2k} x_i}{x_j - x_i} T_{\{1,2,\cdots,r\}} \right) \\
&= r!(n-r)! \sum_{I \subset \{1,2,\cdots,n\}, |I| = r} \left(\prod_{i \in I, j \notin I} \frac{x_j - q^{2k} x_i}{x_j - x_i} \right) T_I
\end{aligned}$$

Here we have used the fact that S_n acts transitively on r-element subsets of $\{1, 2, \cdots, n\}$ with stabiliser of size $r!(n-r)!$. It is easy to see that these commute.

Remark 3.9. *The proof in* [Mac00] *begins by introducing the concept of a "quasi-minuscule weight".*

3.2. Macdonald Polynomials via AHA/DAHA.

We begin by describing a collection of operators that generalise the construction of the first section, without relying on minuscule weights. For the moment, we return to the case of t unspecialised. We recall that $\mathcal{H}_{q,t}$ has a representation $\mathbb{C}_{q,t}[X]$ where π_r acts as π_r and the action of T_i is given by

$$T_i = t_i s_i + (t_i - t_i^{-1}) \frac{s_i - 1}{X^{-\alpha_i} - 1}$$

Here, the meaning of the expression is exactly the same as in previous lectures (the action of $s_i - 1$ on any X^λ yields an element divisible by $X^{-\alpha_i} - 1$). Note that $p \in \mathbb{C}_{q,t}[X]$ is W invariant precisely when it is annihilated by each $T_i - t_i$ (for the action of the latter element is to first apply $s_i - 1$, and then do some divisions and multiplications). This action will be of primary importance for our construction of Macdonald polynomials in $\mathbb{C}_{q,t}[X]^W = \mathbb{C}_{q,t}[P]^W$.

We note also that if we have $T_i = s_i G(\alpha_i)$, for

$$G(\alpha) = t + (t - t^{-1})\frac{1 - s_i}{X^\alpha - 1} = \frac{tX^\alpha - t^{-1}}{X^\alpha - 1} - \frac{(t - t^{-1})s_\alpha}{X^\alpha - 1}$$

Here s_α is the reflection associated to the root α. These $G(\alpha)$ satisfy the property that for $wG(\alpha)w^{-1} = G(w(\alpha))$ for $w \in W^{ae}$. We introduce these elements for the following important triangularity fact.

Definition 3.10. *For $\lambda, \mu \in P^\vee$, say that $\lambda \succ \mu$ if $\lambda^+ > \mu^+$ (i.e. $\lambda^+ - \mu^+ \in Q_+$) or $\lambda^+ = \mu^+$ and $\lambda > \mu$. When we need to indicate lower order terms, we write l.o.t..*

Proposition 3.11. *For $\lambda \in P^\vee$ and $\mu \in P_+$, we have:*

$$Y^\lambda X^\mu = \sum_{\mu \succeq \nu} c_{\nu,\mu} X^\nu$$

Moreover,

$$c_{\mu,\mu} = q^{(\lambda, \mu + k\rho)}$$

Proof. Clearly it is enough to prove this for $\lambda \in P_+^\vee$ for otherwise we may write it as the difference of two dominant coweights, and compose a triangular operator corresponding to one, with the inverse of the triangular operator corresponding to the other. Choose a reduced expression for $t_\lambda \in W^{ae}$ of the form $\pi_r s_{i_1} \cdots s_{i_r}$, so that $Y^\lambda = \pi_r T_{i_1} \cdots T_{i_r}$. Then we have:

$$\begin{aligned} Y^\lambda &= \pi_r T_{i_1} \cdots T_{i_{r-1}} T_{i_r} \\ &= \pi_r s_{i_1} G(\alpha_{i_1}) \cdots s_{i_{r-1}} G(\alpha_{i_{r-1}}) s_{i_r} G(\alpha_{i_r}) \\ &= \pi_r s_{i_1} G(\alpha_{i_1}) \cdots s_{i_{r-1}} s_{i_r} G(s_{i_r}(\alpha_{i_{r-1}})) G(\alpha_{i_r}) \\ &= \cdots \\ &= \pi_r s_{i_1} \cdots s_{i_{r-1}} s_{i_r} G(\alpha^{(1)}) \cdots G(\alpha^{(r)}) \\ &= t_\lambda G(\alpha^{(1)}) \cdots G(\alpha^{(r)}) \end{aligned}$$

Here $t_\lambda(X_\mu) = q^{2(\lambda,\mu)}$, and we see that the $\alpha^{(i)}$ that arise are precisely the positive roots that t_λ^{-1} maps to negative roots, i.e. $\{\alpha + k'\delta \mid \alpha \in R_+, 0 \le k' < (\lambda, \alpha)\}$.

Note that the definition of $G(\alpha + k')$ gives us the following:

$$G(\alpha + k')X^\mu = \begin{cases} tX^\mu + \text{l.o.t.} & (\mu, \alpha^\vee) \ge 0 \\ t^{-1}X^\mu + \text{l.o.t.} & (\mu, \alpha^\vee) < 0 \end{cases}$$

Since $\mu \in P_+$, and $\alpha \in R_+$, we are in the first case. It is clear that the leading order term picks up a factor of $t = q^k$, (λ, α) times for each $\alpha \in R_+$. So the total contribution is $q^{2(\lambda,\rho)}$ which gives the required formula when we include $q^{2(\lambda,\mu)}$ coming from t_λ. $\square$

Proposition 3.12. *If $f(Y) \in \mathbb{C}_{q,t}[Y]^W$, thought of as a central element of the affine Hecke algebra $H^a = H\mathbb{C}_{q,t}[Y]$, then $f(Y)$ preserves the space $\mathbb{C}_{q,t}[X]^W$.*

