LECTURE 2: DOUBLE AFFINE HECKE ALGEBRAS

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Abstract. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

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1. Goals and structure of the talk

This talk introduces one of the main objects of study in our seminar: the double affine Hecke algebra (DAHA). We will make the definitions in great (but not complete) generality, and we will emphasize the $\mathfrak{gl}_n$ case. In the first part of the talk we will briefly recall from Seth's talk the main ingredients for the construction of DAHA: the affine Hecke algebras and Cherednik's basic representation. After this, we will give the definition of DAHA and exhibit an explicit basis of it. We will then spend some time looking at two explicit cases: the DAHA for A_1 and that for $\mathfrak{gl}_n$. In particular, we will give explicit presentations by generators and relations, present their trigonometric and rational degenerations, and exhibit a large group of automorphisms of the DAHA for $\mathfrak{gl}_n$. After that, we will study certain operators on the polynomial representation of DAHA. The importance of these operators is that, first, they can be used to form a big commuting family of difference operators on the group algebra of the weight lattice and, second, they are connected to Macdonald polynomials, which is the topic of a subsequent talk in this seminar. The study of these difference operators naturally leads to the definition of spherical DAHA. We finish the notes with a discussion of trigonometric and rational degenerations in the general setting, the description of (trigonometric, difference-rational and rational) Dunkl operators, and applications to the theory of quantum integrable systems.

2. Double Affine Hecke Algebras

2.1. Reminders.

2.1.1. *Root systems and Weyl groups.* We will work with affine root systems that are of the form R^a, where R is an irreducible finite root system (so, for example, we will ignore the affine root systems of the form $(C_n^\vee, C_n)$, etc.). Throughout these notes, we will use the following notation.

- $\{\alpha_1, \dots, \alpha_n\}$ denotes the set of simple roots and $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ the set of simple co-roots of R.
- $Q, Q^\vee$ denote the root and coroot lattice of R, respectively. Similarly, $P, P^\vee$ denote the weight and coweight lattice of R, respectively.
- $W := \langle s_1, \dots, s_n \rangle$ is the Weyl group of R, where s_i denotes the reflection s_{α_i}.
- $\alpha_0 := -\theta + \delta$, so that $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ forms a set of simple roots for R^a.
- $W^a := \langle s_0, \dots, s_n \rangle$ is the Weyl group of R^a, aka the affine Weyl group. Recall that we have an isomorphism
$$W^a = W \ltimes t(Q^\vee)$$
- $W^{ae} := W \ltimes t(P^\vee)$ is the extended affine Weyl group.
- $\Omega \subseteq W^{ae}$ denotes the subgroup of all elements of length 0. This is a finite subgroup of W^{ae}, acting faithfully on the set of simple roots $\{\alpha_0, \dots, \alpha_n\}$, and it is actually isomorphic to $P^\vee/Q^\vee$.
- We have an isomorphism
$$W^{ae} = \Omega \ltimes W^a$$
where the action of Ω' on W^a is given as follows: if $\pi_r \in \Omega'$ is such that $\pi_r(\alpha_i) = \alpha_j$, then $\pi_r s_i \pi_r^{-1} = s_j$.

2.1.2. *Affine Hecke algebras.* Throughout this talk, $\tau := \{\tau_0, \dots, \tau_n\}$ will denote a collection of *formal variables* such that $\tau_i = \tau_j$ whenever the reflections s_i and s_j are conjugate in W^a, and let $\mathbb{C}_\tau := \mathbb{C}(\tau_0, \dots, \tau_n)$ denote the field of rational functions in these variables. Recall that we have the *affine Hecke algebra* $\mathcal{H}_\tau$ of W, which is a quotient of the group algebra $\mathbb{C}_\tau B^{ae}$, where B^{ae} is the extended affine braid group. We have two presentations of this algebra.

The Coxeter presentation. $\mathcal{H}_\tau = \mathbb{C}_\tau\langle T_0, \dots, T_n, \Omega\rangle$ with the following relations.

(a) $T_iT_j \cdots = T_jT_i \cdots$, where term has m_{ij} factors.
(b) $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$.
(c) $\pi_r T_i \pi_r^{-1} = T_j$, if $\pi_r(\alpha_i) = \alpha_j$.

The Bernstein presentation. $\mathcal{H}_\tau = \mathbb{C}_\tau\langle T_1, \dots, T_n, Y^{P^\vee}\rangle$ with relations:

(a') Relations (1) and (2) above for the T_i.
(b') $Y^\lambda Y^\mu = Y^{\lambda+\mu}$.
(c') $T_iY^\lambda = Y^\lambda T_i$ if $\langle \lambda, \alpha_i\rangle = 0$.
(d') $T_iY^{s_i(\lambda)}T_i = Y^\lambda$ if $\langle \lambda, \alpha_i\rangle = 1$.

2.1.3. *Cherednik's basic representation.* Let us now recall Cherednik's basic representation of the affine Hecke algebra $\mathcal{H}_\tau$. We let q be a variable, and consider the affine Hecke algebra defined over the field $\mathbb{C}_{q,\tau} := \mathbb{C}_\tau(q^{1/e})$, where e is such that $\langle P, P^\vee\rangle = \frac{1}{e}\mathbb{Z}$. Now let $\mathbb{C}_{q,\tau}[X]$ denote the group algebra of P. Note that this contains $\mathbb{C}_\tau[\widehat{X}]$, the group algebra of the affine weight lattice $\widehat{P} := P \oplus \mathbb{Z}\delta$, by setting $X^{\lambda + r\delta} := q^rX^\lambda$. So the extended affine Weyl group W^{ae} acts on $\mathbb{C}_{q,\tau}[X]$ by setting, for $w = t(\lambda)v$, $\lambda \in P^\vee, v \in W$ and $\mu \in P$,

$$w(X^\mu) := X^{w(\mu)} = q^{-\langle \lambda, v(\mu)\rangle} X^{v(\mu)}$$

We have that $\mathbb{C}_{q,\tau}[X]$ becomes a $\mathcal{H}_\tau$-module via the formulas:

$$\pi_r \mapsto \pi_r, \pi_r \in \Omega$$
$$T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1})\frac{s_i - \mathrm{id}}{X^{\alpha_i} - 1}, i = 0, \dots, n$$

Let us be more explicit on the action of T_0. Recall that $\alpha_0 = -\theta + \delta$, and that we are identifying $q = X^\delta$. So it follows that $s_0 X^\mu = X^{\mu - (-\theta+\delta)\langle \mu, -\theta^\vee\rangle} = X^\mu (q^{-1}X^\theta)^{\langle \mu, -\theta^\vee\rangle}$. Thus,

$$T_0 : X^\mu \mapsto \left(\tau_0 (q^{-1}X^\theta)^{\langle \mu, -\theta^\vee\rangle} + (\tau_0 - \tau_0^{-1})\frac{(q^{-1}X^\theta)^{\langle \mu, -\theta^\vee\rangle} - 1}{qX^{-\theta} - 1}\right) X^\mu$$

In particular, if $\langle \mu, -\theta^\vee\rangle = 0$, then $T_0(X^\mu) = \tau_0 X^\mu$, while if $\langle \mu, -\theta^\vee\rangle = 1$, then $T_0(X^\mu) = \tau_0 q^{-1} X^{\theta+\mu} - (\tau_0 - \tau_0^{-1}) q^{-1} X^{\theta+\mu}$.

2.1.4. *The induced representation.* Let us denote by $\mathcal{H}_\tau^X$ the affine Hecke algebra for the root system $(R^\vee)^a$. In particular, we have the Bernstein presentation for this Hecke algebra, which is completely analogous to the Bernstein presentation above. We have the induced representation of $\mathcal{H}_\tau^X$ on $\mathbb{C}_{q,\tau}[X]$, where the X^μ act by multiplication and the T_i act by

$$T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1})\frac{s_i - \mathrm{id}}{X^{\alpha_i} - 1}, i = 1, \dots, n$$

Let us remark that the induced representation of $\mathcal{H}_\tau^X$ on $\mathbb{C}_{q,\tau}[X]$ is obtained by the eponymous representation on $\mathbb{C}_\tau[X]$ by base-change to the field $\mathbb{C}_{q,\tau}$.

2.2. **Double affine Hecke algebras.** We are now ready to define the double affine Hecke algebra for R. The idea here is to glue together the affine Hecke algebras $\mathcal{H}_\tau$ and $\mathcal{H}_\tau^X$ along their common representation $\mathbb{C}_{q,\tau}[X]$.

Definition 2.2.1. *The* double affine Hecke algebra $\mathbb{H} := \mathbb{H}(W)$ *is the $\mathbb{C}_{q,\tau}$-algebra generated by elements $T_0, \dots, T_n, \Omega, X^P$ with relations.*

1. *The relations (a)-(c) above for the affine Hecke algebra between $T_0, \dots, T_n$ and Ω.*
2. *Denote $\alpha_0^\vee := -\theta^\vee$. Then, for $i = 0, \dots, n$:*

$$\begin{aligned} T_i X^\mu &= X^\mu T_i && \text{if } \langle \mu, \alpha_i^\vee\rangle = 0 \\ T_i X^\mu &= X^{s_i(\mu)} T_i^{-1} && \text{if } \langle \mu, \alpha_i^\vee\rangle = 1 \end{aligned}$$

3. $\pi_r X^\mu \pi_r^{-1} = X^{\pi_r(\mu)}$.

Definition 2.2.2. *Note that, by its very definition, the DAHA $\mathbb{H}$ admits a representation on the space $\mathbb{C}_{q,t}[X]$, where X^μ acts by multiplication and both π_r and T_i act as in Cherednik's basic representation. We call this representation the* polynomial representation *of $\mathbb{H}$.*

Note that, by Matsumoto's theorem, if $w = \alpha_{i_1} \dots \alpha_{i_k}$ is a reduced decomposition of $w \in W^a$, then we have a well-defined element $T_w \in \mathbb{H}$.

Theorem 2.2.3 (PBW theorem for DAHA)**.** *Every element $h \in \mathbb{H}$ can be uniquely written in the form*

$$h = \sum_{\substack{\mu \in P \\ \pi_r \in \Omega \\ w \in W^a}} a_{\mu,r,w} X^\mu \pi_r T_w, \quad a_{\mu,r,w} \in \mathbb{C}_{q,\tau}$$

The existence of such an expression for h is a standard exercise. The uniqueness is harder. We will use a standard trick that we have already seen in Seth's lecture: we will write down a representation of $\mathbb{H}$ in a space in which the operators $X^\mu \pi_r T_w$ are linearly independent. It turns out that we already know such a representation: the polynomial representation, cf. Definition 2.2.2.

Theorem 2.2.4. *Consider the polynomial representation $\mathbb{C}_{q,\tau}[X]$ of $\mathbb{H}$. Then, the operators $\{X^\mu \pi_r T_w : \mu \in P, \pi_r \in \Omega, w \in W^a\}$ are linearly independent over the field $\mathbb{C}_{q,\tau}$. In particular, the polynomial representation is faithful.*

Proof. Note that, even though the operators $\pi_r T_w$ are not $\mathbb{C}_{q,\tau}[X]$-linear, we still have an action of $\mathbb{C}_{q,\tau}[X]$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}_{q,\tau}[X])$, $f : \varphi \mapsto (x \mapsto f\varphi(x))$. It clearly suffices to show that the operators $\{\pi_r T_w\}_{\mu \in P, w \in W^a}$ are linearly independent over $\mathbb{C}_{q,\tau}[X]$. In order to do this, we will relate this action to the action of the extended affine Weyl group W^{ae} on $\mathbb{C}_{q,\tau}[X]$, which we know from Seth's talk it is faithful.

Recall that for $i = 0, \ldots, n$, T_i acts via the operator:

$$T_i := \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{s_i - \mathrm{id}}{X^{\alpha_i} - 1} = \left(\tau_i + \frac{\tau_i - \tau_i^{-1}}{X^{\alpha_i} - 1}\right) s_i + \left(\frac{\tau_i^{-1} - \tau_i}{X^{\alpha_i} - 1}\right) \mathrm{id}$$

It follows that, for $\pi_r w \in W^{ae}$, $w \in W^a$, we can write

$$\pi_r T_w = \sum_{w' \leq w} f_{w',w} \pi_r w'$$

where $f_{w',w} \in \mathbb{C}_{q,\tau}(X)$ are rational functions on X and the order on W^a is the usual Bruhat order. Note that $f_{w,w} \neq 0$.

