HALL ALGEBRAS

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1. Introduction

In this seminar the elliptic Hall algebra (EHA) was introduced as the limit of the spherical double affine Hecke algebras of $\mathfrak{gl}_n$ and we have written an explicit presentation in terms of generators and relations [2]. In this talk, we will define it as an algebra that specializes to a certain subalgebra of the Hall algebra of every elliptic curve over a finite field, definition that will explain its name.

The plan for this talk is the following. We begin by defining the Hall algebra, and explaining when one can construct a coproduct, when this algebra is a bialgebra with respect to these operations, or a Hopf algebra. A Hall algebra $\mathbb{H}_{\mathcal{A}}$ can be defined for any abelian category $\mathcal{A}$ with certain finitary properties, but we will see that this algebra has richer properties for global dimension one categories. A natural supply of such categories are the abelian categories of representations of a quiver over a finite field and of coherent sheaves over a projective curve C over a finite field. We are interested in the curves C of genus one, and inside the Hall algebras of these curves we will find specializations of EHA as defined in the previous talks. In the second part of the talk, we focus on the elliptic curve case, for which we show that the derived equivalences of $D^b\mathrm{Coh}\,(X)$ act by algebra automorphisms on the Drinfeld double of $\mathbb{H}_{\mathcal{A}}$. This action will be used in proving a PBW theorem for certain subalgebras of the Hall algebra, and in identifying this algebra with the EHA defined by generators and relations in [2].

2. Hall algebras

2.1. **Definition of the product and of the coproduct.** We start with a small abelian category $\mathcal{A}$ of finite global dimension. We say that $\mathcal{A}$ is finitary if for all objects M and N in $\mathcal{A}$, we have that

$$|\mathrm{Hom}(M,N)|,\ |\mathrm{Ext}^i(M,N)| < \infty.$$

In most examples, these finitary categories will be linear over a finite field k. Examples of such categories are the categories of $k-$representations of a quiver, or of a finite dimensional algebra over k, and the categories of coherent sheaves on any projective smooth scheme over k.

For two objects M and N of $\mathcal{A}$, define

$$\langle M,N\rangle := (\prod_{i=0}^{\infty} |\mathrm{Ext}^i(M,N)^{(-1^i)}|)^{1/2}.$$

This defines the multiplicative Euler form $\langle , \rangle : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{C}^\times$. When $\mathcal{A}$ is $k-$linear, we have that $\langle M,N\rangle = v^{\chi(M,N)}$, where $v^2 = q$ is a square root of q, the number of elements of the field k.

We are now ready to define the Hall algebra $\mathbb{H}_{\mathcal{A}}$. As a vector space,

$$\mathbb{H}_{\mathcal{A}} := \bigoplus_{M \text{ iso class in } \mathcal{A}} \mathbb{C}M.$$

Given M, N, and R three objects in $\mathcal{A}$, we define $P^R_{M,N}$ to be the number of short exact sequences $0 \to N \to R \to M \to 0$, and $a_p := |\mathrm{Aut}\,(P)|$.

We observe that

$$\frac{P^R_{M,N}}{a_M a_N} = |\{L \subset R | L \cong N, R/L \cong M\}|.$$

Proposition 2.1. *(Ringel) The multiplication*

$$[M]\cdot[N] = \langle M,N\rangle \sum_R \frac{P^R_{M,N}}{a_M a_N}[R]$$

defines an associative algebra structure on $\mathbb{H}_{\mathcal{A}}$, with unit $[0]$, zero object of $\mathcal{A}$.

Proof. Because $\mathrm{Ext}^1(M,N)$ is finite, the definition of $[M]\cdot[N]$ is an element of $\mathbb{H}_{\mathcal{A}}$. It is immediate to check that $[0]$ is the unit. For the associativity, direct computations give that, for three objects M, N, and L of $\mathcal{A}$, we have:

$$([M] \cdot [N]) \cdot [L] = \langle M, N\rangle \langle R, L\rangle \langle M, L\rangle \sum_R c^R_{M,N,L}[R],$$

where $c^R_{M,N,L}$ counts the number of elements of the set $\{0 \subset B \subset A \subset R | B \cong L, A/B \cong N, R/A \cong M\}$. Computing $[M] \cdot ([N] \cdot [L])$ gives the same result. □

Observe that the multiplication of a Hall algebra encodes all the ways in which one can extend one object by another object. One can define a Hall algebra for any exact category with the above finitary conditions.