Proof. It is enough to show that for $p \in \mathbb{C}_{q,t}[X]^W$, $(T_i - t_i)f(Y)p = 0$. But $f(Y)$ and $T_i - t_i$ are elements of $H^a = H\mathbb{C}_{q,t}[Y]$ in which the former is central, which means they commute. So $(T_i - t_i)f(Y)p = f(Y)(T_i - t_i)p = f(Y) \cdot 0 = 0$. $\square$

Since $W^{ae} = P^\vee \rtimes W$, the action of any $w \in W^{ae}$ in $\mathbb{C}_{q,t}[X]$ may be written as

$$\sum_{w \in W, \lambda \in P^\vee} g_{\lambda,w} t_\lambda w$$

here we recall that $t_\lambda(X^\mu) = q^{2(\lambda,\mu)}X^\mu$. The $g_{\lambda,w}$ are rational functions in the X^α, whose denominators are products of terms of the form $X^{\alpha_i} - 1$, for α_i simple roots.

Definition 3.13. *Define the restriction of the action of T_w for $w \in W^{ae}$ via*

$$Res(\sum_{w\in W,\lambda\in P^\vee} g_{\lambda,w}t_\lambda w) = \sum_{w\in W,\lambda\in P^\vee} g_{\lambda,w}t_\lambda$$

(that is, omit the Weyl group action in each term). This is clearly a linear operation. For $f(Y) \in \mathbb{C}_{q,t}[Y]^W$, we define $L_f = Res(f)$.

Proposition 3.14. *The L_f , for $f \in \mathbb{C}_{q,t}[Y]^W$ are W-invariant commuting operators on $\mathbb{C}_{q,t}[X]$.*

Proof. The W-invariance is clear because for $f \in \mathbb{C}_{q,t}[Y]^W$, because the action of $w \in W$ on $Res(f)$ can be obtained by restricting the action of w on f (which is invariant).

Consider the action of f as $\sum_{w\in W,\lambda\in P^\vee} f_{\lambda,w}t_\lambda w$, and that of g in the form $\sum_{w'\in W,\mu\in P^\vee} g_{\mu,w'}t_\mu w'$. Thus, since g is W-invariant, fg acts as

$$\sum_{w\in W,\lambda\in P^\vee} f_{\lambda,w}t_\lambda w \sum_{w'\in W,\mu\in P^\vee} g_{\mu,w'}t_\mu w' = \sum_{w\in W,\lambda\in P^\vee} f_{\lambda,w}t_\lambda \sum_{w'\in W,\mu\in P^\vee} g_{\mu,w'}t_\mu w'w$$

Taking the restriction of this, we obtain $L_f Lg$. Hence $L(f)L(g) = L(fg)$. But $fg = gf$, so we also get $L(fg) = L(gf) = L(g)L(f)$ (where the last step uses the W-invariance of f). □

By proposition 3.11, we see that the L_f ($f \in \mathbb{C}_{q,t}[Y]^W$) are actually triangular operators on $\mathbb{C}_{q,t}[X]^W$, with respect to the m_μ basis. In particular, we have $L_f(m_\mu) = f(q^{2(\mu+k\rho)})m_\mu +$ l.o.t.. This notation means that each Y^λ in f should be replaced with the scalar $q^{2(\lambda,\mu+k\rho)}$. One easily checks that the eigenvalues $\sum_{\nu\in W\lambda} q^{2(\nu,\mu)}$ determine μ (as $\lambda \in P^\vee_+$ varies). Since the L_f commute, we easily see they form a family of simultaneously diagonalisable operators. We thus obtain:

Lemma 3.15. *The operators are L_f are diagonalisable triangular operators on $\mathbb{C}_{q,t}[X]^W$, with respect to the m_μ basis. The eigenvalues of L_f are $f(q^{2(\mu+k\rho)})$. Moreover the eigenvector associated to this may be taken to be of the form $m_\mu +$ l.o.t..*

Remark 3.16. *It can be shown that the operators D_π we constructed for minuscule weights π are the L_f corresponding to $\sum_{w\in W} f^{w(\pi)}$ up to a scalar multiple. We describe the proof. Consider the ordering defined by $\mu \sqsubseteq \lambda$ when $\mu^+ < \lambda^+$, or $\mu^+ = \lambda^+$ and $\mu \geq \lambda$ (note that the last inequality is opposite to that in the definition of $\succeq$). The leading order term of Y^λ turns out to be*

$$\left(\prod_{\alpha\in R^a_-\cap t_\lambda R^a_+} \frac{tX^\alpha - t^{-1}}{X^\alpha - 1}\right) t_\lambda$$

One can check that the leading order term of our operator is of the form $g(X)t_{\pi^-}$, where π^- is the anttdominant coweight in the orbit of π. We see that this can only arise from the term Y^{π^-}, where the coefficient is

$$|W_\pi| \prod_{\alpha\in R,(\alpha,\pi^-)=1} \frac{tX^\alpha - t^{-1}}{X^\alpha - 1} t_{\pi^-}$$

Here, W_π is the stabiliser of π. Since π is minuscule, there are no dominant weights below it, so the remainder of the operator is determined by conjugacy considerations. This gives that

$$Res(\sum_{w\in W} Y^{w(\pi)}) = \sum_{w\in W} w \left(\prod_{\alpha\in R,(\alpha,\pi^-)=1} \frac{tX^\alpha - t^{-1}}{X^\alpha - 1} t_\pi\right)$$

This agrees with D_π up to a scalar.

Independently of the previous remark, we may deduce that the eigenvectors of L_f are actually Macdonald polynomials, for which we will use the same self-adjointness argument that we did before. For this, we introduce a bilinear form.