Now assume that we have a linear combination of the form

$$\sum_{\substack{\pi_r \in \Omega \\ w \in W^a}} g_{r,w}(X) \pi_r T_w = 0$$

where $g_{r,w} \in \mathbb{C}_{q,\tau}[X]$ are not all 0. It follows from the above that we get

$$\sum_{\substack{w \in W^a, \pi_r \in \Omega \\ w' \leq w}} g_{r,w}(X) f_{r,w',w}(X) \pi_r w' = 0$$

The operators $\pi_r w$ are all distinct, since the representation of W^a on $\mathbb{C}_{q,\tau}[X]$ is faithful, and can be extended to automorphisms of the field $\mathbb{C}_{q,\tau}(X)$ (= the field of quotients of $\mathbb{C}_{q,\tau}[X]$). It follows that the operators $\pi_r w$ are linearly independent over the field $\mathbb{C}_{q,\tau}(X)$. So for every $\pi_r \in \Omega$, $w \in W^a$ we have

$$\sum_{w' \geq w} g_{r,w'} f_{r,w,w'} = 0$$

If we pick $w_0 \in W^a$ such that w_0 is maximal w.r.t. the Bruhat order in the set $\{w \in W^a : g_{r,w} \neq 0 \text{ for some } \pi_r \in \Omega\}$ then we get $g_{r,w_0} f_{r,w_0,w_0} = 0$. But since $f_{r,w_0,w_0} \neq 0$, this is a contradiction. We are done. $\square$

Corollary 2.2.5. *We define the following subalgebras of $\mathbb{H}$:*

(1) $\mathcal{H}^X := \langle T_1, \ldots, T_n, X^\mu (\mu \in P) \rangle$.
(2) $\mathcal{H}^Y := \langle T_0, \ldots, T_n, \Omega \rangle$.
(3) $H := \langle T_1, \ldots, T_n \rangle$.

Then, $\mathcal{H}^X$ is (isomorphic to) the affine Hecke algebra for the root system $R^\vee$; $\mathcal{H}^Y$ is (isomorphic to) the affine Hecke algebra for the root system R; and H is (isomorphic to) the finite Hecke algebra of W.

Recall from Seth's talk the definition of the elements $Y^\lambda \in \mathcal{H}^Y$, $\lambda \in P^\vee$. Namely, $Y^\lambda := T_{t(\lambda)}$ if $\lambda \in P^\vee_+$, while $Y^\lambda := Y^\mu (Y^\nu)^{-1}$ if $\lambda = \mu - \nu$ with $\mu, \nu \in P^\vee_+$. Since $\mathcal{H}^Y$ is an affine Hecke algebra, the elements Y^λ are well-defined. The following result follows immediately from Seth's talk.

Theorem 2.2.6 (PBW theorem for DAHA, v2)**.** *Every element $h \in \mathbb{H}$ can be uniquely written in the form*

$$h = \sum_{\substack{\lambda \in P^\vee \\ \mu \in P \\ w \in W}} a_{\lambda,\mu,w} X^\mu Y^\lambda T_w, \quad a_{\lambda,\mu,w} \in \mathbb{C}_{q,\tau}$$

Let us remark that the weight and co-weight lattice play a symmetric role in the definition of DAHA. In order to state this precisely, let $\omega_1, \dots, \omega_n$ be the fundamental weights of R, so $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, and denote by $\omega_1^\vee, \dots, \omega_n^\vee$ the fundamental coweights. We will denote $X_i := X^{\omega_i}$, $Y_i := Y^{\omega_i^\vee}$.

Theorem 2.2.7. *The following assignment can be extended to a $\mathbb{C}$-automorphism of $\mathbb{H}$:*

$$X_i \mapsto Y_i, \quad Y_i \mapsto X_i, \quad T_j \mapsto T_j^{-1}, \quad \tau_j \mapsto \tau_j^{-1}, \quad q \mapsto q^{-1}$$

We will not prove Theorem 2.2.7 in full generality. We will show it for specific types of root systems below. Let us remark that a consequence of Theorem 2.2.7 is the following.

Corollary 2.2.8 (PBW theorem for DAHA, v3)**.** *Every element $h \in \mathbb{H}$ can be uniquely written in the form*

$$h = \sum_{\substack{\lambda \in P^\vee \\ \mu \in P \\ w \in W}} a_{\lambda,\mu,w} Y^\lambda X^\mu T_w, \quad a_{\lambda,\mu,w} \in \mathbb{C}_{q,\tau}$$

Note that we could have also defined $\mathbb{H}$ to be an algebra generated by $T_1, \dots, T_{n-1}, X^P, Y^{P^\vee}$ with certain relations. Of course, the relations among (T_i, X^μ) or among (T_i, Y^λ) can be explicitly written - they are just the relations of the affine Hecke algebra. But it is not easy to write the relations among (X^μ, Y^λ). We will give a couple of examples where these relations can actually be written. As we will see, they are topological in nature.

2.3. Example: DAHA for A_1.

2.3.1. *Generators and relations.* We give explicit generators and relations for the DAHA of A_1. So we have that the (co-)root lattice is $Q = Q^\vee = \mathbb{Z}\alpha$ and the (co-)weight lattice is $P = P^\vee = \mathbb{Z}\rho$, with $\rho = \alpha/2$. Let us denote $s = s_\alpha$. We have that $\Omega = \{1, \pi_\rho\}$, we denote $\pi := \pi_\rho = t(\rho)s$. Setting now $X := X^\rho$, we have that the DAHA $\mathbb{H}$ is generated by $T_0, T_1, X^{\pm 1}$ and π. Note, however, that $T_0 = \pi T_1 \pi$, so we may ignore T_0 from our list of generators. Thus, we have

$$\mathbb{H} = \mathbb{C}_{q,\tau}\langle X, T, \pi \rangle \Big/ \left\{ \begin{array}{cc} TXT = X^{-1}, & \pi X \pi^{-1} = qX^{-1}, \\ \pi^2 = 1, & (T-\tau)(T+\tau^{-1}) = 0 \end{array} \right\}$$

Setting $Y := \pi T$, we have the following alternative presentation of $\mathbb{H}$:

$$\mathbb{H} = \mathbb{C}_{q,\tau}\langle X, T, Y \rangle \Big/ \left\{ \begin{array}{cc} TXT = X^{-1}, & Y^{-1}X^{-1}YX = q^{-1}T^{-2}, \\ TY^{-1}T = Y, & (T-\tau)(T+\tau^{-1}) = 0 \end{array} \right\}$$

Note that this presentation reveals a symmetry between X and Y. The following proposition is obvious, note that its second part is a special case of Theorem 2.2.7.

Lemma 2.3.1. *We have a $\mathbb{C}_{q,\tau}$ anti-involution $\phi : \mathbb{H} \to \mathbb{H}^{\mathrm{opp}}$, defined on generators by the following formulas*

$$\phi(X) = Y^{-1}, \quad \phi(Y) = X^{-1}, \quad \phi(T) = T$$

and a $\mathbb{C}$-involution $\varepsilon : \mathbb{H} \to \mathbb{H}$, defined by

$$\varepsilon(X) = Y, \quad \varepsilon(Y) = X, \quad \varepsilon(T) = T^{-1}, \quad \varepsilon(\tau) = \tau^{-1}, \quad \varepsilon(q) = q^{-1}$$

2.3.2. *The polynomial representation.* Let us give formulas for the action of the elements X, Y, T on the polynomial representation $\mathbb{C}_{q,\tau}[X]$. First of all, we have that the action of T is given by

$$\tau s + (\tau - \tau^{-1}) \frac{s - \mathrm{id}}{X^2 - 1}$$

while the action of $t(\rho)$ is given by $t(\rho)(X) = qX$. Moreover, for a Laurent polynomial $f(X) \in \mathbb{C}_{q,\tau}[X]$, we have that $t(\rho)f(X) = f(qX)$, so that $\pi(X) = t(\rho)s(X) = q^{-1}X^{-1}$. Thus, we have

$$\begin{aligned} Y = {} & \pi \left(\tau s + (\tau - \tau^{-1}) \frac{s - \mathrm{id}}{X^2 - 1} \right) \\ = {} & \tau t(\rho) + (\tau - \tau^{-1}) \pi \frac{s - \mathrm{id}}{X^2 - 1} \\ = {} & \tau t(\rho) + (\tau - \tau^{-1}) \frac{t(\rho) - \pi}{q^{-2}X^{-2} - 1} \\ = {} & \tau t(\rho) + (\tau - \tau^{-1}) t(\rho) \frac{\mathrm{id} - s}{X^{-2} - 1} \\ = {} & t(\rho) \left(\tau + (\tau - \tau^{-1}) \frac{\mathrm{id} - s}{X^{-2} - 1} \right) \end{aligned}$$

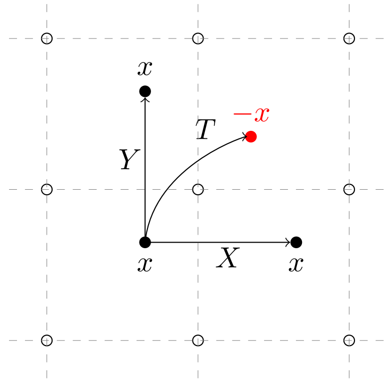
So, for example, we get $Y(X^n) = \tau^{-1}(q^n X^n + q^{n-2}X^{n-2} + \cdots + q^{2-n}X^{2-n})$. The operator Y is known as the *difference-trigonometric Dunkl operator.*

2.3.3. *Topological interpretation.* Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, where we take the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\iota$. Let $0 \in E$ be the zero point, and consider the automorphism $-1 : x \mapsto -x$ of E. Note that $\pi_1((E \setminus \{0\})/\mathbb{Z}_2)$ is trivial, as $(E \setminus \{0\})/\mathbb{Z}_2$ being a disk is contractible. We will consider the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$, where $x \in E \setminus \{0\}$ is a generic point (i.e., not one of the three branching points of $E \setminus \{0\} \to (E \setminus \{0\})/\mathbb{Z}_2$).

Let us recall that the orbifold fundamental group is generated by homotopy classes of paths in $E \setminus \{0\}$ from x to $\pm x$, with multiplication defined by $\gamma_1 \circ \gamma_2$ is γ_2 followed by $-\gamma_1$, if γ_2 connects x to $-x$. So we have an exact sequence

$$1 \to \pi_1(E \setminus \{0\}, x) \to \pi_1^{\mathrm{orb}}((E \setminus \{0\})/\mathbb{Z}_2, x) \to \mathbb{Z}_2 \to 1,$$

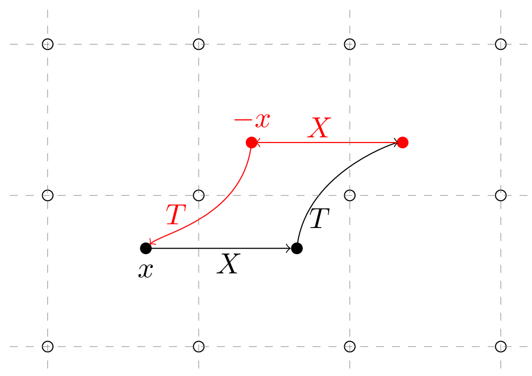
i.e., $\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$ is an extension by $\mathbb{Z}_2$ of the group $\pi_1(E \setminus \{0\}, x)$, the fundamental group of the punctured torus. The latter group has three generators, X (the "horizontal" cycle of the torus), Y (the "vertical" cycle of the torus) and C (a loop around the missing point 0). The orbifold fundamental group $\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$ is then generated by X, Y and an element T (a half-loop around 0) connecting x to $-x$ such that $T^2 = C$.



The elements X, Y, T satisfy the relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad Y^{-1}X^{-1}YXT^2 = 1$$

Let us show, for example, the first relation, which says that $TXTX = 1$. So we have to first go through the loop X, then through the path T, and then through the path $-X$, since the endpoint of T is $-x$. Finally, we go through the path $-T$. It follows from the following picture that this path is null-homotopic.



So we see that $\mathbb{H}$ can be seen as a quotient of the group algebra $\mathbb{C}_{q,\tau}\pi_1^{\mathrm{orb}}((E \setminus \{0\})/\mathbb{Z}_2, x)$, as follows. First of all, recall that out base field $\mathbb{C}_{q,\tau}$ includes $q^{\pm 1/2}$, since $\langle \rho, \rho \rangle = 1/2$. Now set $\widetilde{T} := q^{-1/2}T, \widetilde{X} := q^{1/2}X, \widetilde{Y} := q^{-1/2}Y$, so that $\widetilde{X}, \widetilde{Y}, \widetilde{T}$ satisfy the relations of $\mathbb{H}$ with the exception of the quadratic relation for $\widetilde{T}$. Thus:

$$\mathbb{H} = \mathbb{C}_{q,\tau}\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x) \Big/ ((T - q^{1/2}\tau)(T + q^{1/2}\tau^{-1}))$$

2.3.4. *Trigonometric degeneration.* Now let $\hbar$, c and t be variables. Set $Y := \exp(\hbar\hat{y})$, $q := \exp(t\hbar)$, $\tau := q^c = \exp(\hbar tc)$ and $T := s\exp(\hbar cs)$, where $s \in S_2$ is the non-trivial element. We can consider $\mathbb{H}$ as a $\mathbb{C}[[\hbar, c, t]]$-algebra, with the same generators and relations as above. Then, $\mathbb{H}/\hbar\mathbb{H}$ is generated by $s, \hat{y}$ and X, with relations

$$s^2 = 1, \quad sXs = X^{-1}, \quad s\hat{y} + \hat{y}s = 2c, \quad X^{-1}\hat{y}X - \hat{y} = t - 2cs$$

We call $\mathbb{H}^{\mathrm{trig}} := \mathbb{C}[c,t]\langle s, X, \hat{y}\rangle$ with the relations above the *trigonometric* DAHA of A_1.

