Coherent sheaves on elliptic curves.

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Abstract

We describe the abelian category of coherent sheaves on an elliptic curve, and construct an action of a central extension of $\mathrm{SL}_2(\mathbb{Z})$ on the derived category.

Contents

1 Coherent sheaves on elliptic curve

Definition 1.1. *An elliptic curve* over a field k is a nonsingular projective algebraic curve of genus 1 over k with a fixed k-rational point.

Remark 1.2. If the characteristic of k is neither 2 nor 3, an elliptic curve can be alternatevily defined as the subvariety of $\mathbb{P}_k^2$ defined by an equation $y^2z = x^3 - pxz^2 - qz^3$, where $p, q \in k$, and the polynomial $x^3 - pxz^2 - qz^3$ is square-free. In this case, the fixed point is $(0 : 1 : 0)$.

Remark 1.3. Over the field of complex numbers, there is even a simpler description. An elliptic curve is precisely a quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a nondegenerate lattice $\Lambda \subset \mathbb{C}$ of rank 2.

Remark 1.4. Any elliptic curve carries a structure of a group, with the fixed point being the identity.

Fix an elliptic curve X over a field k. We do not assume that k is algebraically closed, since the main example is the finite field $\mathbb{F}_q$.

Recall that *a coherent sheaf* $\mathcal{F}$ on X is a sheaf of modules over $\mathcal{O}$ such that for every open affine $U \subset X$ the restriction $\mathcal{F}|_U$ is isomorphic to $\widehat{N}$ for some finitely generated $\mathcal{O}(U)$-module N.

Example 1.5. The structure sheaf $\mathcal{O}$ is indeed a coherent sheaf. Also, one can consider the ideal sheaf $\mathfrak{m}_x = \mathcal{O}(-x)$ corresponding to a closed point $x \in X$. Then the cokernel of the inclusion $\mathcal{O}(-x) \to \mathcal{O}$ is the so called *skyscrapper sheaf* $\mathcal{O}_x$, which is coherent as well.

Theorem 1.6. *Coherent sheaves on X form an abelian category $Coh(X)$.*

Theorem 1.7 (Global version of Serre theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k admits a resolution $\mathcal{F}_n \to \mathcal{F}_{n-1} \to \ldots \to \mathcal{F}_1 \to \mathcal{F}_0$ where each $\mathcal{F}_i$ is finitely generated and locally free ($\simeq$ vector bundle).*

Theorem 1.8 (Grothendieck's finiteness theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k has finite dimensional cohomologies over k.*

Corollary 1.9. *For any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ the space* $\operatorname{Hom}(\mathcal{F},\mathcal{G})$ *has finite dimension over k, since* $\operatorname{Hom}(\mathcal{F},\mathcal{G}) = \Gamma(\mathcal{H}om(\mathcal{F},\mathcal{G}), X) = H^0(\mathcal{H}om(\mathcal{F},\mathcal{G}), X)$.

Theorem 1.10 (Grothendieck's vanishing theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k has no i-th cohomologies for $i > n$.*

Definition 1.11. An abelian category $\mathcal{C}$ is called *hereditary* if $\operatorname{Ext}^2(-,-) = 0$.

Corollary 1.12. *The category $Coh(X)$ is hereditary.*

2 (Semi)stable sheaves

To a coherent sheaf we can associate two numbers, the Euler characteristic $\chi(\mathcal{F})$ and the rank $\operatorname{rk}(\mathcal{F})$.

Definition 2.1. The Euler characteristic $\chi(\mathcal{F})$ is the alternating sum $\sum_i (-1)^i \dim_k H^i(\mathcal{F}, X)$. In our case, it is equal to $\dim_k H^0(\mathcal{F}, X) - \dim_k H^1(\mathcal{F}, X)$.

Definition 2.2. The rank $\operatorname{rk}(\mathcal{F})$ is the dimension of the stalk $\mathcal{F}_\xi$ of $\mathcal{F}$ at a generic point ξ of X over the residue field. It is independent of ξ.

Example 2.3. We have $\chi(\mathcal{O}) = 0$, $\operatorname{rk}(\mathcal{O}) = 1$, $\chi(\mathcal{O}_x) = 1$, $\operatorname{rk}(\mathcal{O}_x) = 0$.

Proposition 2.4. *Given a short exact sequence* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, *we have* $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$ *and* $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{F}') + \operatorname{rk}(\mathcal{F}'')$.