Further, observe that $\mathbb{H}_{\mathcal{A}}$ is naturally graded by the classes in the Grothendieck group $K(\mathcal{A})$.

Example. Let $\mathcal{A}$ be a semisimple category with S_i the simple objects. Then for $i \neq j$ we have $[S_i][S_j] = [S_i \oplus S_j] = [S_j][S_i]$, and $[S_i][S_i] = |\mathrm{End}\,(S_i)|^{\frac{1}{2}}(|\mathrm{End}\,(S_i)| + 1)[S_i \oplus S_i]$.

It is natural to ask whether we can define a comultiplication on the vector space $\mathbb{H}_{\mathcal{A}}$. It should involve all possible ways to break an object into two smaller objects in $\mathcal{A}$. Usually we can break an object in infinitely many ways into two objects, so we need to introduce certain completions of the Hall algebra in order to define the coproduct. We will gloss over some of the details, see [4] for full explanations.

For $a, b \in K(\mathcal{A})$, we define

$$\mathbb{H}_{\mathcal{A}}[a] \hat{\otimes} \mathbb{H}_{\mathcal{A}}[b] = \prod_{M \text{of class}\, a, N \text{ of class}\, b} \mathbb{C}[M] \otimes \mathbb{C}[N].$$

Elements of this vector space are simply formal (infinite) linear combinations

$$\sum_{M \text{ of class } a, N \text{ of class } b} c_{M,N}[M] \otimes [N].$$

Further, we define

$$\mathbb{H}_{\mathcal{A}} \hat{\otimes} \mathbb{H}_{\mathcal{A}} := \prod_{a,b} \mathbb{H}_{\mathcal{A}}[a] \hat{\otimes} \mathbb{H}_{\mathcal{A}}[b].$$

Thus, the elements of this completed tensor product are all formal (infinite) linear combinations $\sum_{M,N} c_{M,N}[M] \otimes [N]$.

Proposition 2.2. *(Green) The coproduct*

$$\Delta[R] = \sum_{M,N} \langle M, N\rangle \frac{P^R_{M,N}}{a_R}[M] \otimes [N]$$

puts on $\mathbb{H}_{\mathcal{A}}$ the structure of a (topological) coassociative coalgebra with counit $\varepsilon : \mathbb{H}_{\mathcal{A}} \to \mathbb{C}$ defined by $\varepsilon[M] = \delta_{M,0}$.

Observe that the coproduct takes values in the finite part $\mathbb{H}_{\mathcal{A}} \otimes \mathbb{H}_{\mathcal{A}}$ if and only if for any object R, there exist only finitely many subobjects $N \subset R$. It holds for categories of representations of quivers, but not for categories of coherent sheaves on a projective variety. Indeed, any subrepresentation of the $k-$representation (V_i) of a quiver Q is specified by subspaces of the V_i, of which there are finitely many possibilities, and maps between the corresponding spaces, of which there are finitely many.

It is not clear how to check coassociativity of Δ given this formula, because it is not clear that $(\Delta \otimes 1)\Delta$ makes sense. Fortunately, it makes sense because the only terms in $\mathbb{H}_{\mathcal{A}} \hat{\otimes} \mathbb{H}_{\mathcal{A}}$ that contribute to $[M_1] \otimes [M_2] \otimes [M_3]$ are of the form $[N] \otimes [M_3]$, where N is an extension of M_1 by M_2, of which there are finitely many.

As one last comment about the coproduct, sometimes the Grothendieck group $K(\mathcal{A})$ is not finitely generated, which happens for the category of coherent sheaves on an elliptic curve. It is preferable to work with a smaller $K-$group, like the numerical $K-$group for an arbitrary curve, in these situations. In these cases the definition of the completion $\mathbb{H} \hat{\otimes} \mathbb{H}$ needs to be slightly changed, see [1].