Lemma 2.3.2. *Every element $h \in \mathbb{H}^{\mathrm{trig}}$ can be uniquely written as*

$$h = \sum_{\substack{m \in \mathbb{Z} \\ n \in \mathbb{Z}_{\geq 0} \\ i = 0,1}} a_{m,i,n} X^m s^i \hat{y}^n, \quad a_{m,i,n} \in \mathbb{C}[c,t]$$

The lemma again can be proved using a faithful representation of $\mathbb{H}^{\mathrm{trig}}$. Here the space is $\mathbb{C}[c,t][X^{\pm 1}]$. The element s acts by $X \mapsto X^{-1}$ and X acts by multiplication. To give the action of $\hat{y}$, first define the *trigonometric derivative* by $\partial(X) = X$. Then, $\hat{y}$ acts by the *trigonometric Dunkl operator*

$$D^{\mathrm{trig}} := t\partial - 2c\frac{1}{1-X^{-2}}(\mathrm{id} - s) + c$$

This is known as the *differential polynomial representation* of $\mathbb{H}^{\mathrm{trig}}$. We also have a different polynomial representation of $\mathbb{H}^{\mathrm{trig}}$, which stems from the fact that the variables X and $\hat{y}$ are not symmetric. This is an action of $\mathbb{H}^{\mathrm{trig}}$ on $\mathbb{C}[c,t][\hat{y}]$. Note that we have an action of S_2 on $\mathbb{C}[c,t][\hat{y}]$, where s acts by $y \mapsto -y$. Then, $s \in \mathbb{H}^{\mathrm{trig}}$ acts on $\mathbb{C}[c,t][\hat{y}]$ via the operator

$$S := s - \frac{c}{\hat{y}}(s - \mathrm{id})$$

To define an action of X, let $\pi : \mathbb{C}[c,t][\hat{y}] \to \mathbb{C}[c,t][\hat{y}]$ be defined by $f(\hat{y}) \mapsto f(-\hat{y}+t)$. Then, X acts via the operator πS. This is known as the *difference-rational polynomial representation* of $\mathbb{H}^{\mathrm{trig}}$. The operator πS is known as the *difference-rational Dunkl operator.*

Corollary 2.3.3. *The following are subalgebras of* $\mathbb{H}^{\mathrm{trig}}$*:*

(1) *The group algebra of the extended affine Weyl group for A_1: it is isomorphic to the subalgebra of $\mathbb{H}^{\mathrm{trig}}$ generated by s, X.*
(2) *The degenerate affine Hecke algebra for A_1: it is isomorphic to the subalgebra of $\mathbb{H}^{\mathrm{trig}}$ generated by $s, \hat{y}$.*

2.3.5. *Rational degeneration.* Now in $\mathbb{H}^{\mathrm{trig}}$ set $X = \exp(\hbar x)$ and $y = \hbar\hat{y}$. Then, modulo $\hbar$, the elements s, x, y, satisfy the following relations

$$s^2 = 1, \quad sx = -xs \quad sy = -ys \quad yx - xy = t - 2cs$$

Define the algebra $\mathbb{H}^{\mathrm{rat}} := \mathbb{C}[c,t]\langle s, x, y\rangle$ with the relations above. This is known as the *rational* DAHA of A_1.

Lemma 2.3.4. *Every element $h \in \mathbb{H}^{\mathrm{rat}}$ can be uniquely written in the form*

$$h = \sum_{\substack{m,n\in\mathbb{Z}\\ i=0,1}} a_{m,i,n} x^m s^i y^n, \quad a_{m,i,n} \in \mathbb{C}[c,t]$$

Lemma 2.3.4 may be proven using the *polynomial representation* of $\mathbb{H}^{\mathrm{rat}}$. This is the representation on $\mathbb{C}[c,t][x]$, where s acts by $x \mapsto -x$, x acts by multiplication and y acts by the *rational Dunkl operator*

$$D^{\mathrm{rat}} := t\frac{d}{dx} + c\frac{1}{x}(s - \mathrm{id})$$

2.4. Example: DAHA for $\mathfrak{gl}_n$.

2.4.1. *The affine Hecke algebra for $\mathfrak{gl}_n$, revisited.* We will now define the DAHA for $\mathfrak{gl}_n$, which is different (but closely related to) from the DAHA of type A_n. So the first step is to study the affine Hecke algebra for $\mathfrak{gl}_n$, which has already appeared at the end of Seth's talk. Recall that we denote $\mathbb{C}_\tau := \mathbb{C}(\tau)$, the field of rational functions on the variable τ.

Definition 2.4.1. *Let $n > 0$. The* affine Hecke algebra of $\mathfrak{gl}_n$*, $\mathcal{H}_n$, is the $\mathbb{C}_\tau$-algebra with generators $T_1, \ldots, T_{n-1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ and relations*

(Quadratic relations)	$(T_i - \tau)(T_i + \tau^{-1}) = 0$, *for* $i = 1, \dots, n-1$.
(Braid relations)	$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ *for* $i = 1, \dots, n-2$; $T_iT_j = T_jT_i$ *if* $\|i-j\| > 1$.
(Action relations)	$T_i^{-1}Y_iT_i^{-1} = Y_{i+1}$, $i = 1, \dots, n-1$; $T_jY_i = Y_iT_j$ *if* $j \neq i, i-1$.
(Laurent relations)	$Y_iY_j = Y_jY_i$ *for* $i, j = 1, \dots, n$; $Y_iY_i^{-1} = 1$.

In order to define DAHA, we will need Cherednik's basic representation for $\mathcal{H}_n$. This is more easily given in the Coxeter presentation, so we will need an analogue of the Coxeter presentation of $\mathcal{H}_n$. Let us introduce the following element of $\mathcal{H}_n$:

$$\pi := T_1^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_i \cdots T_{n-1}$$

Note that, thanks to the action relations, the element π is well-defined, i.e., it does not depend on $i = 1, \dots, n$. So, for example, $\pi = Y_1^{-1}T_1 \cdots T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1}Y_n$.

Lemma 2.4.2. *The element π^n is central in $\mathcal{H}_n$.*

Proof. We have

$$\begin{aligned} \pi^n = & \ (Y_1^{-1}T_1 \cdots T_{n-1})(Y_1^{-1}T_1 \cdots T_{n-1}) \cdots (Y_1^{-1}T_1 \cdots T_{n-1}) \\ = & \ Y_1^{-1}Y_2^{-1} \cdots Y_n^{-1} A_1A_2 \cdots A_n \end{aligned}$$

where $A_i := T_1^{-1}T_2^{-1} \cdots T_{n-i}^{-1}T_{n-i+1} \cdots T_{n-1}$, so, for example, $A_1 = T_1^{-1}T_2^{-1} \cdots T_{n-1}^{-1}$ and $A_n = T_1T_2 \cdots T_{n-1}$. We claim that $A_1A_2 \cdots A_n = 1$, note that this will finish the proof of the lemma. Indeed, this can already be seen in the braid group B_n: first of all, the associated permutation in S_n of every A_i's is the cycle $1 \mapsto n \mapsto n-1 \mapsto \cdots \mapsto 2 \mapsto 1$, and the n-th power of this cycle is the identity. So $A_1 \cdots A_n$ is, at least, an element of the pure braid group.

Now note that, in A_i, the strand starting at 1 passes *below* the strands starting at $2, \dots, n-i$ and *above* the strands starting at $n-i+1, \dots, n-1$. So, in the product $A_1 \cdots A_n$, the strand connecting 1 to 1 passes below all other strands; the strand connecting 2 to 2 passes above the strand connecting 1 to 1 and below all other strands and, in general, the strand connecting i to i passes above the strand connecting j to j if $j < i$, and below the strand connecting j to j if $j > i$. So $A_1 \cdots A_n = 1$. $\square$

Lemma 2.4.3. *We have $\pi T_i \pi^{-1} = T_{i+1}$, $i = 1, \dots, n-2$.*

Proof. Here we use $\pi = T_1^{-1} \cdots T_{n-1}^{-1}Y_n$. So

$$\begin{aligned} \pi T_i \pi^{-1} = & \ (T_1^{-1} \cdots T_{n-1}^{-1}Y_n)T_i(Y_n^{-1}T_{n-1} \cdots T_1) \\ = & \ (T_1^{-1} \cdots T_i^{-1})(T_{i+1}^{-1}T_iT_{i+1})(T_i \cdots T_1) \end{aligned}$$

Now we use the identity $T_{i+1}^{-1}T_iT_{i+1} = T_iT_{i+1}T_i^{-1}$, which follows immediately form the braid relation involving i, $i+1$. From here, the result follows easily. $\square$

Theorem 2.4.4. *The affine Hecke algebra $\mathcal{H}_n$ is generated by $T_1, \dots, T_{n-1}, \pi^{\pm 1}$ with relations:*

(1) *The braid and quadratic relations involving the T_i.*
(2) $\pi T_i \pi^{-1} = T_{i+1}$, $i = 1, \dots, n-2$.
(3) π^n *is central.*

Proof. Let $\mathcal{H}'_n$ denote the algebra defined in the statement of the theorem. Define

$$Y_i := T_i \cdots T_{n-1}\pi^{-1}T_1^{-1} \cdots T_{i-1}^{-1}.$$

We have to check that the Y_i's satisfy the action and commutativity relations. Let us check the action relations. First of all, it is clear that $T_i^{-1}Y_iT_i^{-1} = Y_{i+1}$, for $i = 1, \dots, n$. Now, if $j > i$ we have

$$\begin{aligned} T_jY_i = &\ T_jT_iT_{i+1}\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_i\cdots T_jT_{j-1}T_j\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_i\cdots T_{j-1}T_j\cdots T_{n-1}T_{j-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_i\cdots T_{n-1}\pi^{-1}T_jT_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ Y_iT_j \end{aligned}$$

and if $j < i-1$ we have

$$\begin{aligned} T_jY_i = &\ T_i\cdots T_{n-1}T_j\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_i\cdots T_{n-1}\pi^{-1}T_{j+1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_i\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{j+1}T_j^{-1}T_{j+1}^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_i\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_j^{-1}T_{j+1}^{-1}T_jT_{j+2}^{-1}\cdots T_{i-1}^{-1} \\ = &\ Y_iT_j \end{aligned}$$

Let us proceed to the commutation relations. We prove them in several steps.

Step 1: If $Y_1Y_j = Y_jY_1$ *for every* $j = 1, \dots, n$*, then* $Y_iY_j = Y_jY_i$ *for every* $i, j = 1, \dots, n$*.* Indeed, assume that $i < j$. Then, using the action relations that we have already shown:

$$\begin{aligned} Y_iY_j = &\ T_{i-1}^{-1}\cdots T_1^{-1}Y_1T_1^{-1}\cdots T_{i-1}^{-1}Y_j \\ = &\ T_{i-1}^{-1}\cdots T_1^{-1}Y_1Y_jT_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ T_{i-1}^{-1}\cdots T_1^{-1}Y_jY_1T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ Y_jT_{i-1}^{-1}\cdots T_1^{-1}Y_1T_1^{-1}\cdots T_{i-1}^{-1} \\ = &\ Y_jY_i \end{aligned}$$

Step 2: If $Y_1Y_2 = Y_2Y_1$*, then* $Y_1Y_j = Y_jY_1$ *for every* $j = 1, \dots, n$*.* This is done similarly to Step 1.

Step 3: $Y_1Y_2 = Y_2Y_1$. We need to show that $Y_1T_1^{-1}Y_1T_1^{-1} = T_1^{-1}Y_1T_1^{-1}Y_1$. The left-hand side of this equation becomes

$$\begin{aligned} Y_1T_1^{-1}Y_1T_1^{-1} = &\ T_1\cdots T_{n-1}\pi^{-1}T_1^{-1}T_1\cdots T_{n-1}\pi^{-1}T_1^{-1} \\ = &\ T_1\cdots T_{n-1}T_1\cdots T_{n-2}\pi^{-2}T_1^{-1} \end{aligned} \tag{2.4.1}$$

And the right-hand side becomes

$$\begin{aligned} T_1^{-1}Y_1T_1^{-1}Y_1 = &\ T_1^{-1}T_1\cdots T_{n-1}\pi^{-1}T_1^{-1}T_1\cdots T_n\pi^{-1} \\ = &\ T_2\cdots T_{n-1}T_1\cdots T_{n-2}\pi^{-2} \end{aligned} \tag{2.4.2}$$

Now we use that π^n is central in $\mathcal{H}_n'$. Indeed, we have $\pi^{-n}T_1\pi^n = T_1$, which implies that $\pi^{-2}T_1\pi^2 = T_{n-1}$, or $T_1\pi^2 = \pi^2T_{n-1}$, so $\pi^{-2}T_1^{-1} = T_{n-1}^{-1}\pi^{-2}$. We use this on the right-hand side of Equation (2.4.1). Now inductively use the identity $T_{i-1}T_i^{-1} = T_i^{-1}T_{i-1}^{-1}T_iT_{i-1}$, together with the braid relations, to get an equality with (2.4.2). □

We also need an analog of Cherednik's basic representation. This is given by the following.