Definition 2.5. *The slope* $\mu(\mathcal{F})$ of a nontrivial coherent sheaf $\mathcal{F}$ is the quotient $\chi(\mathcal{F})/\operatorname{rk}(\mathcal{F})$. In the case $\operatorname{rk}(\mathcal{F}) = 0$ we set $\mu(\mathcal{F}) = \infty$.

Lemma 2.6. *Given a short exact sequence* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, *we have three options:*

- $\mu(\mathcal{F}') < \mu(\mathcal{F}) < \mu(\mathcal{F}'')$;
- $\mu(\mathcal{F}') = \mu(\mathcal{F}) = \mu(\mathcal{F}'')$;
- $\mu(\mathcal{F}') > \mu(\mathcal{F}) > \mu(\mathcal{F}'')$.

Proof. We have

$$\mu(\mathcal{F}') = \frac{\chi(\mathcal{F}')}{\operatorname{rk}(\mathcal{F}')},$$
$$\mu(\mathcal{F}'') = \frac{\chi(\mathcal{F}'')}{\operatorname{rk}(\mathcal{F}'')},$$
$$\mu(\mathcal{F}) = \frac{\chi(\mathcal{F})}{\operatorname{rk}(\mathcal{F})} = \frac{\chi(\mathcal{F}') + \chi(\mathcal{F}')}{\operatorname{rk}(\mathcal{F}') + \operatorname{rk}(\mathcal{F}'')}$$

Since both $\operatorname{rk}(\mathcal{F}')$ and $\operatorname{rk}(\mathcal{F}'')$ are nonnegative, we indeed get the lemma. □

Definition 2.7. A coherent sheaf $\mathcal{F}$ is called *stable (resp. semistable)* if for any nontrivial short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have $\mu(\mathcal{F}') < \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$).

General theory gives us the following

Theorem 2.8 ([1] Harder-Narasimhan filtration)**.** *For a coherent sheaf $\mathcal{F}$, there is a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} = \mathcal{F}$$

such that all $\mathcal{A}_i = \mathcal{F}_{i+1}/\mathcal{F}_i$ are semistable and $\mu(\mathcal{A}_i) > \mu(\mathcal{A}_{i+1})$ for each i.

In our case, we can derive much stronger proposition. Before stating it, note two useful statements.

Proposition 2.9. *If $\mathcal{F}$ and $\mathcal{G}$ are semistable sheaves, and $\mu(\mathcal{F}) > \mu(\mathcal{G})$, then $Hom(\mathcal{F}, \mathcal{G}) =$ 0.*

Proof. Suppose we have a nontrivial map $f \colon \mathcal{F} \to \mathcal{G}$. Then $\mu(\mathcal{F}) \leq \mu(\mathcal{F}/\ker f) = \mu(\operatorname{im} f) \leq \mu(\mathcal{G})$. Contradiction. □

Another property of $Coh(X)$ we will need is

Proposition 2.10 (Calabi-Yau property)**.** *For any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is an isomorphism* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \simeq \operatorname{Ext}^1(\mathcal{G}, \mathcal{F})^*$.

Proof. From Remark 1.4 we know that the canonical bundle K is trivial, $K \simeq \mathcal{O}$. Also by Serre duality we get

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \operatorname{Ext}^0(\mathcal{F}, \mathcal{G}) \simeq \operatorname{Ext}^1(\mathcal{G}, \mathcal{F} \otimes K)^* = \operatorname{Ext}^1(\mathcal{G}, \mathcal{F})^*.$$

□

We are ready to prove

Theorem 2.11. *Any nontrivial coherent sheaf is a direct sum of indecomposable semistable sheaves.*

Proof. We only need to prove that any indecomposable sheaf is semistable. Suppose some indecomposable sheaf $\mathcal{F}$ is not semistable. Then the Harder-Narasimhan filtration of $\mathcal{F}$ is nontrivial. Consider only the case of length 1 filtration, it captures the main idea. So, we have a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, where both $\mathcal{F}'$ and $\mathcal{F}''$ are semistable, and $\mu(\mathcal{F}') > \mu(\mathcal{F}'')$. By Proposition 2.9 we get $\mathrm{Hom}(\mathcal{F}', \mathcal{F}'') = 0$. By Proposition 2.10 we obtain $\mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}') = \mathrm{Hom}(\mathcal{F}', \mathcal{F}'')^* = 0$. Therefore the exact sequence splits, contradiction with the assumption that $\mathcal{F}$ is indecomposable. $\square$

Definition 2.12. Denote the full subcategory of semistable coherent sheaves on X of slope μ by C_μ.