Definition 3.17. *Define the $\mathbb{C}_{q,t}$-linear involution ι of $\mathbb{C}_{q,t}(q)$ by $\iota(q) = q^{-1}$, and extend it to $\mathbb{C}_{q,t}[X]$ by declaring $\iota(X^\mu) = X^\mu$. Then, definite the bilinear form $\langle -, \rangle'_k$ on $\mathbb{C}_{q,t}[X]$ as follows.*

$$\langle f, g\rangle'_k = [f\mu_k\iota(\bar{g})]_0$$

Similarly to before, $[h]_0$ is the coefficient of $X^0 = 1$ in h, written in the X^μ basis, and μ_k is defined as

$$\mu_k = \prod_{\alpha\in R_+}\prod_{i=1-k}^{k}(q^iX^{\alpha/2} - q^{-i}X^{-\alpha/2})$$

This inner product is neither symmetric, nor W-invariant, but it happens to be exactly the right definition to help prove the Macdonald conjecture.

Definition 3.18. *We define the following quantity, which will be important.*

$$\varphi_k = \prod_{\alpha\in R_+}(q^kX^{\alpha/2} - q^{-k}X^{-\alpha/2})$$

Note that $\varphi_0 = \delta$, the Weyl denominator.

This means that we have

$$\mu_k = (-1)^{k|R_+|}q^{-k(k-1)|R_+|}\Delta_k\frac{\varphi_k}{\delta}$$

Here $\Delta_k = \Delta_{q,t}$ (from the first section, but we identify e^α with X^α), but we emphasise the dependence on k. We make some observations

(1) $\bar{\mu_k} = \iota(\mu_k)$, so $\langle f, g\rangle'_k = \iota(\langle g, f\rangle'_k)$.
(2) At $q = 1$, $\langle -, -\rangle'_k = \pm\langle -, -\rangle_{q,t}$, in particular, the form $\langle -, -\rangle'_k$ is generically non degenerate.

Proposition 3.19. *If $f, g \in \mathbb{C}_{q,t}[X]^W$, then*

$$\langle f, g\rangle'_k = (-1)^{k|R_+|}q^{-k(k-1)|R_+|}\langle f, \iota(g)\rangle_{q,t}d_k$$

where

$$d_k = q^{k|R_+|}\sum_{w\in W}q^{-2k|R_+\cap w^{-1}(R_-)|}$$

Proof. We observe that $[h]_0 = [w(h)]_0$ for any $h \in \mathbb{C}_{q,t}[X]$. It's enough to compute

$$\begin{aligned}
[f\overline{\iota(g)}\mu_k]_0 &= [f\overline{\iota(g)}\Delta_k\frac{\varphi_k}{\delta}]_0 \\
&= \frac{1}{|W|}\sum_{w\in W}[w(f\overline{\iota(g)}\Delta_k\frac{\varphi_k}{\delta})]_0 \\
&= \frac{1}{|W|}\sum_{w\in W}[f\overline{\iota(g)}\Delta_k w(\frac{\varphi_k}{\delta})]_0
\end{aligned}$$

Here we have used W-invariance of f, g, Δ_k. Note that $\sum_{w\in W}w(\frac{\varphi_k}{\delta}) = \frac{1}{\delta}\sum_{w\in W}\varepsilon(w)w(\varphi_k)$. It is clear that multiplying by δ gives a W-antiinvariant polynomial, hence something divisible by δ. Therefore this quantity is a polynomial. It is

$$\sum_w \in W\prod_{\alpha\in R_+}\frac{q^{k\epsilon_\alpha}X^{\alpha/2} - q^{-k\epsilon_\alpha}X^{-\alpha/2}}{X^{\alpha/2} - X^{-\alpha/2}}$$

where ϵ_α is 1 if $w(\alpha) \in R_+$ and -1 otherwise. The leading order term of this is a constant multiple of X^0, so the whole quantity must be scalar. That scalar is

$$\sum_{w\in W}\prod_{\alpha\in R_+}q^{k\epsilon_\alpha} = q^{k|R_+|}\sum_{w\in W}q^{-2k|R_+\cap w^{-1}(R_-)|}$$

Combining this with the scalar factors relating μ_k and Δ_k gives the desired result. ☐

We now prove some statements about Macdonald polynomials before we embark on the proof of their existence.

Proposition 3.20. *We have:*

(1) $\iota(P_\lambda) = P_\lambda$
(2) $\langle P_\mu, P_\nu\rangle'_k = 0$ *if* $\mu \neq \nu$
(3) *The Macdonald polynomials are uniquely defined by* $P_\lambda = m_\lambda + l.o.t.$ *and the above orthogonality property.*

Proof.

$$\Delta_k = \prod_{\alpha\in R_+}\prod_{i=0}^{k-1}(-q^{2i}X^\alpha + 1 + q^{4i} - q^{2i}X^{-\alpha})$$

This immediately implies that $\iota(\Delta_k) = q^{-4k(k-1)|R_+|}\Delta_k$, which gives that $[P_\mu\Delta_k P_\nu]_0 = 0$ implies $[\iota(P_\mu)\Delta_k\iota(P_\nu)]_0 = 0$. So, $\iota(P_\lambda)$ satisfy the definition of the Macdonald polynomials, hence they equal P_λ by uniqueness (so the first statement follows). Since $\langle -,-\rangle'_k$ and $\langle -,-\rangle_{q,t}$ agree up to a scalar, the second statement is true. The third statement follows from the nondegeneracy of $\langle -,-\rangle'_k$. □

Definition 3.21. *For an operator h from $\mathbb{C}_{q,t}$ to itself, define its adjoint h^* (with respect to $\langle -,-\rangle'_k$) in the usual way:*

$$\langle hf, g\rangle'_k = \langle f, h^* g\rangle'_k$$

We also define $h^\dagger$ via $[h(f)\overline{\iota(g)}]_0 = [f\overline{\iota(h^\dagger(g))}]_0$.