Theorem 2.4.5. *The following assignment defines a representation of* $\mathcal{H}_n$ *on the space* $\mathbb{C}_{q,\tau}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$*:*

$$T_i \mapsto \tau s_i + (\tau - \tau^{-1})\frac{s_i - \mathrm{id}}{1 - X_iX_{i+1}^{-1}}$$
$$\pi(X_1^{a_1}\cdots X_n^{a_n}) = q^{-a_n}X_1^{a_n}X_2^{a_1}\cdots X_n^{a_{n-1}}$$

Proof. We need to check that these operators satisfy the relations of $\mathcal{H}_n$. That the T_i satisfy the braid and quadratic relations is very similar to what Seth has already done. Note also that $\pi^n(X_1^{a_1}\cdots X_n^{a_n}) = q^{-\sum a_i}X_1^{a_1}\cdots X_n^{a_n}$. Since the operators T_i preserve the grading, it follows that they commute with π^n. The only relation we need to check now is that $\pi T_i = T_{i+1}\pi$ for $i = 1,\ldots,n-2$. This is clear. ☐

Let us examine the relations between the operators X_i (multiplication) and π. First of all, it is clear that for $i = 1,\ldots,n-1$, we have that $\pi X_i = X_{i+1}\pi$. For $i = n$, we get $\pi X_n = q^{-1}X_1\pi$. And since π^n is a grading operator, we get that $\pi^n X_i = q^{-1}X_i\pi^n$.

2.4.2. *Generators and relations.* The DAHA for $\mathfrak{gl}_n$ is the $\mathbb{C}_{q,\tau}$-algebra generated by the operators $T_j, j = 1,\ldots,n-1$, $X_i, i = 1,\ldots,n$ and π. Let us give a precise definition by generators and relations.

Definition 2.4.6. *The DAHA $\mathbb{H}_n$ is the $\mathbb{C}_{q,\tau}$-algebra generated by $T_1,\ldots,T_{n-1},X_1^{\pm1},\ldots,X_n^{\pm1},\pi^{\pm1}$ with relations:*

(1) *The quadratic relations for $T_1,\ldots,T_{n-1}$: $(T_i-\tau)(T_i+\tau^{-1}) = 0$.*
(2) *The braid relations for $T_1,\ldots,T_{n-1}$: $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $T_iT_j = T_jT_i$ if $|i-j|>1$.*
(3) *The Laurent relations for $X_1^{\pm},\ldots,X_n^{\pm}$: $X_iX_j = X_jX_i$, $X_iX_i^{-1} = X_i^{-1}X_i = 1$.*
(4) *The action relations involving T_i, X_j: $T_iX_iT_i = X_{i+1}$ if $i = 1,\ldots,n-1$; $T_iX_j = X_jT_i$ if $i \neq j, j-1$.*
(5) *$\pi X_i = X_{i+1}\pi, i = 1,\ldots,n-1$; ; $\pi^n X_i = q^{-1}X_i\pi^n$.*
(6) *$\pi T_i = T_{i+1}\pi$, $i = 1,\ldots,n-2$; $\pi^nT_i = T_i\pi^n$, $i = 1,\ldots,n-1$.*

Remark 2.4.7. *Let us remark that the relations $\pi X_n = q^{-1}X_1\pi$ and $\pi^2T_{n-1} = T_1\pi^2$ are formal corollaries of the relations (5), (6) in Definition 2.4.6.*

Now set

$$Y_i := T_i\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \in \mathbb{H}_n$$

It is clear that the Y_i's satisfy the Laurent relations, as well as the relations

$$T_i^{-1}Y_iT_i^{-1} = Y_{i+1}, \quad T_iY_j = Y_jT_i \text{ if } i \neq j, j-1$$

Let us examine the relations of Y_i with X_j. First of all, since $Y_1\cdots Y_n = \pi^{-n}$, we get

$$\widetilde{Y}X_j = qX_j\widetilde{Y} \tag{2.4.3}$$

where $\widetilde{Y} := Y_1\cdots Y_n$. Now, setting $\widetilde{X} := X_1\cdots X_n$, we have that $\widetilde{X}$ commutes with all the T_i's while we have that $\pi\widetilde{X} = q^{-1}\widetilde{X}\pi$. This easily implies that

$$\widetilde{X}Y_j = q^{-1}Y_j\widetilde{X}. \tag{2.4.4}$$

Finally, we have the following relation.

$$\begin{aligned} Y_2^{-1}X_1Y_2X_1^{-1} = &\ (T_1\pi T_{n-1}^{-1}\cdots T_2^{-1})X_1(T_2\cdots T_{n-1}\pi^{-1}T_1^{-1})X_1^{-1}\\ = &\ T_1\pi T_{n-1}^{-1}\cdots T_2^{-1}T_2\cdots T_{n-1}X_1\pi^{-1}T_1^{-1}X_1^{-1}\\ = &\ T_1(\pi X_1\pi^{-1})T_1^{-1}X_1^{-1}\\ = &\ T_1X_2(T_1^{-1}X_1^{-1}T_1^{-1})T_1\\ = &\ T_1(X_2X_2^{-1})T_1\\ = &\ T_1^2. \end{aligned} \tag{2.4.5}$$

Theorem 2.4.8. *The DAHA $\mathbb{H}_n$ is isomorphic to the $\mathbb{C}_{q,\tau}$-algebra generated by $T_1, \ldots, T_{n-1}$, $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ subject to the following relations.*

1. *The quadratic and braid relations for $T_1, \ldots, T_{n-1}$.*
2. *The Laurent relations for $\{X_1^{\pm 1}, \ldots, X_n^{\pm 1}\}$ and for $\{Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}\}$.*
3. *The action relations for (T_i, X_j) and for (T_i, Y_j).*
4. *Relations (2.4.3), (2.4.4) and (2.4.5).*

Proof. Let $\mathbb{H}'_n$ denote the algebra defined in the statement of the theorem. Define π by

$$\pi := T_1^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_i \cdots T_{n-1}$$

Thanks to the action relations involving T and Y, π is independent of i. We need to check that $T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}, \pi$ satisfy the relations of $\mathbb{H}_n$. Note that we only need to check the relations involving π and X_i. Moreover, since $\pi^n = Y_1^{-1} \cdots Y_n^{-1}$, we only need to check that $\pi X_i = X_{i+1}\pi$ for $i = 1, \ldots, n-1$. Furthermore, note that because of the action relations involving T, X and the relations $\pi T_i = T_{i+1}\pi$, we only need to check the relation $\pi X_1 = X_2 \pi$. Using the relation (2.4.5) we have

$$\begin{aligned}
\pi X_1 = {} & T_1^{-1} Y_2^{-1} T_2 \cdots T_{n-1} X_1 \\
= {} & T_1^{-1} (Y_2^{-1} X_1) T_2 \cdots T_{n-1} \\
= {} & T_1^{-1} (T_1^2 X_1 Y_2^{-1}) T_2 \cdots T_{n-1} \\
= {} & T_1 X_1 (T_1 T_1^{-1}) Y_2^{-1} T_2 \cdots T_{n-1} \\
= {} & (T_1 X_1 T_1) T_1^{-1} Y_2^{-1} T_2 \cdots T_{n-1} \\
= {} & X_2 \pi
\end{aligned}$$

and the result follows. □

Just as in the A_1 case, the T, X, Y presentation of the DAHA $\mathbb{H}_n$ has the advantage of revealing a symmetry between the X and Y parameters.

Lemma 2.4.9. *The following defines a $\mathbb{C}_{q,\tau}$-linear anti-involution of $\mathbb{H}_n$*

$$\phi(X_i) = Y_i^{-1}, \quad \phi(Y_i) = X_i^{-1}, \quad \phi(T_j) = T_j, \quad 1 \le i \le n, 1 \le j \le n-1$$

and the following defines a $\mathbb{C}$-linear involution of $\mathbb{H}_n$:

$$\varepsilon(X_i) = Y_i, \quad \varepsilon(Y_i) = X_i, \quad \varepsilon(T_j) = T_j^{-1}, \quad \varepsilon(\tau) = \tau^{-1}, \quad \varepsilon(q) = q^{-1}, \quad 1 \le i \le n, 1 \le j \le n-1$$

Proof. For the first statement, we only need to check that the relation (2.4.5) is self-dual with respect to ϕ. Note that we can write this relation as:

$$\begin{aligned}
1 = {} & T_1^{-1} Y_2^{-1} X_1 Y_2 X_1^{-1} T_1^{-1} \\
= {} & (T_1^{-1} Y_2^{-1} T_1^{-1})(T_1 X_1 T_1)(T_1^{-1} Y_2 T_1^{-1})(T_1 X_1^{-1} T_1) T_1^{-2} \\
= {} & Y_1^{-1} X_2 (T_1^{-2} Y_1 T_1^{-2})(T_1^2 X_2^{-1} T_1^2) T_1^{-2} \\
= {} & Y_1^{-1} X_2 T_1^{-2} Y_1 X_2^{-1}
\end{aligned} \tag{2.4.6}$$

which we can rewrite as $T_1^2 = Y_1 X_2^{-1} Y_1^{-1} X_2$, and so (2.4.5) is self-dual with respect to ϕ. Note that this is also the equation required to prove that ε extends to a morphism $\mathbb{H} \to \mathbb{H}$. This finishes the proof. □

Exercise 2.4.10. *The following relations hold in $\mathbb{H}_n$:*

$$Y_{i+1}^{-1} X_i Y_{i+1} X_i^{-1} = T_i^2, \quad Y_{j+1}^{-1} X_i Y_{j+1} X_i^{-1} = T_j \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_j^{-1}, j > i$$

2.4.3. *Topological interpretation.* Let E be a 2-torus. Consider the n-fold product E^n, and let $(E^n)^{reg} := \{(x_1, \dots, x_n) \in E^n : x_i \neq x_j \text{ if } i \neq j\}$, $C := (E^n)^{reg}/S_n$. The fundamental group $\pi_1(C)$ is known as the *elliptic braid group.*

Lemma 2.4.11. *We have* $\pi_1(C) = \langle T_1, \dots, T_{n-1}, X_1, \dots, X_n, Y_1, \dots, Y_n\rangle$ *with relations*

$$\begin{array}{ccc} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & T_iT_j = T_jT_i \text{ if } |i-j|>1, & X_iX_j = X_jX_i \\ Y_iY_j = Y_jY_i & T_iX_iT_i = X_{i+1} & T_i^{-1}Y_iT_i^{-1} = Y_{i+1} \\ T_iX_j = X_jT_i, i \neq j, j-1 & T_iY_j = Y_jT_i, i \neq j, j-1 & Y_2^{-1}X_1Y_2X_1^{-1} = T_1^2 \\ (Y_1\cdots Y_n)X_j = X_j(Y_1 \cdots Y_n) & & (X_1\cdots X_n)Y_j = Y_j(X_1\cdots X_n) \end{array}$$

In the previous lemma, the generator X_i corresponds to the i-th point going around a loop in the "horizontal" direction on E; Y_i corresponds to the i-th point going around in the "vertical" direction on E; while T_i corresponds to the transposition of the i-th and $(i+1)$-th points. Let us remark that, unlike the A_1 case, it is *not* possible to renormalize the generators so that the DAHA $\mathbb{H}_n$ becomes an honest quotient of the group algebra of $\pi_1(C)$. However, one may form a *twisted group algebra*, which is a deformation of the group algebra $\pi_1(C)$ arising from a central extension of $\pi_1(C)$ (so that the central element z becomes q in the twisted group algebra) and we indeed have

$$\mathbb{H}_n = \mathbb{C}^{tw}_{q,\tau}\pi_1(C)/((T_i - \tau)(T_i + \tau^{-1}))_{i=1,\dots,n-1}$$

2.4.4. *From* $\mathfrak{gl}_n$ *to* $\mathfrak{sl}_n$. Let us explain how to recover the DAHA for the root system A_{n-1} from $\mathbb{H}_n$. First of all, in the lattice generated by Y_i we must have $Y_1 \cdots Y_n = 1$. Thus, we pass to the algebra

$$\widetilde{\mathbb{H}}_n := \mathbb{H}_n/(\pi^n - 1)$$

In this algebra, we take the subalgebra generated by $T_1, \dots, T_{n-1}$, the elements $\overline{Y}_i := Y_1 \cdots Y_i$, $i = 1, \dots, n-1$ and their multiplicative inverses, and the elements $\overline{X}_i := X_1 \cdots X_i(\widetilde{X})^{-i/n}$. We remark that the element $\widetilde{X}$ does have an n-root in $\mathbb{H}_n$ (this can be seen, for example, using the automorphism ε defined in Lemma 2.4.9 and using the fact that $\widetilde{Y} = \pi^{-n}$) so this expression makes sense. We also take $\overline{X}_i^{-1}$. This subalgebra is isomorphic to $\mathbb{H}(A_{n-1})$.