Proposition 2.13. *The category C_μ is abelian, artinian, and closed under extensions. The simple objects in C_μ are stable sheaves of slope μ.*

Corollary 2.14. *$Coh(X)$ is the direct sum of all C_μ (on the level of objects).*

3 Euler form

Since rk and χ are well defined on $K_0(Coh(X))$, we can consider

Definition 3.1. *The Euler form* $\langle \mathcal{F}, \mathcal{G} \rangle$ of two elements $\mathcal{F}, \mathcal{G} \in K_0(Coh(X))$ is equal to $\dim \mathrm{Hom}(\mathcal{F}, \mathcal{G}) - \dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$.

Proposition 3.2. *We have $\langle \mathcal{F}, \mathcal{G} \rangle = \mathrm{rk}(\mathcal{F})\chi(\mathcal{G}) - \chi(\mathcal{F})\,\mathrm{rk}(\mathcal{G})$.*

Proof. First notice that the RHS only depends on the classes of $\mathcal{F}$ and $\mathcal{G}$ in the Grothendieck group $K_0(Coh(X))$. Therefore it is sufficient to check the equality for some generators of the Grothendieck group, for example, for locally free sheaves. If $\mathcal{F}$ is locally free, the LHS reduces to $\chi(\mathcal{F}^\vee \otimes \mathcal{G})$. Note that in the case of elliptic curve, the Hirzebruch-Riemann-Roch theorem gives us that $\chi(\mathcal{E}) = \deg(\mathcal{E})$ for any coherent sheaf $\mathcal{E}$. Applying it here, we get

$$\begin{aligned} LHS = \chi(\mathcal{F}^\vee \otimes \mathcal{G}) = \deg(\mathcal{F}^\vee \otimes \mathcal{G}) = \mathrm{rk}(\mathcal{F}) \deg(\mathcal{G}) - \deg(\mathcal{F})\,\mathrm{rk}(\mathcal{G}) = \\ \mathrm{rk}(\mathcal{F})\chi(\mathcal{G}) - \chi(\mathcal{F})\,\mathrm{rk}(\mathcal{G}) = RHS. \end{aligned}$$

$\square$

Definition 3.3. *The charge map* is

$$Z = (\mathrm{rk}, \chi) \colon K_0(Coh(X)) \to \mathbb{Z}^2.$$

It is surjective, since we have both $(1, 0)$ and $(0, 1)$ in the image. We have a canonical nondegenerate volume form on $\mathbb{Z}^2$, $\langle (a, b), (c, d) \rangle = ad - bc$, and it is equal to the push-forward of the Euler form.

Proposition 3.4. *The kernel of the Euler form coincides with the kernel of Z, equivalently, $K_0(Coh(X))/\ker \langle , \rangle \simeq \mathbb{Z}^2$.*

Also we can now write some relations between different C_μ and $C_{\mu'}$.

Proposition 3.5. *Suppose $\mathcal{F}$ and $\mathcal{F}'$ are indecomposable, and $Z(\mathcal{F}) = (r,\chi)$, $Z(\mathcal{F}') = (r',\chi')$.*

- *If $\chi/r > \chi'/r'$, then $\mathrm{Hom}(\mathcal{F},\mathcal{F}') = 0$, $\dim \mathrm{Ext}^1(\mathcal{F},\mathcal{F}') = \chi r' - \chi' r$;*
- *If $\chi/r < \chi'/r'$, then $\dim \mathrm{Hom}(\mathcal{F},\mathcal{F}') = \chi' r - \chi r'$, $\mathrm{Ext}^1(\mathcal{F},\mathcal{F}') = 0$.*

Proof. By Proposition 2.10 and Proposition 2.9 we know that

- if $\chi/r > \chi'/r'$, then $\mathrm{Hom}(\mathcal{F},\mathcal{F}') = 0$;
- if $\chi/r < \chi'/r'$, then $\mathrm{Ext}^1(\mathcal{F},\mathcal{F}') = 0$.

Proposition 3.2 concludes the proof. $\square$

4 Derived category of coherent sheaves

Let us show that Corollary 1.12 implies a neat description of the derived category $D^b(Coh(X))$ of bounded complexes of coherent sheaves on X.