Proposition 3.22. *We have the following:*

(1) $h^* = \mu_k^{-1}h^\dagger\mu_k$.
(2) *If $p \in \mathbb{C}_{q,t}[X]$ is identified with the operator of multiplication by p, then $p^\dagger = \overline{\iota(p)}$.*
(3) *For $w \in W^{ae}$, $w^\dagger = w^{-1}$.*
(4) $T_i^\dagger = T_i^{-1}$
(5) $(Y^\lambda)^\dagger = Y^{-\lambda}$

Proof. The first statement is clear from the definition of the bilinear form. The second statement is trivial. The third statement follows from W-invariance of X^0. To prove the fourth statement, we note that $\overline{\iota(t)} = t^{-1}$, since $\iota(q) = q^{-1}$ and $t = q^k$. As $T_i^{-1} = T_i - (t - t^{-1})$, it is enough to show $(T_i - t)^* = T_i - t$. For this, we have:

$$s_i^* = \mu_k^{-1}s_i\mu_k = -\varphi_k s_i\varphi_k = \frac{q^{-k}X^{\alpha_i/2} - q^kX^{-\alpha_i/2}}{q^kX^{\alpha_i/2} - q^{-k}X^{-\alpha_i/2}}s_i$$

where we have used the fact that s_i permutes the set of positive roots different from α_i, so that only the factor corresponding to α_i in the definition of ϕ_k is relevant. One easily checks that

$$T_i - t = \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}}(s_i - 1)$$

This is something we know how to take the adjoint of

$$\begin{aligned}
(T_i - t)^* &= (s_i^* - 1)\left(\frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}}\right)^* \\
&= \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{t^{-1}X^{-\alpha_i/2} - tX^{\alpha_i/2}}\frac{tX^{\alpha_i/2} - t^{-1}X^{-\alpha_i/2}}{X^{\alpha_i/2} - X^{-\alpha_i/2}}s_i - \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}} \\
&= \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}}(s_i - 1) \\
&= T_i - t
\end{aligned}$$

To prove the fifth part, it is enough to consider $\lambda \in P_+^\vee$, for which the previous part suffices, together with $\pi_r^* = \pi_r^{-1}$ (as Y^λ is a product of π_r and T_i). We already know that $\pi_r^\dagger = \pi_r^{-1}$, so this is equivalent to the statement that π_r preserves μ_k. □

Corollary 3.23. *The mapping from $\mathbb{C}_{q,t}[Y]^W$ to itself defined by $f(q)Y^\lambda \mapsto \iota(f(q))Y^{-\lambda}$ is an involution which agrees with the $*$-adjoint. Therefore $h \mapsto h^*$ is an involution on $\mathbb{C}_{q,t}[Y]^W$.*

Theorem 3.24. *For $f \in \mathbb{C}_{q,t}[Y]^W$, the Macdonald polynomial P_λ is an eigenvector of L_f with eigenvalue $f(q^{2(\lambda+k\rho)})$.*

Proof. We already know eigenvectors with the stated eigenvalues exist, and it suffices to check that they satisfy the definition of Macdonald polynomials. By proposition 3.20, it is enough to check orthogonality (we have already seen triangularity). Observe that for $f = m_\nu \in \mathbb{C}_{q,t}[Y]^W$, $f^* = m_{-\nu^-} = m_{-w_0\nu}$. We check

$$\begin{aligned}\sum_{\eta\in W\nu} q^{2(\eta,\lambda+k\rho)}\langle P_\lambda, P_\nu\rangle'_k &= \langle L_f P_\lambda, P_\nu\rangle'_k \\ &= \langle P_\lambda, L_f^* P_\nu\rangle'_k \\ &= \iota(\sum_{\eta\in -Ww_0\nu} q^{2(\eta,\mu+k\rho)})\langle P_\lambda, P_\nu\rangle'_k \\ &= \sum_{\eta\in W\nu} q^{2(\eta,\mu+k\rho)}\langle P_\lambda, P_\nu\rangle'_k\end{aligned}$$

Since we already know these eigenvalues are distinct (for suitably chosen f), this shows that the P_λ are orthogonal, and the L_f are self-adjoint. □

Corollary 3.25. *The eigenvectors of the L_f operators ($f \in \mathbb{C}_{q,t}[Y]^W$), P_λ, satisfy the definition of Macdonald polynomials. Therefore Macdonald polynomials exist.*

4. The Macdonald Conjecture

This section is dedicated to the proof the following theorem (the Macdonald Conjecture), using DAHA.

Theorem 4.1. *Let P_λ be the Macdonald polynomial associated to $\lambda \in P$, and let $t_\alpha = q^k$. Then*

$$\langle P_\lambda, P_\lambda\rangle_{q,t} = \prod_{\alpha\in R_+}\prod_{i=1}^{k-1}\frac{1-q^{2(\alpha^\vee,\lambda+k\rho)+2i}}{1-q^{2(\alpha^\vee,\lambda+k\rho)-2i}}$$

If we write $[n] = \frac{q^n-q^{-n}}{q-q^{-1}}$ for the "quantum number n", this may also be written

$$q^{|R|\frac{k(k-1)}{2}}\prod_{\alpha\in R_+}\prod_{i=1}^{k-1}\frac{[(\alpha^\vee,\lambda+k\rho)+i]}{[(\alpha^\vee,\lambda+k\rho)-i]}$$

Remark 4.2. *If $k = 1$, the $P_\lambda = \chi_\lambda$ (the Weyl character); we already know these are orthonormal with respect to $\langle -,-\rangle_{q,t}$ (see example 3.4), which is consistent with the theorem.*

Remark 4.3. *The case where $\lambda = 0$, i.e.*

$$\frac{1}{|W|}[\prod_{\alpha\in R}\prod_{i=0}^{k-1}(1-q^{2i}e^\alpha)]_0 = q^{|R|\frac{k(k-1)}{2}}\prod_{\alpha\in R_+}\prod_{i=1}^{k-1}\frac{[(\alpha^\vee,k\rho)+i]}{[(\alpha^\vee,k\rho)-i]}$$

is known as the Macdonald constant term conjecture. This special case of the conjecture was not known until the general case was resolved.