2.4.5. *Trigonometric degeneration.* Let us introduce the trigonometric degeneration of the DAHA $\mathbb{H}_n$. This is done completely analogously to the A_1 case. So the first thing we need to do is to think of $\mathbb{H}_n$ as a $\mathbb{C}[t,c][[\hbar]]$-algebra. Set

$$Y_i := e^{\hbar\hat{y}_i}, \quad q := e^{t\hbar}, \quad \tau := e^{\hbar c}, \quad T_j := s_je^{\hbar c s_j}, \quad i = 1, \dots, n, \;\; j = 1, \dots, n-1$$

where $s_i \in S_n$ is the transposition $(i, i+1)$. We have $\mathbb{H}_n^{\mathrm{trig}} := \mathbb{H}_n/\hbar\mathbb{H}_n$. So $\mathbb{H}^{\mathrm{trig}}$ is generated by $s_i, i = 1, \dots, n-1$, $X_i^{\pm 1}$, $i = 1, \dots, n$, and $\hat{y}_i$, $i = 1, \dots, n$. We have $s_i^2 = 1$. From the identity $T_iX_iT_i = X_{i+1}$, we get $s_iX_is_i = X_{i+1}$. Let us now examine the identity $T_i^{-1}Y_iT_i^{-1} = Y_{i+1}$, we have

$$(s_i - \hbar c s_i^2 + \hbar^2\frac{c^2s_i^3}{2!} + \dots)(1 + \hbar\hat{y}_i + \hbar^2\frac{\hat{y}_i^2}{2!} + \dots)(s_i - \hbar c s_i^2 + \hbar^2\frac{c^2s_i^3}{2!} + \dots) = 1 + \hbar\hat{y}_{i+1} + \hbar^2\frac{\hat{y}_{i+1}^2}{2!} + \dots$$

looking at the coefficient of $\hbar$ we get the identity $s_i\hat{y}_is_i - 2cs_i = \hat{y}_{i+1}$ or, equivalently,

$$s_i\hat{y}_i - \hat{y}_{i+1}s_i = 2c.$$

Similarly, we have the following relations:

$$\begin{aligned}&(\hat{y}_1 + \cdots + \hat{y}_n)X_j = X_j(t + \hat{y}_1 + \cdots + \hat{y}_n),\\ &X_1 \cdots X_n \hat{y}_j = (-t + \hat{y}_j)X_1 \cdots X_n,\\ &X_1 \hat{y}_2 X_1^{-1} - \hat{y}_2 = 2cs_1\end{aligned}$$

Definition 2.4.12. *The* trigonometric double affine Hecke algebra *for* $\mathfrak{gl}_n$, $\mathbb{H}_n^{\mathrm{trig}}$, *is the* $\mathbb{C}[c,t]$-*algebra with generators* $s_1, \ldots, s_{n-1}, \hat{y}_1, \ldots, \hat{y}_n, X_1^{\pm}, \ldots, X_n^{\pm}$ *subject to the relations*

(1) $s_i^2 = 1, i = 1, \ldots, n-1$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$; $s_i s_j = s_j s_i$ *if* $|i-j| > 1$.
(2) $s_i X_i s_i = X_{i+1}$; $s_i X_j = X_j s_i$ *if* $i \neq j, j-1$.
(3) $s_i \hat{y}_i - \hat{y}_{i+1} s_i = 2c$, $s_i \hat{y}_j = \hat{y}_j s_i$ *if* $i \neq j, j-1$.
(4) $(\hat{y}_1 + \cdots + \hat{y}_n)X_j = X_j(t + \hat{y}_1 + \cdots + \hat{y}_n)$.
(5) $X_1 \cdots X_n \hat{y}_j = (\hat{y}_j - t)X_1 \cdots X_n$.
(6) $X_1 \hat{y}_2 X_1^{-1} - \hat{y}_2 = 2cs_1$

Lemma 2.4.13. *Every element* $h \in \mathbb{H}_n^{\mathrm{trig}}$ *can be uniquely written as a sum*

$$h = \sum_{\substack{P \in \mathbb{C}[X_i^{\pm 1}] \\ w \in S_n \\ f \in \mathbb{C}[\hat{y}_i]}} a_{P,w,f} P(X) w f(\hat{y}), \quad a_{P,w,f} \in \mathbb{C}[c,t]$$

Of course, the lemma is proven by means of an action of $\mathbb{H}_n^{\mathrm{trig}}$ on its *polynomial representation* $\mathbb{C}[c,t][X_1^{\pm 1}, \ldots, X_n^{\pm n}]$, where s_i acts by transposing the i-th and $(i+1)$-th variables, X_i acts by multiplication. Now, for $i = 1, \ldots, n$, define the trigonometric derivative by $\partial_i(X_j) = \delta_{ij} X_j$. So $\hat{y}_i$ acts by the *trigonometric Dunkl operator*

$$D_i^{\mathrm{trig}} := t\partial_i + 2c \sum_{i \neq j} \frac{1}{1 - X_i X_j^{-1}} (\mathrm{id} - s_{ij}) - 2cis_i$$

Corollary 2.4.14. *The following are subalgebras of* $\mathbb{H}_n^{\mathrm{trig}}$:

(1) *The group algebra of the extended affine Weyl group of* $\mathfrak{gl}_n$, *which is isomorphic to the subalgebra of* $\mathbb{H}_n^{\mathrm{trig}}$ *generated by* $s_1, \ldots, s_{n-1}$, $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$.
(2) *The degenerate affine Hecke algebra of* $\mathfrak{gl}_n$, *which is isomorphic to the subalgebra of* $\mathbb{H}_n^{\mathrm{trig}}$ *generated by* $s_1, \ldots, s_{n-1}, \hat{y}_1, \ldots, \hat{y}_n$.

Let us remark that we also have a *difference-rational polynomial representation* of $\mathbb{H}_n^{\mathrm{trig}}$. This is the representation of $\mathbb{H}_n^{\mathrm{trig}}$ on the space $\mathbb{C}[c,t][\hat{y}_1, \ldots, \hat{y}_n]$ which is defined as follows. The element $\hat{y}_i$ just acts by multiplication, the element s_i acts by the *Demazure-Lusztig operator*:

$$S_i := \tilde{s}_i - 2c \frac{1}{\hat{y}_i - \hat{y}_{i+1}} (\tilde{s}_i - \mathrm{id})$$

where $\tilde{s}_i$ is the operator on $\mathbb{C}[c,t][\hat{y}_1, \ldots, \hat{y}_n]$ that transposes the variables $\hat{y}_i$ and $\hat{y}_{i+1}$. To state the action of X_i, first define the operator $\pi : \mathbb{C}[c,t][\hat{y}_1, \ldots, \hat{y}_n] \to \mathbb{C}[c,t][\hat{y}_1, \ldots, \hat{y}_n]$ by

$$\pi(f(\hat{y}_1, \ldots, \hat{y}_n)) = f(\hat{y}_2, \ldots, \hat{y}_n, \hat{y}_1 - t)$$

And now define the action of X_i to be by the operator $S_{i-1} \cdots S_1 \pi S_{n-1} \cdots S_i$. This is known as the *difference-rational Dunkl operator*.

2.4.6. *Rational degeneration.* Now we define the rational degeneration for the DAHA $\mathbb{H}_n$. Similarly to what was done for the case of A_1, let $X_i = \exp(\hbar x_i)$, $y_i := \hbar \hat{y}_i$. Then $s_1, \ldots, s_{n-1}, y_1, \ldots, y_n, x_1, \ldots, x_n$ satisfy the following relations modulo $\hbar$.

$$s_i x_i s_i = x_{i+1}, \quad s_i y_i s_i = y_{i+1}, \quad s_i x_j = x_j s_i (i \neq j, j-1), \quad s_i y_j = y_j s_i (i \neq j, j-1),$$
$$y_i x_j - x_j y_i = 2c s_{ij}, i \neq j \quad (y_1 + \cdots + y_n) x_j = t + x_j (y_1 + \cdots + y_n)$$

Note that, in view of all the other relations, the last relation is equivalent to $y_i x_i - x_i y_i = t - 2c \sum_{j \neq i} s_{ij}$. These are the defining relations for the *rational* DAHA of $\mathfrak{gl}_n$, $\mathbb{H}_n^{\mathrm{rat}}$.

Lemma 2.4.15. *Every element $h \in \mathbb{H}_n^{\mathrm{rat}}$ can be written as*

$$h = \sum_{\substack{f \in \mathbb{C}[x] \\ w \in S_n \\ g \in \mathbb{C}[y]}} a_{f,w,g} f(x) w g(y), \quad a_{f,w,g} \in \mathbb{C}[c, t]$$

We have the polynomial representation $\mathbb{C}[c,t][x_1, \ldots, x_n]$ of $\mathbb{H}_n^{\mathrm{rat}}$. Here, S_n acts by permutation of the indices, x_i acts by multiplication, and y_i acts via the rational Dunkl operator

$$D_i^{\mathrm{rat}} := t \frac{d}{dx_i} - 2c \sum_{j \neq i} \frac{1}{x_i - x_j} (\mathrm{id} - s_{ij})$$

Let us remark that similar degenerations $\mathbb{H}^{\mathrm{trig}}, \mathbb{H}^{\mathrm{rat}}$ exist for a general root system R. This will be the subject of Section 4.

2.4.7. *Braid group action.* The main goal of this section is to produce a braid group action on $\mathbb{H}_n$ by algebra automorphisms.

Lemma 2.4.16. *The following assignment can be extended to an automorphism of $\mathbb{H}_n$:*

$$\rho_1(T_i) = T_i, i = 1, \ldots, n-1, \quad \rho_1(X_j) = X_j, j = 1, \ldots, n, \quad \rho_1(\pi) = X_1^{-1} \pi \tag{2.4.7}$$

Proof. The only relation that is not immediate to check that it is preserved is $\pi^n T_i = T_i \pi^n$. Using the relations $\pi X_i^{-1} = X_{i+1}^{-1} \pi$ if $i < n$

$$\begin{aligned} (X_1^{-1}\pi) \cdots (X_1^{-1}\pi) = & \quad X_1^{-1} (\pi X_1^{-1}) \cdots (\pi X_1^{-1}) \pi \\ = & \quad X_1^{-1} X_2^{-1} \cdots X_n^{-1} \pi^n \end{aligned}$$

Which is the product of a symmetric polynomial in the $X_i's$ and π^n. Both terms commute with all T_i. From here, the result follows. ∎

For completeness, let us give a formula for $\rho_1(Y_i)$. Recall that we have $Y_1 = T_1 \cdots T_{n-1} \pi^{-1}$, so $\rho_1(Y_1) = T_1 \cdots T_{n-1} \pi^{-1} X_1 = Y_1 X_1$. Now, using the fact that $T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$ we get:

$$\rho_1(Y_i) = Y_i X_i (T_{i-1}^{-1} \cdots T_1^{-1})(T_1^{-1} \cdots T_{i-1}^{-1}) \tag{2.4.8}$$

The following lemma can be checked similarly to Lemma 2.4.16.

Lemma 2.4.17. *The following assignment can be extended to an automorphism of $\mathbb{H}_n$:*

$$\rho_2(T_i) = T_i, i = 1, \ldots, n-1; \quad \rho_2(Y_j) = Y_j, \quad \rho_2(X_j) = X_j Y_j (T_{j-1} \cdots T_1)(T_1 \cdots T_{j-1}). \tag{2.4.9}$$

Remark 2.4.18. *Note that $\rho_2 = \varepsilon \rho_1 \varepsilon$, where $\varepsilon : \mathbb{H} \to \mathbb{H}$ is the $\mathbb{C}$-linear involution defined in Lemma 2.4.9.*

Let us give a formula for ρ_2^{-1}. We have that $\rho_2^{-1}(T_i) = T_i, \rho_2^{-1}(Y_j) = Y_j$, while

$$\rho_2^{-1}(X_j) = X_j(T_{j-1}^{-1} \cdots T_1^{-1})(T_1^{-1} \cdots T_{j-1}^{-1})Y_j^{-1}$$

Lemma 2.4.19. *Consider the braid group on three strands, $B_3 := \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. The assignment*

$$\sigma_1 \mapsto \rho_1, \quad \sigma_2 \mapsto \rho_2^{-1}$$

gives an action of B_3 on $\mathbb{H}_n$.

Proof. We need to check that ρ_1, ρ_2^{-1} satisfy the relation:

$$\rho_1\rho_2^{-1}\rho_1 = \rho_2^{-1}\rho_1\rho_2^{-1} \tag{2.4.10}$$

It is obvious that when we evaluate both sides on T_i we just get T_i. Then, thanks to the action relations and the fact that both sides of (2.4.10) are automorphisms, we just need to check that $\rho_1\rho_2^{-1}\rho_1(X_1) = \rho_2^{-1}\rho_1\rho_2^{-1}(X_1)$, and a similar equation for Y_1. We have:

$$\rho_1\rho_2^{-1}\rho_1(X_1) = \rho_1\rho_2^{-1}(X_1) = \rho_1(X_1Y_1^{-1}) = X_1X_1^{-1}Y_1^{-1} = Y_1^{-1}$$
$$\rho_2^{-1}\rho_1\rho_2^{-1}(X_1) = \rho_2^{-1}(Y_1^{-1}) = Y_1^{-1}$$

It is similarly easy to check that $\rho_1\rho_2^{-1}\rho_1(Y_1) = \rho_2^{-1}\rho_1\rho_2^{-1}(Y_1) = Y_1X_1Y_1^{-1}$. The lemma follows. □

Let us remark that the automorphisms ρ_1, ρ_2 descend to the rational degeneration of $\mathbb{H}_n$.