Theorem 4.1. *Suppose $\mathcal{C}$ is a hereditary abelian category. Then any object $L \in D^b(\mathcal{C})$ is isomorphic to the sum of its cohomologies, i. e. $L = \bigoplus_i H^i L[-i]$.*

Proof. Let L be a complex $\ldots \xrightarrow{d^{i-1}} L^i \xrightarrow{d^i} L^{i+1} \xrightarrow{d^{i+1}} \ldots$. Fix any i. We have a short exact sequence $0 \to \ker d^{i-1} \to L^{i-1} \to \mathrm{im}\, d^{i-1} \to 0$. Apply $\mathrm{RHom}(H^i L, -)$. This gives rise to an exact sequence $\mathrm{Ext}^1(H^i L, L^{i-1}) \to \mathrm{Ext}^1(H^i L, \mathrm{im}\, d^{i-1}) \to \mathrm{Ext}^2(H^i L, \ker d^{i-1})$. Since $Coh(X)$ is hereditary, we obtain a surjection from $\mathrm{Ext}^1(H^i L, L^{i-1})$ to $\mathrm{Ext}^1(H^i L, \mathrm{im}\, d^{i-1})$. In particular, there exists M^i such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & L^{i-1} \arrow[r] \arrow[d] & M^i \arrow[r] \arrow[d] & H^i L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{im}\, d^{i-1} \arrow[r] & \ker d^i \arrow[r] & H^i L \arrow[r] & 0
\end{tikzcd}$$

Then the following morhphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^i L \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & L^{i-1} \arrow[r] \arrow[u] & M^i \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

of complexes is a quasi-isomorphism. If we compose its inverse with the morphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & L^{i-1} \arrow[r] \arrow[d, equal] & M^i \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & L^{i-2} \arrow[r] & L^{i-1} \arrow[r] & L^i \arrow[r] & L^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

we get a morphism $H^i L[-i] \to L$ in $D^b(Coh(X))$ which is isomorphism in the i-th cohomology, and zero elsewhere. Therefore, if we sum up all this morphisms, we obtain an isomorphism $\bigoplus_i H^i L[-i] \to L$. $\square$

Corollary 4.2. *The derived category $D^b(Coh(X))$ is the direct sum of $\mathbb{Z}$ copies of $Coh(X)$, a sheaf $\mathcal{F}$ in the i-th copy goes to $\mathcal{F}$.*

Since $K_0(D^b(Coh(X))) = K_0(Coh(X))$, Z is defined on $K_0(D^b(Coh(X)))$ as well. Note that $Z(\mathcal{F}[i]) = (-1)^i Z(\mathcal{F})$.

Remark 4.3. The corollary works for any smooth projective curve X. Another example of a hereditary category is the category of representations of a quiver.

5 $\mathrm{SL}_2(\mathbb{Z})$ action

Proposition 3.2 suggests to define $\langle L, M\rangle = \sum_i (-1)^i \dim \mathrm{Hom}(L, M[i])$ for any two objects $L, M \in D^b(Coh(X))$. Therefore the Euler form is preserved by any autoequivalence of $D^b(Coh(X))$. In other words, any autoequivalence $f \in Aut(D^b(Coh(X)))$ gives a corresponding automorphism of $\mathbb{Z}^2$ preserving the volume form, i. e. gives an element $\pi(f) \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 5.1. Say that an object $\mathcal{E} \in D^b(Coh(X))$ is *spherical* if $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = k$ (and consequently $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[1]) = k$).

Example 5.2. The structure sheaf $\mathcal{O}$ and the skyscrapper sheaf at a rational k-point are spherical.

Definition 5.3. *A Forier-Mukai transform* with a kernel $\mathcal{L} \in D^b(Coh(X \times Y))$ is a functor $\Phi_{\mathcal{L}} \colon D^b(Coh(X)) \to D^b(Coh(Y))$ which sends an object $\mathcal{F} \in D^b(Coh(X))$ to $R\pi_{2*}(\pi_1^* \mathcal{F} \otimes^L \mathcal{L})$, where $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to X$ are the natural projections.

Definition 5.4. For a spherical object $\mathcal{E} \in D^b(Coh(X))$, which is a complex of locally free sheaves, we can define *a twist functor* $T_{\mathcal{E}} \colon D^b(Coh(X)) \to D^b(Coh(X))$ to be equal to a Forier-Mukai transform with the kernel $cone(\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta) \in D^b(Coh(X \times X))$.