Next, we investigate when these two operations define a bialgebra. One cannot take the product of two elements in the completed product of the Hall algebra, but one can take the product if they are in the image of the comultiplication Δ, see [4] for more details.

In order to state the next theorem, which gives an answer to when these two operations put a bialgebra structure on $\mathbb{H}_{\mathcal{A}}$, we need to twist the multiplication, or, alternatively, we need to add a degree zero piece to $\mathbb{H}_{\mathcal{A}}$. Let $K = \mathbb{C}[K(\mathcal{A})]$ be the group algebra of the Grothendieck group of $\mathcal{A}$, and denote by k_a the class of the element $a \in K(\mathcal{A})$. Define the vector space $\mathbb{H}'_{\mathcal{A}} = \mathbb{H}_{\mathcal{A}} \otimes K$. We want to put an algebra structure on this space extending the algebra structure on the two factors of the tensor product. We only need to explain how k_a and $[M]$ commute, for which we introduce the relation

$$k_a[M]k_a^{-1} = \langle a, M\rangle\langle M, a\rangle[M].$$

We can also extend the comultiplication as follows: $\Delta(k_a) = k_a \otimes k_a$ and

$$\Delta([R]k_a) = \sum_{M,N} \langle M, N\rangle \frac{P^R_{M,N}}{a_R}[M]k_{N+a} \otimes [N]k_a.$$

For the next theorem, we need to assume that the global dimension gldim $(\mathcal{A}) \leqslant 1$. Recall that the global dimension n of an abelian category is the smallest integer n with the property that $\mathrm{Ext}^{n+1}(A, B) = 0$ for all objects $M, N \in \mathcal{A}$. Recall that for $\mathcal{A}$ a category with enough injectives and projectives, this number is the same as the supremum after all projective dimensions of elements in $\mathcal{A}$ and the same as the supremum after all injective resolutions of elements in $\mathcal{A}$.

Theorem 2.3. *The comultiplication map*

$$\Delta : \mathbb{H}'_{\mathcal{A}} \to \mathbb{H}'_{\mathcal{A}} \otimes \mathbb{H}'_{\mathcal{A}}$$

is an algebra morphism.

We can also extend the counit map to $\varepsilon([M]k_a) = \delta_{M,0}$. Then the theorem says that $(\mathbb{H}'_{\mathcal{A}}, i, m, \varepsilon, \Delta)$ is a (topological) bialgebra.

Remember that if $\mathcal{A}$ satisfies the finite subobject condition, there is no need to introduce the completion. Further, if the symmetrized Euler product $\langle M, N\rangle\langle N, M\rangle = 1$, then there is no need to introduce the factor $\mathbb{C}[K(\mathcal{A})]$. For a $k-$linear category $\mathcal{A}$, this happens when $\chi(M, M) = 0$, for all $M \in \mathcal{A}$, for example for $\mathcal{A} = \mathrm{Coh}\,(X)$, where X is an elliptic curve. A final observation is that even if one can define Hall algebras for exact categories, Green's theorem holds for abelian categories only.

It is natural to ask when this bialgebra is actually a Hopf algebra. For this, we would need to construct an antipode map which satisfies the axioms of a Hopf algebra. Xiao managed to construct one such antipode map $S : \mathbb{H}'_{\mathcal{A}} \to \mathbb{H}'_{\mathcal{A}}$ for categories of representations of a quiver.

The Hall algebras for categories of $\mathrm{gldim}\,(\mathcal{A}) \leqslant 1$ come with (at least) one other piece of extra structure: a nondegenerate Hopf pairing:

Proposition 2.4. *The scalar product* $(,) : \mathbb{H}_{\mathcal{A}} \otimes \mathbb{H}_{\mathcal{A}} \to \mathbb{C}$ *defined by*

$$([M]k_a, [N]k_b) = \frac{\delta_{M,N}}{a_M}\langle a, b\rangle\langle b, a\rangle$$

is a non-degenerate Hopf pairing, that is, it satisfies $(xy, z) = (x \otimes y, \Delta(z))$ *for any* $x, y, x \in \mathbb{H}'_{\mathcal{A}}$.