Lemma 2.4.20. *The following define automorphisms of $\mathbb{H}_n^{\mathrm{rat}}$:*

$$\begin{array}{ccc} \rho_1(s_i) = s_i & \rho_1(x_j) = x_j & \rho_1(y_j) = y_j + x_j \\ \rho_2(s_i) = s_i & \rho_2(x_j) = x_j + y_j & \rho_2(y_j) = y_j \end{array}$$

We still have the relations $\rho_1\rho_2^{-1}\rho_1 = \rho_2^{-1}\rho_1\rho_2^{-1}$. Moreover, $(\rho_1\rho_2^{-1}\rho_1)^4 = \mathrm{id}$, so that we have an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}_n^{\mathrm{rat}}$, cf. Lemma 3.3.5.

It is interesting to note that, according to [C2, 2.12.4], the automorphisms ρ_1, ρ_2 have no trigonometric analogue.

3. The polynomial representation

3.1. **Upper triangularity of Y^λ.** We study the polynomial representation more carefully. Our first goal is to see that the operators Y^λ are upper triangular with respect to a certain partial order on P. First of all, recall that we have the partial order $<$ on P^+, which is defined by $\nu > \mu$ if $\nu - \mu \in P^+$. We extend this order to P.

Definition 3.1.1. *For $\mu \in P$, let $\mu^+ \in P^+$ be the dominant weight lying in the orbit $W\mu$. Define a partial order on the weight lattice P as follows: $\nu \prec \mu$ if $\nu^+ < \mu^+$, or $\nu^+ = \mu^+$ and $\nu > \mu$ (note the change of signs!)*

Let us give some properties of the order $\prec$ that will be useful later.

Lemma 3.1.2. *Let $\mu \in P$ and let $\alpha \in R_+$*

1. *If $\langle \mu, \alpha^\vee \rangle = r > 0$, then $s_\alpha(\mu) \succ \mu$, while $\mu - \alpha, \dots, \mu - (r-1)\alpha \prec \mu$.*
2. *If $\langle \mu, \alpha^\vee \rangle = -r < 0$, then $\mu + \alpha, \dots, \mu + (r-1)\alpha, s_\alpha(\mu) \prec \mu$.*

Proof. See e.g. [M, Section 2.6]. □

Recall now from Seth's talk that, if $\lambda \in P_+^\vee$ is such that $t(\lambda) = \pi_r s_{i_\ell} \cdots s_{i_1}$ is a reduced expression, then $Y^\lambda = \pi_r T_{i_l} \cdots T_{i_1}$. We would like to use this to obtain some information about the operator Y^λ. First of all, if $a \in R^a$ is a root, define the operator

$$G(a) := \tau_a + (\tau_a - \tau_a^{-1})\frac{\mathrm{id} - s_a}{X^{-a} - 1}$$

where $\tau_a = \tau_i$ if $w(a) = a_i$ for some element $w \in W^a$. Note that $T_i = s_i G(a_i)$ and, if $w \in W^a$, $G(w(a)) = wG(a)w^{-1}$. In particular, if s is a reflection we have $G(a)s = sG(s(a))$. Thus, for $\lambda \in P_+^\vee$ we have

$$Y^\lambda = \pi_r s_{i_\ell} G(a_{i_\ell}) s_{i_{\ell-1}} G(a_{i_{\ell-1}}) \cdots s_{i_1} G(a_{i_1}) = t(\lambda) G(a^{(\ell)}) \cdots G(a^{(1)}),$$

where $a^{(j)} = s_{i_1} \cdots s_{i_{j-1}}(a_{i_j})$.

Theorem 3.1.3. *For $\lambda \in P^\vee, \mu \in P$ we have*

$$Y^\lambda(X^\mu) = \sum_{\nu \preceq \mu} c_{\mu,\nu} X^\nu$$

with $c_{\mu,\nu} \in \mathbb{C}_{q,\tau}$.

Proof. Assume first that $\lambda \in P_+^\vee$, so that in particular $Y_\lambda = t(\lambda)G(a^{(\ell)}) \cdots G(a^{(1)})$ with $a^{(i)} = \alpha_i + k_i\delta$, $\alpha_i \in R^+$. So let $a = \alpha + k\delta$ with $\alpha \in R^+$. Then we have:

$$G(a)X^\mu = \tau_a X^\mu + (\tau_a - \tau_a^{-1})\frac{X^{s_a(\mu)} - X^\mu}{1 - X^{-a}}$$

Now assume that $\langle \alpha^\vee, \mu \rangle = r > 0$, so $s_a(\mu) = \mu - ra$. Thus, we have

$$G(a)X^\mu = \tau_a X^\mu - (\tau_a - \tau_a^{-1})(X^\mu + X^{\mu-a} + \cdots + X^{\mu-(r-1)a}) = \tau_\alpha^{-1} X^\mu + \ldots$$

where the ellipsis stands for lower order terms, see Lemma 3.1.2. The case $\langle \alpha^\vee, \mu \rangle < 0$ is similar, for $\langle \alpha^\vee, \mu \rangle = 0$ we just have $G(a)X^\mu = \tau_a X^\mu$. Since $t(\lambda)$ is diagonal, we have that Y^λ is a composition of upper triangular operators and the result follows.

Now if $\lambda = \lambda' - \lambda''$ with $\lambda', \lambda'' \in P_+^\vee$, then $Y^\lambda = Y^{\lambda'}(Y^{\lambda''})^{-1}$. Since the inverse of an upper triangular operator is again upper triangular, the result follows. □

3.2. **Difference operators.** The goal of this section is to produce some difference operators on the space $\mathbb{C}_{q,\tau}[X]$ using the representation theory of DAHA. Recall from the proof of Theorem 2.2.4 that for every $w \in W^{ae}$, the extended affine Weyl group, the action of T_w on $\mathbb{C}_{q,\tau}[X]$ may be written as

$$T_w = \sum_{\substack{\lambda \in P^\vee \\ w \in W}} g_{\lambda,w} t(\lambda) w, \qquad g_{\lambda,w} \in \mathbb{C}_{q,\tau}(X) \tag{3.2.1}$$

so in particular the same is true for $Y^\lambda \in \mathcal{H}^Y \subseteq \mathbb{H}$. Recall that the center of $\mathcal{H}^Y$ is precisely $\mathbb{C}_{q,\tau}[Y]^W$.

Lemma 3.2.1. *Let $f(Y) \in \mathbb{C}_{q,\tau}[Y]^W$. Then, the action of $f(Y)$ on $\mathbb{C}_{q,\tau}[X]$ preserves the space of W-invariants $\mathbb{C}_{q,\tau}[X]^W$.*

Proof. Note that, from the formula for the action of T_i, $i = 1, \ldots, n$, on $\mathbb{C}_{q,\tau}[X]$ it follows that $p(X) \in \mathbb{C}_{q,\tau}[X]$ is W-invariant if and only if

$$T_i p(X) = \tau_i p(X), i = 1, \ldots, n$$

From here, the result follows easily using the fact that $f(Y)$ commutes with T_i. □

Now let f be an operator on $\mathbb{C}_{q,\tau}[X]$ of the form (3.2.1). We define its restriction by

$$\operatorname{Res}(f) := \sum_{\substack{\lambda\in P^\vee \\ w\in W}} g_{\lambda,w}\tau(\lambda)$$

In particular, $\operatorname{Res}(f)$ is a difference operator on $\mathbb{C}_{q,\tau}[X]$ and $f|_{\mathbb{C}_{q,\tau}[X]^W} = \operatorname{Res}(f)|_{\mathbb{C}_{q,\tau}[X]^W}$. For $f \in \mathbb{C}_{q,\tau}[Y]^W$, we denote $L_f := \operatorname{Res}(f)$. This is a difference operator on $\mathbb{C}_{q,\tau}[X]$ preserving the space of invariants $\mathbb{C}_{q,\tau}[X]^W$.

Corollary 3.2.2. *The operators $L_f, f \in \mathbb{C}_{q,\tau}[Y]^W$, are pairwise commutative and W-invariant.*

Proof. Let $f, g \in \mathbb{C}_{q,\tau}[Y]^W$. Then $L_fL_g = \operatorname{Res}(f)\operatorname{Res}(g)$. Since g is W-invariant, $\operatorname{Res}(f)\operatorname{Res}(g) = \operatorname{Res}(fg) = \operatorname{Res}(gf)$. Since now f is W-invariant, we get $\operatorname{Res}(gf) = \operatorname{Res}(g)\operatorname{Res}(f) = L_gL_f$. □

Remark 3.2.3. *It follows from Theorem 3.1.3 that the operators $L_f : \mathbb{C}_{q,\tau}[X]^W \to \mathbb{C}_{q,\tau}[X]^W$ are upper triangular with respect to the basis formed by $\{x_\lambda := \sum_{\lambda'\in W\lambda} X^{\lambda'}\}_{\lambda\in P^+}$ and the dominance ordering on P^+.*

The operators L_f are intimately related to the theory of Macdonald's polynomials. This will be the subject of a subsequent talk.

3.2.1. *Example: A_1.* Let us consider the example of a root system of type A_1. We keep the notation of Section 2.3.2, with one small caveat. Now we set $X := X^\alpha$, so that $\mathbb{C}_{q,\tau}[X]$ is the algebra of polynomials in $X^{\pm 1/2}$. With this convention, the action of T is given by

$$\tau s + (\tau - \tau^{-1})\frac{s - \mathrm{id}}{X - 1}$$

While the action of $t(\rho)$ is $t(\rho)(X) = q^2X$, so $t(\rho)f(X) = f(q^2X)$ and, in particular, $\pi_\rho(X) = t(\rho)s(X) = q^{-2}X^{-1}$. Note that we have

$$Y^\rho = \quad t(\rho)\left(\tau + (\tau - \tau^{-1})\frac{\mathrm{id} - s}{X^{-1} - 1}\right)$$

Let us now deal with $Y^{-\rho} = T^{-1}\pi_\rho$. Here we will use the reflection s_0 on the affine Weyl group: it is easy to check that we have a relation $\pi_\rho T^{-1}\pi_\rho = T_0^{-1}$, so that $Y^{-\rho} = \pi_\rho T_0^{-1} = \pi_\rho(T_0 + (\tau^{-1} - \tau))$. Thus,

$$\begin{aligned} Y^{-\rho} = &\quad \pi_\rho\left(\tau_{s_0} + (\tau - \tau^{-1})\frac{s_0 - \mathrm{id}}{q^{-2}X^{-1} - 1} + (\tau^{-1} - \tau)\right) \\ = &\quad t(\rho)s\left(\tau t(\alpha)s + (\tau - \tau^{-1})\frac{t(\alpha)s - q^{-2}X^{-1}}{q^{-2}X^{-1} - 1}\right) \\ = &\quad \tau t(-\rho) + (\tau - \tau^{-1})t(\rho)\frac{t(-\alpha) - q^{-2}Xs}{q^{-2}X - 1} \\ = &\quad \tau t(-\rho) + (\tau - \tau^{-1})\frac{t(-\rho) - Xt(\rho)s}{X - 1} \end{aligned}$$

Thus, $\operatorname{Res} Y^\rho = \tau t(\rho)$, $\operatorname{Res} Y^{-\rho} = \dfrac{\tau X - \tau^{-1}}{X - 1}t(-\rho) + (\tau - \tau^{-1})\dfrac{1}{X^{-1} - 1}t(\rho)$. So

$$\operatorname{Res}(Y^\rho + Y^{-\rho}) = \frac{\tau X^{-1} - \tau^{-1}}{X^{-1} - 1}t(\rho) + \frac{\tau X - \tau^{-1}}{X - 1}t(-\rho). \tag{3.2.2}$$

This is (a scalar multiple of) Macdonald's difference operator for A_1. The symmetric polynomials here are spanned by binomials of the form $X^{i/2} + X^{-i/2}, i \geq 0$. It is an easy exercise to check that the operator (3.2.2) indeed preserves the space of symmetric polynomials, and that it is upper triangular with respect to the basis $x_i := X^{i/2} + X^{-i/2}$.

3.3. **Spherical DAHA.** We have seen that the operators $L_f, f \in \mathbb{C}_{q,\tau}[X]^W$ define difference operators on the space of W-invariant polynomials on X. We can actually define a smaller algebra than the DAHA $\mathbb{H}$ which includes all the operators L_f and which acts on $\mathbb{C}_{q,\tau}[X]^W$. This is known as the *spherical* DAHA and it is constructed as follows.