Theorem 5.5 ([2])**.** *For a spherical object $\mathcal{E} \in D^b(Coh(X))$ the twist functor $T_{\mathcal{E}}$ is an exact equivalence which sends an object $\mathcal{F}$ to $cone(\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E} \xrightarrow{ev_{\mathcal{F}}} \mathcal{F})$.*

Remark 5.6. The evaluation works by applying $ev \colon \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-i] \to \mathcal{F}$ on each grading.

Let us see how $T_{\mathcal{E}}$ acts on Grothendieck group.

Proposition 5.7. *The action of $T_{\mathcal{E}}$ on $K_0(D^b(Coh(X)))$ is given by $[\mathcal{F}] \mapsto [\mathcal{F}] - \langle \mathcal{E}, \mathcal{F}\rangle [\mathcal{E}]$.*

Proof. Indeed, $[T_{\mathcal{E}}(\mathcal{F})] = [\mathcal{F}] - [\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E}] = [\mathcal{F}] - \langle \mathcal{E}, \mathcal{F}\rangle [\mathcal{F}]$. □

Corollary 5.8. $\pi(T_{\mathcal{O}}) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, $\pi(T_{\mathcal{O}_x}) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

Proof. Since $\mathbb{Z}^2$ are generated by the charges of $\mathcal{O}$ and $\mathcal{O}_x$, we can check this on $\mathcal{O}$ and $\mathcal{O}_x$ only.

$$
\begin{aligned}
T_{\mathcal{O}}([\mathcal{O}]) &= [\mathcal{O}] - \langle \mathcal{O}, \mathcal{O}\rangle [\mathcal{O}] = [\mathcal{O}],\\
T_{\mathcal{O}}([\mathcal{O}_x]) &= [\mathcal{O}_x] - \langle \mathcal{O}, \mathcal{O}_x\rangle [\mathcal{O}] = [\mathcal{O}_x] - [\mathcal{O}],\\
T_{\mathcal{O}_x}([\mathcal{O}]) &= [\mathcal{O}] - \langle \mathcal{O}_x, \mathcal{O}\rangle [\mathcal{O}_x] = [\mathcal{O}] + [\mathcal{O}_x],\\
T_{\mathcal{O}_x}([\mathcal{O}_x]) &= [\mathcal{O}_x] - \langle \mathcal{O}_x, \mathcal{O}_x\rangle [\mathcal{O}_x] = [\mathcal{O}_x].
\end{aligned}
$$

□

Proposition 5.9. *$T_{\mathcal{O}_x}$ is in fact just the tensor product with $\mathcal{O}(x)$.*

Proof. The formula for the adjoint of a Forier-Mukai transform gives that the inverse of $T_{\mathcal{O}_x}$ is the Forier-Mukai transform with the kernel $cocone(\mathcal{O}_\Delta \to \mathcal{O}_{(x,x)})$. The map inside a cocone is nonzero. But any nonzero map $\mathcal{O}_\Delta \to \mathcal{O}_{(x,x)}$ is a nonzero multiple of the natural surjection $\mathcal{O}_\Delta \to \mathcal{O}_{(x,x)}$. Therefore the cocone is equal to the kernel of this map, or just $\mathcal{O}_\Delta \otimes \pi_1^*(\mathcal{O}(-x))$. Now note that the sheaf $\mathcal{O}_\Delta$ in the kernel trivializes all pullbacks and pushforwards we do to the identity maps between sheaves on X and on $\Delta \simeq X$. The proposition follows. $\square$

The matrices $\pi(T_{\mathcal{O}})$ and $\pi(T_{\mathcal{O}_x})$ generate $\mathrm{SL}_2(\mathbb{Z})$, therefore, $\pi\colon Aut(D^b(Coh(X))) \to \mathrm{SL}_2(\mathbb{Z})$ is surjective.