Firstly, note that

$$\begin{aligned}\langle P_\lambda, P_\lambda\rangle_{q,t} &= d_k(-1)^{k|R_+|}q^{-k(k-1)|R_+|}\langle P_\lambda, \iota(P_\lambda)\rangle'_k \\ &= d_k(-1)^{k|R_+|}q^{-k(k-1)|R_+|}\langle P_\lambda, P_\lambda\rangle'_k\end{aligned}$$

so it is sufficient to find $\langle P_\lambda, P_\lambda\rangle'_k$. Actually we will do this inductively with k, so it will be necessary to distinguish P_λ associated to different k. Therefore we will write $P_\lambda^{(k)}$ for the Macdonald polynomial indexed by λ for $t = q^k$.

Definition 4.4. *We need the following quantities.*

(1) $\mathcal{X} = \varphi_{-k} = \prod_{\alpha\in R_+}(q^{-k}X^{\alpha/2} - q^kX^{-\alpha/2})$
(2) $\mathcal{Y} = \varphi_{-k}^\vee = \prod_{\alpha\in R_+}(q^{-k}Y^{\alpha^\vee/2} - q^kY^{-\alpha^\vee/2})$
(3) $\hat{\mathcal{Y}} = \varphi_k^\vee = \prod_{\alpha\in R_+}(q^kY^{\alpha^\vee/2} - q^{-k}Y^{-\alpha^\vee/2})$
(4) $G = \mathcal{X}^{-1}\mathcal{Y}$

(5) $\hat{G} = \hat{\mathcal{Y}}\mathcal{X}$
(6) $\mathcal{P} = \frac{1}{|W|}\sum_{w\in W} w$
(7) $\mathcal{P}_- = \frac{1}{|W|}\sum_{w\in W} \varepsilon(w)w$
(8) $\mathcal{P}_-^q = \frac{1}{\sum_{w\in W} t^{-2l(w)}}\sum_{w\in W}(-t)^{-l(w)}T_w$

Lemma 4.5. *We have the following (checking these is trivial):*

(1) $\iota(\mathcal{X}) = (-1)^{|R_+|}\overline{\mathcal{X}} = \varphi_k$
(2) $\mathcal{X}^* = (-1)^{|R_+|}\mathcal{X}$
(3) $\mathcal{Y}^* = (-1)^{|R_+|}\mathcal{Y}$
(4) $\hat{\mathcal{Y}}^* = (-1)^{|R_+|}\hat{\mathcal{Y}}$
(5) $\mathcal{P}_-^+|_{q=t=1} = \mathcal{P}_-$

We first prove the statement that $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ vanishes on $\mathbb{C}_{q,t}[X]^W$.

4.1. **Properties of Symmetrisers and Antisymmetrisers.**

Lemma 4.6. *Let V be a finite dimensional representation of W and let $V' = \sum \ker(1 - s_i)$. Then V' is W-invariant.*

Proof. It is enough to check that $s_i \ker(1 - s_j) \subseteq \ker(1 - s_i) + \ker(1 - s_j)$. If $v \in s_i \ker(1 - s_j)$. then $s_i v \in \ker(1 - s_j)$, so in particular, $s_j(s_i v) = s_i v$. Writing $v_\pi = \frac{1}{2}(v \pm s_i v)$, so that $v = v_+ + v_-$, $s_i v = v_+ - v_- \in \ker(1 - s_j)$. But $v_+ \in \ker(1 - s_i)$, so we are done. ☐

Corollary 4.7. *In the previous lemma, V' is the sum of isotypic components of non-sign representations of W.*

Proof. It is enough to check this on irreducible representations, in which case V' is either zero or all of V. It is zero if and only if each $\ker(1 - s_i)$ is empty, i.e. each s_i acts with eigenvalue -1 only (since $s_i^2 = 1$, the eigenvalues of its action must be ± 1). But then s_i acts as minus the identity, so V is a direct sum of sign representations of W. ☐

Corollary 4.8. *We conclude that* $\ker(\mathcal{P}_-) = \sum_i \ker(1 - s_i)$.

Proposition 4.9. *We have that $\mathcal{P}_-$ is divisible by $T_i - t$ on both sides. Also, when acting on $\mathbb{C}_{q,t}[X]$, $\ker(\mathcal{P}_-^q) = \ker(\mathcal{P}_-)$, and $Im(\mathcal{P}_-^q) = \mathbb{C}_{q,t}[X]^{-W}$ (antiinvariants). Analogous statements are true for the action on $\mathbb{C}_{q,t}[Y]$.*