Let $\mathbb{C}_\tau$ be the 1-dimensional (over $\mathbb{C}_\tau$) representation of the finite Hecke algebra H where T_i acts by τ_i, $i = 1, \dots, n$. We can realize this representation as $\mathbb{C}_\tau = H\mathbf{e}$, where $\mathbf{e} \in H$ is an idempotent which is constructed as follows. For $w \in W$ let $\tau_w := \tau_{i_1} \cdots \tau_{i_k}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition. Note that τ_w is well-defined, since it is the scalar by which w acts on the representation $\mathbb{C}_\tau$. Now define $\widetilde{\mathbf{e}} := \sum_{w \in W} \tau_w T_w$.

Lemma 3.3.1. *For $i = 1, \dots, n$, we have $T_i \widetilde{\mathbf{e}} = \tau_i \widetilde{\mathbf{e}}$.*

Proof. We will do a direct calculation. We will need the following equation that we have already seen in Seth's talk. In the finite Hecke algebra H:

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ T_{s_i w} + (\tau_i - \tau_i^{-1}) T_w & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

where the length ℓ is the usual one in W, i.e., the length of a reduced expression of w. Thus, we have:

$$T_i \widetilde{\mathbf{e}} = \sum_{\substack{w \in W \\ \ell(s_i w) > \ell(w)}} \tau_w T_{s_i w} + \sum_{\substack{w \in W \\ \ell(s_i w) < \ell(w)}} \tau_w (T_{s_i w} + (\tau_i - \tau_i^{-1}) T_w)$$

Now we find the coefficient of T_w in the previous expression. We have two cases. If $\ell(s_i w) < \ell(w)$, then we have that the coefficient of T_w is $\tau_{s_i w} + \tau_w(\tau_i - \tau_i^{-1}) = \tau_i \tau_w$, since $\tau_w = \tau_i \tau_{s_i w}$. If $\ell(s_i w) > \ell(w)$, then the coefficient of T_w is simply $\tau_{s_i w} = \tau_i \tau_w$. We are done. $\square$

Remark 3.3.2. *Similarly, we can see that $\widetilde{\mathbf{e}} T_i = \tau_i \widetilde{\mathbf{e}}$ for $i = 1, \dots, n$.*

Thanks to the previous lemma, $\widetilde{\mathbf{e}}^2 = \sum_{w \in W} \tau_w T_w \widetilde{\mathbf{e}} = \sum_{w \in W} \tau_w^2 \widetilde{\mathbf{e}}$. Thus

$$\mathbf{e} := \left(\sum_{w \in W} \tau_w^2 \right)^{-1} \widetilde{\mathbf{e}}$$

is an idempotent.

Definition 3.3.3. *Define the* spherical DAHA *as $\mathbb{SH} := \mathbf{e}\mathbb{H}\mathbf{e}$. This is a non-unital subalgebra of $\mathbb{H}$, with unit $\mathbf{e}$.*

Remark 3.3.4. *In the $\mathfrak{gl}_n$ case, note that the automorphisms ρ_1, ρ_2 of $\mathbb{H}_n$ preserve the idempotent $\mathbf{e}$, hence they also preserve the spherical subalgebra. So we have an action of B_3 on $\mathbb{SH}_n$.*

The following result will be important to connect DAHA's to EHA's, which is one of the objectives of the course. First, we recall a well-known result. For a proof, see e.g. [KT, Appendix A].

Lemma 3.3.5. *The group $\mathrm{SL}_2(\mathbb{Z})$ is a quotient of the braid group on three strands $B_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. The quotient map $B_3 \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z})$ is given by*

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

the kernel of this map is generated by $(\sigma_1\sigma_2\sigma_1)^4$.

Theorem 3.3.6. *The braid group action on $\mathbb{SH}_n$ factors through $\mathrm{SL}_2(\mathbb{Z})$, that is, $(\rho_1 \rho_2^{-1} \rho_1)|_{\mathbb{SH}_n}^4 = \mathrm{id}_{\mathbb{SH}_n}$.*

Proof. According to [C2, 3.2.2], $(\rho_1\rho_2^{-1}\rho_1)^4$ is conjugation by $T_{w_0}^{-2}$, where w_0 is the longest element of S_n. Since $T_{w_0}\mathbf{e} = \tau_{w_0}\mathbf{e}$, the result follows easily. $\square$

Note that, if M is a $\mathbb{H}$-module, then $\mathbf{e}M$ becomes a $\mathbb{SH}$-module. For the polynomial representation we have:

$$\mathbf{e}\mathbb{C}_{q,\tau}[X] = \{f \in \mathbb{C}_{q,\tau}[X] : T_i f = \tau_i f, i = 1, \dots, n\} = \mathbb{C}_{q,\tau}[X]^W$$

Now note that, for $f \in \mathbb{C}_{q,\tau}[Y]^W$, we have that $\mathbf{e}L_f\mathbf{e}|_{\mathbb{C}_{q,\tau}[X]^W} = L_f|_{\mathbb{C}_{q,\tau}[X]^W}$. Thus, the action of the spherical DAHA $\mathbb{SH}$ on $\mathbb{C}_{q,\tau}[X]^W$ already includes the operators L_f defined above.

4. Degenerations

In this section, we give definitions that generalize the degenerate (trigonometric and rational) DAHA's from Sections 2.3.4, 2.3.5, 2.4.5, 2.4.6. These algebras can be obtained from $\mathbb{H}$ in a very similar manner to what was done there.

4.1. **Trigonometric degeneration.**

Let us first define the trigonometric DAHA. In order to do this, let c_i, $i = 0, \dots, n$ be formal variables such that $c_i = c_j$ whenever s_i and s_j are conjugate. We will also take commuting variables $\hat{y}_1, \dots, \hat{y}_n$ and, for $b \in P^\vee$, we will denote

$$\hat{y}_b := \sum \langle b, \alpha_j \rangle \hat{y}_j.$$

Let us remark that the extended affine Weyl group $W^{ae} = W \ltimes t(P)$ acts on the space $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$ by algebra automorphisms. Indeed, we need to define the action of $s_1, \dots, s_n$ and $t(\lambda)$, $\lambda \in P$ on elements of the form $\hat{y}_b, b \in P^\vee$. We have that $s_i\hat{y}_b = \hat{y}_{s_i(b)}$ for $i = 1, \dots, n$, while $t(\lambda)\hat{y}_b = \hat{y}_b - \langle \lambda, b \rangle t$.

Definition 4.1.1. *The* trigonometric DAHA, $\mathbb{H}^{\mathrm{trig}}$ *is the* $\mathbb{C}[c,t]$*-algebra generated by the extended affine Weyl group* W^{ae} *and pairwise commuting variables* $\hat{y}_1, \dots, \hat{y}_n$*, subject to the following relations.*

$$s_i\hat{y}_b - \hat{y}_{s_i(b)}s_i = -c_i\langle b, \alpha_i \rangle, \quad s_0\hat{y}_b - s_0(\hat{y}_b)s_0 = c_0\langle b, \theta \rangle, \quad \pi_r\hat{y}_b = \hat{y}_{\pi_r(b)}\pi_r \tag{4.1.1}$$

for $i = 1, \dots, n$*,* $b \in P^\vee$*, and* $\pi_r \in \Omega'(\cong P/Q)$*.*

Let us remark that the variable t appears in disguise in the second relation of (4.1.1).

Since, unlike the nondegenerate and rational cases, the variables X, $\hat{y}$ are not symmetric, the algebra $\mathbb{H}^{\mathrm{trig}}$ admits more than one polynomial representation. First, we have the *differential polynomial representation*, which is given in terms of trigonometric differential Dunkl operators. In order to do this, for $b \in P^\vee$, define the following derivation on the group algebra $\mathbb{C}[c,t][X]$ of the weight lattice P:

$$\partial_b(X^a) = \langle b, a \rangle X^a$$

We have then that $\mathbb{H}^{\mathrm{trig}}$ acts on $\mathbb{C}[c][X]$. The group W acts naturally and y_b acts via the *trigonometric differential Dunkl operator*

$$D_b^{\mathrm{trig}} := t\partial_b + \sum_{\alpha \in R^+} \frac{c_\alpha \langle b, \alpha^\vee \rangle}{1 - X^{-\alpha}}(\mathrm{id} - s_\alpha) - \langle \rho_c, b \rangle$$

where ρ_c is the formal expression $\rho_c := \frac{1}{2}\sum_{\alpha \in R^+} c_\alpha \alpha.$

We also have the *difference-rational polynomial representation*, on the algebra $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$. Recall that the extended affine Weyl group W^{ae} acts on this space by algebra automorphisms. We deform this action by the *Demazure-Lusztig operators*:

$$S_i := s_i + \frac{c_i}{\hat{y}_{\alpha_i}}(s_i - \mathrm{id}), i = 0, \dots, n$$

where $y_{\alpha_0} := -y_\theta + t$. And define, for $w \in W^{ae}$ with $w = \pi_r s_{i_1} \cdots s_{i_\ell}$ a reduced expression, $S_w := \pi_r S_{i_1} \cdots S_{i_\ell}$. According to [C2, 1.6], this still defines an action of W^{ae} on $\mathbb{C}[c,t][\lambda]$. Here we only check that $S_i^2 = \mathrm{id}$. Indeed, we have for $i \neq 0$

$$\begin{aligned} S_i \hat{y}_a = & \left(s_i + \frac{c_i}{\hat{y}_{\alpha_i}}(s_i - \mathrm{id}) \right) \hat{y}_a \\ = & \ \hat{y}_{s_i(a)} + \frac{c_i}{\hat{y}_{\alpha_i}} \left(\hat{y}_{s_i(a)} - \hat{y}_a \right) \\ = & \ \hat{y}_{s_i(a)} + \frac{c_i}{\hat{y}_{\alpha_i}} \left(\hat{y}_a - \hat{y}_{\langle \alpha_i^\vee, a \rangle \alpha_i} - \hat{y}_a \right) \\ = & \ \hat{y}_{s_i(a)} + \frac{c_i}{\hat{y}_{\alpha_i}} \left(-\langle \alpha_i^\vee, a \rangle \hat{y}_{\alpha_i} \right) \\ = & \ \hat{y}_{s_i(a)} - c_i \langle \alpha_i^\vee, a \rangle .1 \end{aligned} \tag{4.1.2}$$

Thanks to (4.1.2), we have that $S_i(\hat{y}_{s_i(a)}) = \hat{y}_a + c_i\langle \alpha_i^\vee, a\rangle.1$. It follows from (4.1.2) again that $S_i^2 = \mathrm{id}$. Let us now treat the case $i = 0$. First of all, note that $s_0 \hat{y}_a = \hat{y}_a + \langle a, \theta^\vee \rangle (t - \hat{y}_\theta)$. Then, we have:

$$\begin{aligned} S_0 \hat{y}_a = & \left(s_0 + \frac{c_0}{t - \hat{y}_\theta}(s_0 - \mathrm{id}) \right) \hat{y}_a \\ = & \ \hat{y}_a + \langle a, \theta^\vee \rangle (t - \hat{y}_\theta) + \frac{c_0}{t - \hat{y}_\theta} \langle a, \theta^\vee \rangle (t - \hat{y}_\theta) \\ = & \ \hat{y}_a + \langle a, \theta^\vee \rangle (t - \hat{y}_\theta) + c_0 \langle \theta^\vee, a \rangle .1 \end{aligned} \tag{4.1.3}$$

It follows from (4.1.3), the fact that S_0 clearly fixes c, t and 1, and that $\langle \theta, \theta^\vee \rangle = 2$, that $S_0^2 = \mathrm{id}$.

Theorem 4.1.2 (See e.g. Proposition 1.6.3 in [C2])**.** *The algebra $\mathbb{H}^{\mathrm{trig}}$ acts on the space $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$, where elements of the group W act via S_w, and $\hat{y}_b$ acts by multiplication. This representation is faithful and it is known as the* difference-rational polynomial representation*.*

For $b \in P$, the operators $S_{t(b)}$ are known as the *difference-rational Dunkl operators*.

Corollary 4.1.3. *The following are subalgebras of $\mathbb{H}^{\mathrm{trig}}$:*

1. *The group algebra of W, in a natural way.*
2. *The degenerate affine Hecke algebra for W, which is the algebra generated by W and $\hat{y}_1, \dots, \hat{y}_n$.*

4.2. **Rational degeneration.** We also have a rational degeneration. Here, we substitute the group algebras of the lattices P and $P^\vee$ by the vector spaces $V^* \cong V$ where our root systems R, $R^\vee$ are defined.