2.2. **Examples.** We have already computed the product of some elements in a semisimple category $\mathcal{A}$. The next easiest example is provided by categories of nilpotent $k-$representations of quiver $\mathrm{Rep}\,(Q)$. Recall that k is a finite field. For us, a quiver will be allowed to have multiple edges or cycles, but no loops. Therefore $\mathrm{gldim}\,\mathrm{Rep}(Q) \leqslant 1$. Such a quiver has simple objects S_i with a one dimensional vector space k in the ith vertex, and zero everywhere else.

Let's compute relations for the A_2 quiver. There are only two simple objects S_1 and S_2, and they have a unique nontrivial extension I_{12} corresponding to $\mathrm{Ext}^1(S_1, S_2) = k$. Observe that $\mathrm{Ext}^1(S_2, S_1) = 0$. Further, the only indecomposable objects are S_1, S_2, and I_{12}. We can prove the following relations in the Hall algebra of this quiver:

(1) $[S_1][S_2] = v^{-1}([S_1 \oplus S_2] + [I_{12}])$ because there is only one subobject isomorphic to S_1 in both $S_1 \oplus S_2$ and I_{12},
(2) $[S_2][S_1] = [S_1 \oplus S_2]$ because there are no extensions of S_1 by S_2,

(3) $[S_2][S_1]^2 = v(v^2+1)[S_2][S_1^2] = v(v^2+1)[S_1^2 \oplus S_2]$ is similar to item (2),

(4) $[S_1]^2[S_2] = v(v^2+1)[S_1^2][S_2] = v^{-1}(v^2+1)([S_1^2 \oplus S_2] + [S_1 \oplus I_{12}])$ is similar to item (1),

(5) $[S_1][S_2][S_1] = [S_1][S_1 \oplus S_2] = (v^2+1)[S_1^2 \oplus S_2] + [S_1 \oplus I_{12}]$ because there are $\mathbb{P}^1(\mathrm{End}\,(S_1, S_1))$ subobjects $S_1 \subset S_1 \oplus S_1$.

Putting the last three relations together, we get that

(6) $[S_1]^2[S_2] - (v+v^{-1})[S_1][S_2][S_1] + [S_2][S_1]^2 = 0.$

Similarly we can also prove that

(7) $[S_2]^2[S_1] - (v+v^{-1})[S_2][S_1][S_2] + [S_1][S_2]^2 = 0.$

In fact, any relation satisfied by $[S_1]$ and $[S_2]$ is a consequence of one of the two above relations. Indeed, relations (5) and (6) provide us with an algebra morphism

$$\Phi : U_v(\mathfrak{b}) \to \mathbb{H}_Q,$$

where in this case Q is the A_2 quiver and $\mathfrak{b} \subset \mathfrak{sl}_2$ is the positive Borel subalgebra− we recall the structure of the quantum group in the next paragraph. The map Φ is automatically a surjection because the Hall algebra is generated by the classes of the simple objects $[S_1]$ and $[S_2]$. To show injectivity, observe that $U_v(\mathfrak{b}) = U_v(\mathfrak{n}) \otimes \mathbb{C}[K_1^{\pm}, K_2^{\pm}]$ and $\mathbb{H}' = \mathbb{H} \otimes \mathbb{C}[k_{S_1}^{\pm}, k_{S_2}^{\pm}]$. The dimensions of the $(n,m)-$graded piece of the quantum group $U_v(\mathfrak{n})$ can be computed by the PBW theorem as the number of ways to write (n,m) as the sum $a_1(1,0) + a_2(0,1) + a_3(1,1)$, where $(1,0), (0,1)$, and $(1,1)$ are the dimension vectors of the positive roots. The dimension of the $(n,m)-$graded piece of $\mathbb{H}_Q$ is the same number, because $(1,0), (0,1)$, and $(1,1)$ are the dimension vectors for the indecomposable representations of Q.