Proof. To see the first statement, break W into cosets of $\{1, s_i\}$ (left or right cosets, according to which divisibility condition one wishes to prove). Choosing minimal length coset representatives T_w, we obtain that $\mathcal{P}_-^q$ is a linear combination of terms of the form $(-t)^{-l(w)}T_w(1 - t^{-1}T_i)$ (this is the right divisibility case). This is clearly divisible by $T_i - t$. To see the kernel and image of $\mathcal{P}_-^q$, first observe that $\mathbb{C}_{q,t}[X]$ is filtered by finite dimensional spaces (this easily follows from the fact that W-orbits on P are finite). Note that the kernel contains the sum of the kernels of $T_i - t$, and that is the same as the sum of the kernels of $s_i - 1$ (recall that we observed that $\ker(T_i - t) = \ker(s_i - 1)$ when we showed that the intersection of these is the space of invariants). But when $q = 1$, $t = 1$ and $\mathcal{P}_-^q$ becomes $\mathcal{P}_-$ and we have equality of spaces. We must therefore have equality of spaces generically. To see the image statement, observe that because we have left divisibility, and noting that $T_i - t$ is left divisible by $s_i - 1$, multiplying by s_i changes the sign. Therefore the image of $\mathcal{P}_-^q$ is contained in $\mathbb{C}_{q,t}[X]^{-W}$ with equality at $q = 1$ (as in the kernel case). Therefore we have equality generically. The $\mathbb{C}_{q,t}[Y]$ case is essentially the same. ☐

Corollary 4.10. *The operator $\mathcal{P}_-^q$ is a projector. On its image (antiinvariants) T_i acts as $-t^{-1}$. Therefore T_w acts as $(-t)^{-l(w)}$. In particular, $\sum_{w\in W}(-t)^{-l(w)}T_w$ acts as $\sum_{w\in W} t^{-2l(w)}$. Dividing through by that scalar shows that $\mathcal{P}_-^q$ acts as the identity on antiinvariants.*

Theorem 4.11. *For $f \in \mathbb{C}_{q,t}[X]^W$, $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})f = 0$.*

Proof. We have shown that for $f \in \mathbb{C}_{q,t}[X]$, $\mathcal{P}_-^q(f) = 0$ if and only if $\mathcal{P}_-(f) = 0$. So it is enough to prove the statement with $\mathcal{P}_-$ replaced by $\mathcal{P}_-^q$. Let $R = \mathcal{P}_-^q(\mathcal{Y} - \hat{\mathcal{Y}}) \in H^{ae}$. Since $\mathbb{C}_{q,t}[X]$ is a faithful representation of H^{ae}, the statement that R annihilates all invariants is equivalent to R being of the form $\sum_i h_i(T_i - t)$ (for some $h_i \in H^{ae}$). This is because there is a PBW-style theorem for H^{ae} which implies that any element may be written as a linear combination of terms, each of which is Y^λ followed by a product of T_i. Since we can replace T_i by $(T_i - t) + t$, we may rewrite our element at Y^λ followed by a product of $T_i - t$. It is clear that a nonzero sum of Y^λ (with no $T_i - t$ factors following it) does not annihilate invariants. To prove this decomposition in H^{ae}, we may work in any faithful representation of H^{ae} and we choose to do so in $\mathbb{C}_{q,t}[Y]$. We now employ the same reasoning in reverse. The stated decomposition is equivalent to R annihilating $\mathbb{C}_{q,t}[Y]^W$. This is true if and only if $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ annihilates invariants. But $\mathcal{P}_-$ is divisible by $(1 + (-1)^{|R_+|}w_0)$ on both sides (recall that w_0 is the longest element of W which sends positive roots to negative roots and vice versa). It is easy to check that the following identities hold:

$$\begin{aligned} w_0(\mathcal{Y}) &= (-1)^{|R_+|}\hat{\mathcal{Y}} \\ w_0(\hat{\mathcal{Y}}) &= (-1)^{|R_+|}\mathcal{Y} \end{aligned}$$

Then it immediately follows that

$$(1 + (-1)^{|R_+|}w_0)(\mathcal{Y} - \hat{\mathcal{Y}}) = 0$$

This implies that $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ annihilates invariants, which is what we needed to prove. □

4.2. **The Proof.**

Lemma 4.12. *We have the following (as actions on $\mathbb{C}_{q,t}[X]$):*

$$\begin{aligned} (T + t^{-1})\mathcal{X} &= \frac{t^{-1}X^{-\alpha_i/2} - tX^{\alpha_i/2}}{t^{-1}X^{\alpha_i/2} - tX^{-\alpha_i/2}}\mathcal{X}(T_i - t) \\ (T + t^{-1})\mathcal{Y} &= \frac{t^{-1}Y^{-\alpha_i^\vee/2} - tY^{\alpha_i^\vee/2}}{t^{-1}Y^{\alpha_i^\vee/2} - tY^{-\alpha_i^\vee/2}}\mathcal{Y}(T_i - t) \\ (T - t)\hat{\mathcal{Y}} &= \frac{tY^{-\alpha_i^\vee/2} - t^{-1}Y^{\alpha_i^\vee/2}}{tY^{\alpha_i^\vee/2} - t^{-1}Y^{-\alpha_i^\vee/2}}\hat{\mathcal{Y}}(T_i + t^{-1}) \end{aligned}$$

Proof. Recall that T_i acts as $ts_i + (t - t^{-1})\frac{s_i - 1}{X^{-\alpha_i} - 1}$, which is of the form $As_i + B$ (where A, B commute with the X^λ). It is clear that these terms preserve $\prod_{\alpha \in R_+, \alpha \neq \alpha_i}(q^{-k}X^{\alpha_i} - q^kX^{-\alpha_i})$, so it is enough to consider the term $(q^{-k}X^{\alpha_i} - q^kX^{-\alpha_i})$ (the only factor of $\mathcal{X}$ not appearing in the previously mentioned product). One can verify the following equations.