Definition 4.2.1. *The* rational DAHA*, $\mathbb{H}^{\mathrm{rat}}$, is the $\mathbb{C}[c,t]$-algebra generated by $\mathbb{C}[V], \mathbb{C}[V^*]$ and the group W subject to the relations*

$$wx = w(x)w, \quad wy = w(y)w \quad [y,x] = t\langle y,x\rangle - \textstyle\sum_{\alpha \in R^+} c_\alpha \langle y, \alpha \rangle \langle \alpha^\vee, x \rangle s_\alpha, \quad w \in W, x \in V^*, y \in V$$

The algebra $\mathbb{H}^{\mathrm{rat}}$ admits a polynomial representation on the space $\mathbb{C}[V]$. Here, W acts in a natural way, and $x \in V^*$ acts by multiplication. Now recall that $y \in V$ defines a derivation on $\mathbb{C}[V]$, by setting $\partial_y(x) = \langle y, x\rangle$, $x \in V^*$. Then, we define the *rational Dunkl operator*

$$D_y^{\mathrm{rat}} := t\partial_y - \sum_{\alpha \in R^+} c_\alpha \frac{\langle \alpha, y \rangle}{\alpha} (\mathrm{id} - s_\alpha)$$

Theorem 4.2.2. *The assignment $w \mapsto w$, $x \mapsto x$, $y \mapsto D_y^{\mathrm{rat}}$ defines a representation of $\mathbb{H}^{\mathrm{rat}}$ on $\mathbb{C}[c,t][V]$. This is known as the* polynomial representation*, and it is a faithful representation of $\mathbb{H}^{\mathrm{rat}}$.*

Remark 4.2.3. *The hard part of the previous theorem is to prove that the Dunkl operators commute.*

Corollary 4.2.4. *The algebras $\mathbb{C}[c,t][V], \mathbb{C}[c,t][V^*], \mathbb{C}[c,t]W$ sit naturally as subalgebras of $\mathbb{H}^{\mathrm{rat}}$.*

Let us remark that, unlike $\mathbb{H}$ and $\mathbb{H}^{\mathrm{trig}}$, the definition of the rational DAHA $\mathbb{H}^{\mathrm{rat}}$ can be generalized to the case where W is a group generated by complex reflections acting on a vector space V (so W is not necessarily the Weyl group of a root system). This has been done in [EG].

Remark 4.2.5. *We can also define spherical subalgebras $\mathbb{SH}^{\mathrm{trig}}$, $\mathbb{SH}^{\mathrm{rat}}$ of the degenerate DAHAs. There are defined as $e\mathbb{H}^{\mathrm{trig}}e$, $e\mathbb{H}^{\mathrm{rat}}e$, respectively, where the idempotent e now is the trivial idempotent of the group W, that is, $e = \dfrac{1}{|W|}\sum_{w\in W} w$.*

Remark 4.2.6. *Let us remark that, just as we did in Section 3.2, we can use the representation theory of DAHA to define a large family of commuting differential (resp. difference) operators on $\mathbb{C}[c,t][X]$ or $\mathbb{C}[c,t][x_1,\dots,x_n]$ (resp. on $\mathbb{C}[c,t][\lambda]$) that restrict to differential (resp. difference) operators on the W-invariant subalgebras. These operators are given by elements in $\mathbb{C}[c,t][\hat{y}]^W$, $\mathbb{C}[c,t][y]^W$ and $\mathbb{C}[c,t][X]^W$, respectively.*

4.3. **Integrable systems.** The degenerate DAHA are connected to the theory of the Olshanetsky-Perelomov integrable systems, aka generalized Calogero-Moser integrable systems. In this section we elaborate on this connection. Here, we treat the differential case (i.e., rational DAHA,) the difference (i.e., trigonometric) case can be done by similar methods, see e.g. [C1]. Recall that we have a root system $R \subseteq V^* \cong V$, where V is a vector space with nondegenerate form $\langle\cdot,\cdot\rangle$. For the rest of these notes, we specialize to $t = 1$.

Definition 4.3.1. *The* quantum Olshanetsky-Perelomov Hamiltonian *of R is the differential operator*

$$H := \Delta_V - \sum_{\alpha\in R^+} \frac{c_\alpha(c_\alpha+1)\langle\alpha,\alpha\rangle}{\alpha^2}$$

where Δ_V is the Laplace operator on V, and $c_\alpha \in \mathbb{C}$ are such that $c_\alpha = c_{w(\alpha)}$ for every $w \in W$.

Example 4.3.2. *Perhaps, the quantum Olshanetsky-Perelomov Hamiltonian has the clearest physical meaning in type A. Here (taking $V = \mathbb{C}^n$ instead of $\mathbb{C}^{n-1}$) we have*

$$H = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{1\leq i<j\leq n} \frac{2c(c+1)}{(x_i-x_j)^2}$$

which is the quantum Hamiltonian for a system of n particles on the line interacting with potential $c(c+1)/(x_i-x_j)^2$.

Our goal is to see that the quantum system defined by the Olshanetsky-Perelomov Hamiltonian is completely integrable. Let us be a bit more explicit about this. Consider the action of the Weyl group W on the symmetric algebra $S(V)$. According to the Chevalley-Shepard-Todd theorem, the algebra of invariants $S(V)^W$ is polynomial, with algebraically homogeneous generators $P_1,\dots,P_n$ of degrees $d_1,\dots,d_n$, respectively. Recall also that we have the symbol map, $D(V) \to S(V^*)\otimes S(V)$, that to each differential operator associates its symbol. Note, however, that we need a slight extension of this: the hamiltonian H does not belong to $D(V)$. We can consider the principal open subset V^{reg} that is the complement of the union of the hyperplanes $\langle\alpha,\cdot\rangle = 0$. Then, we have a symbol map $\sigma : D(V^{reg}) \to \mathbb{C}[V^{reg}]\otimes S(V)$. For example, $\sigma(H) = P$, where $P(p) = \langle p,p\rangle$, and we use the inner product on V^* that is dual to the inner product on V. Note also that $\sigma(H) \in S(V)^W$.

In the sequel, we will assume that V is an irreducible representation of W. So $\sigma(H) = P_1$ where, recall, we denote $P_1, \ldots, P_n$ the algebraically independent homogeneous generators of $S(V)^W$.

Theorem 4.3.3. *The system defined by the quantum Olshanetsky-Perelomov Hamiltonian is completely integrable. More precisely, there exist algebraic differential operators $H_1, \ldots, H_n$ on V^{reg} such that:*

1. $H_1 = H$.
2. $\sigma(H_i) = P_i$.
3. $[H_i, H_j] = 0$.

Remark 4.3.4. *If we do not assume that V is an irreducible representation of W, then Theorem 4.3.3 is still valid with the exception that (1) should be replaced by $H_2 = H$, see e.g. Example 4.3.2, where we have $P_i = \sum_{i=1}^n x^i$.*

The idea to prove this theorem is similar to what we have done in Section 3.2. So, first of all, if $f = \sum_{w\in W} f_w w$ is an operator on V^{reg}, where $f_i \in D(V^{reg})$, define

$$\mathrm{Res}(f) = \sum_{w\in W} f_w$$

So that $\mathrm{Res}(f)$ is a differential operator. Note that if g is W-invariant, then $\mathrm{Res}(fg) = \mathrm{Res}(f)\,\mathrm{Res}(g)$ for any operator f of a similar form. Now let $y_1, \ldots, y_n$ be an orthonormal basis of V. So, considering the algebra $\mathbb{C}[y_1, \ldots, y_n] \subseteq \mathbb{H}^{\mathrm{rat}}$ as an algebra of operators on V^{reg}, which we can do thanks to the Dunkl representation, we have the following result, which is proven similarly to the results in Subsection 3.2.

Lemma 4.3.5. *For every $f \in \mathbb{C}[y_1, \ldots, y_n]^W$, denote $L_f := \mathrm{Res}(f)$. Then, $\{L_f : f \in \mathbb{C}[y_1, \ldots, y_n]^W\}$ form a commuting family of differential operators with coefficients being rational functions on V regular on V^{reg}. Moreover, $\sigma L_{P_i} = P_i$.*

So what remains to do is to relate the operator H to L_{P_1}.

Proposition 4.3.6. *We have*

$$L_{P_1} = \Delta_V - \sum_{\alpha\in R^+} \frac{c_\alpha\langle\alpha,\alpha\rangle}{\alpha}\partial_{\alpha^\vee}$$

Proof. We need to compute $\mathrm{Res}(\sum_{i=1}^n D_{y_i}^2)$, where we denote $D_{y_i} := D_{y_i}^{\mathrm{rat}}$. First of all, note that $\mathrm{Res}(D_{y_i}^2) = \mathrm{Res}(D_{y_i}\partial_{y_i})$. Now, for every $y \in V$ we have

$$\begin{aligned} D_y\partial_y = &\ \partial_y^2 - \textstyle\sum_{\alpha\in R^+} c_\alpha \frac{\langle\alpha,y\rangle}{\alpha}(\mathrm{id} - s_\alpha)\partial_y \\ = &\ \partial_y^2 - \textstyle\sum_{\alpha\in R^+} \frac{\langle\alpha,y\rangle}{\alpha}(\partial_y(\mathrm{id} - s_\alpha) + [\partial_y, s_\alpha]) \\ = &\ \partial_y^2 - \textstyle\sum_{\alpha\in R^+} \frac{\langle\alpha,y\rangle}{\alpha}(\partial_y(\mathrm{id} - s_\alpha) + \langle\alpha,y\rangle\partial_{\alpha^\vee}s) \end{aligned}$$

From where the result follows easily. □

Let us denote $\overline{H} := L_{P_1}$. It is not the quantum OP Hamiltonian, but we can get H via an automorphism $\varphi : D(V^{reg}) \to D(V^{reg})$, which is defined by $\varphi(f) = f$, $f \in \mathbb{C}[V^{reg}]$, $\varphi(\partial_y) = \partial_y - \sum_{\alpha\in R^+} c_\alpha\frac{\langle y,\alpha\rangle}{\alpha}$. It is an exercise to check that φ indeed defines an automorphism of $D(V^{reg})$. The next result finishes the proof of Theorem 4.3.3.

Lemma 4.3.7. *We have $\varphi(H) = \overline{H}$.*

Proof. We have

$$\begin{aligned}\varphi(\partial_{y_i}^2) = & \left(\partial_{y_i} - \sum_{\alpha\in R^+} c_\alpha \frac{\langle \alpha, y_i\rangle}{\alpha}\right)^2 \\ = & \ \partial_{y_i}^2 - \sum_{\alpha\in R^+} c_\alpha \langle\alpha, y\rangle(\partial_{y_i}\alpha^{-1} + \alpha^{-1}\partial_{y_i}) + \sum_{\alpha,\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha, y_i\rangle\langle\alpha', y_i\rangle}{\alpha\alpha'} \\ = & \ \partial_{y_i}^2 - 2\sum_{\alpha\in R^+} c_\alpha\langle\alpha, y_i\rangle\alpha^{-1}\partial_{y_i} + \sum_{\alpha\in R^+} c_\alpha\frac{\langle\alpha, y_i\rangle^2}{\alpha^2} + \sum_{\alpha,\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha, y_i\rangle\langle\alpha', y_i\rangle}{\alpha\alpha'}\end{aligned}$$

So it follows that

$$\begin{aligned}\varphi(\Delta_V) = & \ \Delta_V - \sum_{\alpha\in R^+}\frac{c_\alpha}{\alpha}\sum_{i=1}^n 2\langle\alpha, y_i\rangle\partial_{y_i} + \sum_{\alpha\in R^+}\frac{c_\alpha}{\alpha^2}\sum_{i=1}^n\langle\alpha, y_i\rangle^2 + \sum_{\alpha,\alpha'\in R^+}\frac{c_\alpha c_{\alpha'}}{\alpha\alpha'}\sum_{i=1}^n\langle\alpha, y_i\rangle\langle\alpha', y_i\rangle \\ = & \ \Delta_V - \sum_{\alpha\in R^+}\frac{c_\alpha}{\alpha}\langle\alpha,\alpha\rangle\partial_{\alpha^\vee} + \sum_{\alpha\in R^+}\frac{c_s(c_s+1)\langle\alpha,\alpha\rangle}{\alpha^2} + \sum_{\alpha\neq\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha,\alpha'\rangle}{\alpha\alpha'}\end{aligned}$$

Thus, $\varphi(H) = \overline{H} + \sum_{\alpha\neq\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha,\alpha'\rangle}{\alpha\alpha'}$, and to prove the lemma (and hence Theorem 4.3.3) we just need to show that this last term, which we denote by P, is 0. First of all, note that the term is clearly W-invariant. Now denote

$$\delta := \prod_{\alpha\in R^+}\alpha$$

which is sign-invariant. So δP is sign-invariant. This is a polynomial of degree $n-2$. But the smallest degree of a nonzero sign-invariant element in $S(V)$ is n. Thus, $\delta P = 0$, and so $P = 0$. ☐

References

[C1] I. Cherednik, *A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras.* Invent. Math. **106** (1991) no. 1, 411-431

[C2] I. Cherednik, *Double affine Hecke algebras.* London Mathematical Society Lecture Series **319**, Cambridge University Press, Cambridge, 2005.

[EG] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism.* Invent. Math. **147** (2002) no. 2, 243-348

[KT] C. Kassel, V. Turaev, *Braid groups.* Graduate Texts in Mathematics **247**. Springer, New York, 2008.

[Ki] A. A. Kirillov, Jr. *Lectures on the affine Hecke algebras and Macdonald conjectures.* `https://arxiv.org/abs/math/9501219`

[M] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials.* Cambridge Tracts in Mathematics **157**, Cambridge University Press, Cambridge, 2003.

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