The above connection between a quantum group and the $A-2$ quiver is far from being isolated. Recall that for $\mathfrak{g}$ a Kac-Moody algebra associated to the matrix A, we can define a quantum group $U_v(\mathfrak{g}')$ with positive Borel part $U_v(\mathfrak{b}'_+)$. Here $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The quantum group $U_v(\mathfrak{b}'_+)$ is generated by E_i, K_i, K_i^{-1}, for $i \in I$, with the relations

- $K_i K_i^{-1} = K_i^{-1} K_i = 1,$
- $K_i K_j = K_j K_i,$
- $K_i E_j K_i^{-1} = v^{a_{ij}} E_j$, for all $i, j \in I$,
- the quantum Serre relation involving the E_is, see [4].
- the coproduct is defined via $\Delta(K_i) = K_i \otimes K_i$ and $\Delta(E_i) = E_1 \otimes 1 + K_i \otimes E_i$,
- the antipode is defined by $S(K_i) = K_i^{-1}$, $S(E_i) = -K_i^{-1} E_i$.

Theorem 2.5. *(Ringel, Green) The assignment $E_i \to [S_i]$, $k_i \to k_{S_i}$ for $i \in I$ defines an embedding of Hopf algebras*

$$\Phi : U_v(\mathfrak{b}'_+) \to \mathbb{H}'_Q.$$

The map Φ is an isomorphism if and only if Q is of finite type, or, equivalently, if $\mathfrak{g}$ is a simple Lie algebra.

One can prove that the coefficients $P_{M,N}^{R}$ are polynomials in v with rational coefficients. Using these polynomials as structure constants for multiplication and comultiplication, we can define a universal (or generic) version $\mathbb{H}_Q$ of the Hall algebra over $\mathbb{C}[t^{1/2}, t^{-1/2}]$, which recovers the Hall algebra of a quiver over k when $t = q$ the number of elements of the field k.

Before we start discussing examples coming from geometry, we need to discuss one other quiver example, the Jordan quiver. The reason is the following: the torsion category of sheaves on a smooth projective curve X splits as follows

$$\mathrm{Tor}\,(X) = \prod_{x \in X} \mathrm{Tor}_x.$$

Each category Tor_x is the category of torsion sheaves supported at x, which is the same as the category of finite dimensional modules over the discrete valuation ring $\mathcal{O}_{x,X}$. This category if the same as nilpotent representations of the Jordan quiver over the finite field $k_x = \mathcal{O}_{x,X}/m_{x,X}$.

For k a finite field, denote by N_k the category of nilpotent representations of the Jordan quiver over k. There exists exactly one indecomposable object $I_{(r)}$ of any length $r \in \mathbb{N}$. Further, all objects are of the form $I_\lambda = I_{(\lambda_1)} \oplus \cdots \oplus I_{(\lambda_s)}$ for a partition $\lambda = (\lambda_1, \cdots, \lambda_s)$.

Denote by Λ_t the (Macdonald) ring of symmetric functions over $\mathbb{Q}[t, t^{-1}]$, and by e_λ and p_λ the elementary symmetric functions and the power sum symmetric functions, respectively. Recall the Macdonald ring is defined via the projective limit $\Lambda = \lim \mathbb{C}[t, t^{-1}, x_1, \cdots, x_n]^{\Sigma(n)}$ where the maps between consecutive rings send the biggest index variable to zero, and the multiplication is induced from the multiplication on each of the individual polynomial rings. The coproduct is defined via the inclusion

$$\mathbb{C}[t, t^{-1}x_1, \cdots, x_{2n}]^{\Sigma(2n)} \hookrightarrow \mathbb{C}[t, t^{-1}, x_1, \cdots, x_n]^{\Sigma(n)} \otimes \mathbb{C}[t, t^{-1}, x_{n+1}, \cdots, x_{2n}]^{\Sigma(n)}.$$

Then the following theorem gives us a very explicit description of the Hall algebra associated to N_k.

Theorem 2.6. *(Macdonald)*

The assignment $[I_{(1)^r}] \to u^{r(r-1)} e_r$ extends to a bialgebra isomorphism

$$\Phi_k : \mathbb{H}_{N_k} \to \Lambda_t|_{t=u^2}.$$

Set $F_r := \Phi_l^{-1}(p_r)$. Then:

(i) $F_r = \sum_{|\lambda|=r} n_u(l(\lambda) - 1)[I_\lambda]$, where $n_u(l) := \prod_{i=1}^{l}(1 - u^{-2i})$,

(ii) $\Delta(F_r) = F_r \otimes 1 + 1 \otimes F_r$,

(iii) $(F_r, F_s) = \delta_{rs} \frac{ru^r}{u^{-r} - u^r}$.