$$\begin{aligned} T_iX^{\alpha_i/2} &= X^{-\alpha_i/2}T_i + (t - t^{-1})X^{\alpha_i/2} \\ X^{\alpha_i/2}T_i &= T_iX^{-\alpha_i/2} + (t - t^{-1})X^{\alpha_i/2} \end{aligned}$$

hence,

$$\begin{aligned} T_i(q^{-k}X^{\alpha_i} - q^kX^{-\alpha_i}) &= (q^{-k}X^{-\alpha_i} - q^kX^{\alpha_i})T_i + (q^{2k} - q^{-2k})X^{\alpha_i/2} \\ t^{-1}(q^{-k}X^{\alpha_i} - q^kX^{-\alpha_i}) &= q^{-2k}X^{\alpha_i} - X^{-\alpha_i} \end{aligned}$$

Summing the last two equations gives

$$(T + t^{-1})(q^{-k}X^{\alpha_i} - q^kX^{-\alpha_i}) = (q^{-k}X^{-\alpha_i} - q^kX^{\alpha_i})(T_i - t)$$

This proves the first equation. The others can be proven using similar computations. □

Corollary 4.13. *Since $\mathbb{C}_{q,t}[X]^W$ is the intersection of the kernels of $(T_i - t)$, and $\mathbb{C}_{q,t}[X]^{-W}$ is the the intersection of the kernels of $(T_i + t^{-1})$, the preceding lemma implies the following.*

(1) $\mathcal{X}\mathbb{C}_{q,t}[X]^W = \mathbb{C}_{q,t}[X]^{-W}$
(2) $\mathcal{Y}\mathbb{C}_{q,t}[X]^W \subseteq \mathbb{C}_{q,t}[X]^{-W}$
(3) $\hat{\mathcal{Y}}\mathbb{C}_{q,t}[X]^{-W} \subseteq \mathbb{C}_{q,t}[X]^W$

Proof. The only thing that has not yet been explained is why there is an equality instead of a one-way inclusion in the first statement. The reason for this is that when $q = 1$ we have equality (this reduces to the fact that every antisymmetric polynomial is divisible by the Weyl denominator), hence we have equality generically. $\square$

The point of the previous arguments is to deduce the following.

Corollary 4.14. *The operators $G = \mathcal{X}^{-1}\mathcal{Y}$ and $\hat{G} = \hat{\mathcal{Y}}\mathcal{X}$ preserve $\mathbb{C}_{q,t}[X]^W$.*

Proposition 4.15. *If $f, g \in \mathbb{C}_{q,t}[X]^W$, $\langle Gf, g\rangle'_{k+1} = \frac{d_{k+1}}{d_k}\langle f, \hat{G}g\rangle'_k$.*

Proof. Note that $\mu_{k+1} = \varphi_{k+1}\varphi_{-k}\mu_k = \varphi_{k+1}\mathcal{X}\mu_k$. Since $\mu_k\mathcal{X} = \prod_{\alpha\in R_+}\prod_{i=-k}^{k}(q^iX^{\alpha/2} - q^{-i}X^{-\alpha/2})$, we see that $\mu_k\mathcal{X}$ is antisymmetric (the $i = 0$ term is the Weyl denominator which is antisymmetric, but the product of the $i = \pm j$ gives a symmetric quantity). So,

$$\begin{aligned}
\mathcal{P}(\mu_{k+1}) &= \frac{1}{|W|}\sum_{w\in W} w(\varphi_{k+1}\varepsilon(w)\mathcal{X}\mu_k \\
&= \mathcal{P}_-(\varphi_{k+1})\mathcal{X}\mu_k \\
&= \frac{1}{|W|}d_{k+1}\delta\mathcal{X}\mu_k
\end{aligned}$$

Similar methods show

$$\mathcal{P}(\mathcal{X}^2\mu_k) = \frac{d_k}{|W|}\delta\mathcal{X}\mu_k$$

This gives the (important) equation

$$\mathcal{P}(\mu_{k+1}) = \frac{d_{k+1}}{d_k}\mathcal{P}(\mathcal{X}^2\mu_k)$$

We note that $[f]_0 = [\mathcal{P}f]_0$. Therefore, as $G(f)$ and $\overline{\iota(g)}$ are invariant,

$$\begin{aligned}
\langle Gf, g\rangle'_{k+1} &= [G(f)\overline{\iota(g)}\mu_{k+1}]_0 \\
&= [G(f)\overline{\iota(g)}\mathcal{P}(\mu_{k+1})]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{X}^{-1}\mathcal{Y}(f)\overline{\iota(g)}\mathcal{P}(\mathcal{X}^2\mu_k)]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{P}(\mathcal{X}^{-1}\mathcal{Y}(f)\overline{\iota(g)}\mathcal{X}^2\mu_k)]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{P}_-(\mathcal{Y}(f))\overline{\iota(g)}\mathcal{X}\mu_k]_0
\end{aligned}$$

In this last step we used the fact that $\mathcal{X}\mu_k$ is antisymmetric. We now use the fact that $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ vanishes on invariants. We may therefore replace the $\mathcal{Y}$ with $\hat{\mathcal{Y}}$.

$$\begin{aligned}
&= \frac{d_{k+1}}{d_k}[\mathcal{P}_-(\hat{\mathcal{Y}}(f))\overline{\iota(g)}\mathcal{X}\mu_k]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{P}(\hat{\mathcal{Y}}(f)\overline{\iota(g)}\mathcal{X}\mu_k)]_0 \\
&= \frac{d_{k+1}}{d_k}[\hat{\mathcal{Y}}(f)\overline{\iota(g)}\mathcal{X}\mu_k]_0 \\
&= \frac{d_{k+1}}{d_k}\langle\mathcal{X}\hat{\mathcal{Y}}f, g\rangle'_k \\
&= \frac{d_{k+1}}{d_k}\langle f, \hat{\mathcal{Y}}\mathcal{X}g\rangle'_k \\
&= \langle f, \hat{G}g\rangle'_k
\end{aligned}$$