6 Classification of indecomposable sheaves

Note that indecomposable torsion sheaves lie in C_∞, and generate C_∞. Moreover, we have

Theorem 6.1. *Indecomposable torsion sheaves are parametrized by a positive integer $s > 0$ and a closed point $x \in X$. The corresponding torsion sheaf is $\mathcal{O}/\mathcal{O}(-sx)$.*

Proof. Indeed, we reduce to the case of one point, then the local ring is PID, and the claim follows. $\square$

In addition to that, $\mathrm{SL}_2(\mathbb{Z})$ action allows us to prove

Theorem 6.2. *For each $\mu \in \mathbb{Q}$ we have a canonical isomorphism $C_\mu \simeq C_\infty$.*

Proof. Indeed, let μ be equal to a/b for coprime a and b. Choose some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ which sends (a,b) to $(0,1)$, and lift it to an autoequivalence $\tilde{f} \in Aut(D^b(Coh(X)))$ of the derived category. Take any indecomposable sheaf $\mathcal{F} \in C_\mu$. Then $\tilde{f}(\mathcal{F})$ is an indecomposable object in $D^b(Coh(X))$ with the slope ∞. Therefore, it is of form $\mathcal{G}[k]$, where $\mathcal{G}$ is a torsion sheaf, and k is some integer. Denote by $\overline{f}\colon C_\mu \to C_\infty$ a map which sends an indecomposable sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ defined in this way. It is easy to see that if we begin with the inverse matrix f^{-1}, then we get a map $\overline{f^{-1}}\colon C_\infty \to C_\mu$ which is inverse to $\overline{f}$. Also $\overline{f}$ does not depend on a lift $\tilde{f}$. So C_μ and C_∞ are canonically isomorphic. $\square$

Summarizing, we have

Theorem 6.3. *Indecomposable sheaves are parametrized by a pair (rk, χ) in the right half of $\mathbb{Z}^2$ and a closed point $x \in X$.*

Let us show how this describes indecomposable sheaves with charges $(1,1)$ and $(1,0)$.

Proposition 6.4.

$$T_{\mathcal{O}}(\mathcal{O}) = \mathcal{O}, \quad T_{\mathcal{O}}(\mathcal{O}(x)) = \mathcal{O}_x,$$
$$T_{\mathcal{O}_x}(\mathcal{O}) = \mathcal{O}(x), \quad T_{\mathcal{O}_x}(\mathcal{O}_x) = \mathcal{O}_x.$$

Proof. The second line is a consequence of Proposition 5.9. The first line is an easy computation based on Theorem 5.5. □

Proposition 6.5. *The indecomposable sheaves of charge* $(1,1)$ *are the sheaves* $\mathcal{O}(x)$*. The indecomposable sheaves of charge* $(1,0)$ *are the sheaves* $\mathcal{O}(x-y)$*.*

Proof. The autoequivalence $T_{\mathcal{O}}^{-1}$ maps the charge $(0,1)$ to $(1,1)$, so we can use it to obtain the indecomposables of charge $(1,1)$. Given an indecomposable $\mathcal{O}_x$ of charge $(0,1)$, its image is $\mathcal{O}(x)$ by Proposition 6.4. The first part follows.

Then we can apply $T_{\mathcal{O}_y}^{-1}$ to the latter indecomposables. We get that the indecomposables of charge $(1,0)$ are $\mathcal{O}(x-y)$. □

7 Braid group relations

For matrices $A = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ we have the following relations

$$ABA = BAB$$
$$(AB)^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

We expect similar relations to hold for $T_{\mathcal{O}}$ and $T_{\mathcal{O}_x}$.

Theorem 7.1 ([2])**.**

$$T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}} \simeq T_{\mathcal{O}_x}T_{\mathcal{O}}T_{\mathcal{O}_x}$$
$$(T_{\mathcal{O}}T_{\mathcal{O}_x})^3 \simeq i^*[1],$$

where $i\colon X \to X$ *is the inverse map of* X*.*

We can prove the braid relation using the following

Proposition 7.2 ([2])**.** *Given two spherical objects* E_1 *and* E_2*, we have*

$$T_{E_1}T_{E_2} = T_{T_{E_1}(E_2)}T_{E_1}$$

Proof. Using the computations in Proposition 6.4, we can write

$$T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}} = T_{\mathcal{O}}T_{T_{\mathcal{O}_x}(\mathcal{O})}T_{\mathcal{O}_x} = T_{\mathcal{O}}T_{\mathcal{O}(x)}T_{\mathcal{O}_x} = T_{T_{\mathcal{O}}(\mathcal{O}(x))}T_{\mathcal{O}}T_{\mathcal{O}_x} = T_{\mathcal{O}_x}T_{\mathcal{O}}T_{\mathcal{O}_x}.$$

□

This shows that $T_{\mathcal{O}}$ and $T_{\mathcal{O}_x}$ generate the group $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ in $Aut(D^b(Coh(X)))$, the central extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$.

References

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