2.3. **Hall algebras for projective curves.** Let x be a closed point of the (smooth) projective curve X over a finite field k. denote by $\deg(x)$ the degree of the finite extension k_x/k, where $k_x = \mathcal{O}_{x,X}/m_{x,X}$. Recall that N_{k_x} is equivalent to Tor_x. Let

$$\Phi_{k_k} : \mathbb{H}_{\mathrm{Tor}_x} \to \Lambda_y|_{t=v^{2\deg(x)}}$$

be the isomorphism provided by Macdonald's theorem, where $v^2 = k^{-1}$.

For r a natural number, define

$$\frac{T^{(\infty)}_{r,x}}{[r]} = \frac{\deg(x)}{r} \Phi^{-1}_{k_x}(p_{\frac{r}{\deg(x)}}) \text{ if } \deg(x)|r$$

and by zero otherwise. We put $T^{(\infty)} = \sum_x T^{(\infty)}_{r,x}$ which is a finite sum, as there are only finitely many points on X of a given degree.

For $\mathbb{P}^1$, Kapranov proved a Ringel-Green style theorem, comparing the Hall algebra with the positive part $U_v(L\mathfrak{b}_+) \subset U_v(L\mathfrak{sl}_2)$ of the quantum loop algebra of $\mathfrak{sl}_2$. For more details, see [4][page 64].

Theorem 2.7. *(Kapranov) There exists an embedding of algebras*

$$\Phi : U_v(L\mathfrak{b}_+) \to \mathbb{H}_{\mathbb{P}^1}.$$

2.4. **The Drinfeld double construction.** We have seen in the above examples that the Hall algebra (as defined in the beginning of these notes) recovers the positive nilpotent part of a quantum group. Adding the group algebra of the Grothendieck group $\mathbb{C}[K(\mathcal{A})]$ corresponds to adding the Cartan part to the quantum group. It is natural to try to construct the full quantum group this way. A possible idea is to change the category we are looking at. We know that we should get two copies of the Hall algebra, so we would like to replace the abelian category $\mathcal{A}$ with a variant that contains two copies of $\mathcal{A}$. However, there exists a completely algebraic procedure from which one can double a quantum group, which we will explain in this section.

In our case, we can start with the (topological) bialgebra $\mathbb{H}_X$ and construct another (topological) bialgebra $\mathbb{DH}_X$ which is generated by the Hall algebra $\mathbb{H}_X$ and its dual $\mathbb{H}_X^*$ with opposite coproduct. In our case, we can identify the dual of the Hall algebra with the Hall algebra via the Hopf pairing. Thus $\mathbb{DH}_X$ will be generated by two copies $\mathbb{H}_X^+$ and $\mathbb{H}_X^-$ of the Hall algebra. The set of relations we impose is the following: for any pair (h, g) of elements in $\mathbb{H}_X$, there is one relation $R(h, g)$ given by

$$\sum_{i,j} h_i^{(1)-} g_j^{(2)+} (h_i^{(2)}, g_j^{(1)}) = \sum_{i,j} g_j^{(1)+} h_i^{(2)-} (h_i^{(1)}, g_j^{(2)}),$$

where $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$.

In our case, even if the coproduct takes values in the completed tensor product, the relations $R(h,g)$ actually contain finitely many terms. It is worth noticing that one can prove a PBW theorem for the Drinfeld double, saying that the multiplication map

$$m : \mathbb{H}_X^- \otimes \mathbb{H}_X^+ \to \mathbb{DH}_X$$

is a vector space isomorphism.

Drinfeld used this technique to construct the full quantum group $U_v(\mathfrak{g})$ as a quotient of Drinfeld double of the quantum group $U_v(\mathfrak{b}_+)$.