In the second last step we used the adjoint formulae that we had noted previously in lemma 4.5. $\square$

Theorem 4.16. *We have the following equalities.*

$$G(P^{(k)}_{\lambda+\rho}) = q^{k|R_+|}c_k(\lambda)P^{(k+1)}_{\lambda}$$

(this is taken to be zero if $\lambda+\rho \notin P_+$)

$$\hat{G}(P^{(k+1)}_{\lambda}) = q^{-k|R_+|}\hat{c}_k(\lambda)P^{(k+1)}_{\lambda+\rho}$$

where

$$c_k(\lambda) = \prod_{\alpha\in R_+}(q^{-k+(\alpha^\vee,\lambda+(k+1)\rho)} - q^{k-(\alpha^\vee,\lambda+(k+1)\rho)})$$

$$\hat{c}_k(\lambda) = \prod_{\alpha\in R_+}(q^{k+(\alpha^\vee,\lambda+(k+1)\rho)} - q^{-k-(\alpha^\vee,\lambda+(k+1)\rho)})$$

Proof. Recalling that $Y^\lambda X^\mu = q^{2(\lambda,\mu+k\rho)}X^\mu + \text{l.o.t.}$, it is easy to check the triangularity condition $GP^{(k)}_{\lambda+\rho} = q^{k|R_+|}c_k(\lambda)m_\lambda + \text{l.o.t.}$. So, it suffices to check the triangularity condition $\langle GP^{k}_{\lambda+\rho}, m_\mu\rangle'_{k+1} = 0$ for $\mu < \lambda$. This is equivalent to $\langle P^{(k)}_{\lambda+\rho}, \hat{G}m_\mu\rangle'_k = 0$. But, similarly, we see that $\hat{G}m_\mu$ is a constant multiple of $m_{\mu+\rho}$ plus lower order terms, whence the conclusion follows from earlier properties of Macdonald polynomials. The second statement is proved similarly. $\square$

Definition 4.17. *Let*

$$M'_k(\lambda) = \langle P^{(k)}_\lambda, P^{(k)}_\lambda\rangle'_k$$

Also let

$$M_k(\lambda) = \langle P^{(k)}_\lambda, P^{(k)}_\lambda\rangle_{q,t} = d_k^{-1}(-1)^{k|R_+|}q^{|R_+|k(k-1)}M'_k(\lambda)$$

Note that we have the following (where we make use of the triangularity properties of G and $\hat{G}$):

$$\begin{aligned}
M'_{k+1}(\lambda) &= \frac{1}{c_k(\lambda)\iota(c_k(\lambda))}\langle GP^{(k)}_{\lambda+\rho}, GP^{(k)}_{\lambda+\rho}\rangle'_k \\
&= \frac{1}{c_k(\lambda)\iota(c_k(\lambda))}\frac{d_{k+1}}{d_k}\langle P^{(k)}_{\lambda+\rho}, \hat{G}GP^{(k)}_{\lambda+\rho}\rangle'_k \\
&= \frac{\iota(\hat{c}_k(\lambda))\iota(c_k(\lambda))}{c_k(\lambda)\iota(c_k(\lambda))}\frac{d_{k+1}}{d_k}\langle P^{(k)}_{\lambda+\rho}, P^{(k)}_{\lambda+\rho}\rangle'_k \\
&= (-1)^{|R_+|}\frac{d_{k+1}}{d_k}\frac{\hat{c}_k(\lambda)}{c_k(\lambda)}M'_k(\lambda+\rho)
\end{aligned}$$

Translating this into $M_k(\lambda)$, we obtain:

$$M_{k+1}(\lambda) = \left(\prod_{\alpha\in R_+}\frac{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)+2k}}{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)-2k}}\right)M_k(\lambda+\rho)$$

We iteratively apply this equation $k-1$ times to reduce to the case where $k=1$ where we already know that the Macdonald polynomials are orthonormal (as they are Weyl characters in that case). Note that at each step, the loss of a ρ due to k decreasing is compensated for by λ incrementing by ρ. This gives completes the proof of the Macdonald conjecture. We obtain

$$\langle P^{(k)}_\lambda, P^{(k)}_\lambda\rangle_{q,t} = \prod_{\alpha\in R_+}\prod_{i=1}^{k-1}\frac{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)+2i}}{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)-2i}}$$

5. Closing Remarks

Whilst we worked with $t_i = t = q^k$, we could actually have taken $t_i = q^{k_i}$ where $k_i \in \mathbb{Z}_{\geq 0}$. This would have involved introducing several different shift operators (our G, $\hat{G}$) to control the the different parameters separately. In that case, we would replace $k\rho$ with $\rho_k = \sum_{\alpha \in R_+} \frac{k_\alpha}{2}\alpha$, where k_α is equal to k_i for a simple root α_i in the orbit of α.

It is in fact true that (appropriately understood), Macdonald polynomials in type A define symmetric functions (i.e. they have suitable restriction properties for A_n as n decreases). In [Mac95], it is shown that if $\lambda = a_1\omega_1 + \cdots a_{n-1}\omega_{n-1}$ is written as a partition $\lambda = (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \cdots, a_{n-1})$, then

$$\langle P_\lambda, P_\lambda \rangle_{q,t} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{l(s)}}{1 - q^{a(s)} t^{l(s)+1}}$$

Here $s \in \lambda$ means s is a box in the Young diagram of λ. Also $a(s)$ is the arm length of s, namely the number of boxes to the right of s (not including s), and $l(s)$ is the leg length of s, namely the number of boxes below s (not including s).

References

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 2nd edition, 1995.

[Mac00] I. G. Macdonald. Orthogonal polynomials associated with root systems. *Séminaire Lotharingien de Combinatoire*, 45:B45a, 2000.