3. The Hall algebra of an elliptic curve

3.1. Coherent sheaves on an elliptic curve. In this subsection, we recap some of the material from the previous talk [3]. Let X be an elliptic curve over an arbitrary field k. The slope of a sheaf $F \in \mathrm{Coh}\,(X)$ is defined as

$$\mu(F) = \frac{\deg(F)}{\mathrm{Rank}\,(F)} \in \mathbb{Q} \cup \{\infty\}.$$

A sheaf F is called stable/ semistable if for all proper subsheaves $G \subset F$, we have $\mu(G) < (\leqslant)\mu(F)$. Also, any sheaf F has a unique Harder-Narasimhan filtration, that is, a filtration by subsheaves such that the quotients are semistable of strictly increasing slope. Define the category C_μ as the full subcategory of $\mathrm{Coh}\,(X)$ of semistable sheaves of slope μ [1]; one can show C_μ is an abelian subcategory of $\mathrm{Coh}\,(X)$. For two slopes $b < a$, we define $C[b,a]$ to be the full subcategory of $\mathrm{Coh}\,(X)$ whose objects are the elements of C_μ for $b \leqslant \mu \leqslant a$.

Theorem 3.1. *(Atiyah) The following hold:*

(i) the Harder-Narasimhan filtration of any coherent sheaf splits (non canonically). In particular, any indecomposable coherent sheaf is stable.

(ii) the stable sheaves of slope μ are the simple objects in the category C_μ.

(iii) there exist equivalences (canonical using the chosen rational point on the elliptic curve X) of abelian categories

$$\varepsilon_{a,b} : C_a \to C_b$$

for any $a, b \in \mathbb{Q} \cup \{\infty\}$.

To define the extended Hall algebra, we added as a degree zero part the algebra of $K_0(X)$. In our case, the symmetrized Euler form vanishes, so it is not necessary to add this part to the Hall algebra. Also, it is preferable in some situations to work with the numerical $K-$group, which is finitely generated:

$$K_0(X) \to K_0'(X) = \mathbb{Z}^2,$$

which sends a sheaf F to $(\text{rank}\,(F), \deg(F))$.

Recall from last time the discussion about derived equivalences of $D^b\text{Coh}\,(X)$: examples include the shift functor [1], automorphism induced by automorphisms of the curve X itself, tensoring with line bundles, and the Seidel-Thomas autoequivalences. The Seidel-Thomas equivalences are defined as follows: given a spherical object object $E \in \text{Coh}\,(X)$, namely a sheaf E with $\text{Hom}\,(E,E) = \text{Hom}\,(E, E[1]) = k$, define $T_E : D(X) \to D(X)$ by

$$T_E(F) = \text{cone}\,(RHom(E,F) \otimes E \to F).$$

The autoequivalences $T_{\mathcal{O}}$ and $T_{\mathcal{O}_\S}$ satisfy the braid group relation

$$T_{\mathcal{O}_x} T_{\mathcal{O}} T_{\mathcal{O}_x} = T_{\mathcal{O}} T_{\mathcal{O}_x} T_{\mathcal{O}}.$$

The group generated by $T_{\mathcal{O}}, T_{\mathcal{O}_x}$, and [1] is the universal covering $\widetilde{SL(2,\mathbb{Z})}$ of $SL(2,\mathbb{Z})$ [1]. The only derived equivalences we will be interested in will be elements of this group. It is important to keep in mind that given a spherical object E, the autoequivalence T_E descends to an automorphism of $K_0'(X) = \mathbb{Z}^2$, and that this automorphism can be written explicitly, see [1][page 1177] for more details.

The image of the abelian category $\text{Coh}\,(X)$ under a Seidel-Thomas derived equivalence T_E, for E a spherical object, is given by a tilted heart $\text{Coh}_v(X)$; we will first state the definition of $\text{Coh}_v(X)$ and then we will explain how to compute v from E. Recall that $\text{Coh}_{\leqslant v}(X)$ is the full subcategory of $\text{Coh}\,(X)$ consisting of sheaves F whose all direct summands have slope $\leqslant v$; $\text{Coh}_{>v}(X)$ is defined in a similar manner. Now, the tilted heart $\text{Coh}_v(X)$ is the full subcategory of $D^b\text{Coh}\,(X)$ with objects the complexes $F \oplus G[1]$, where $F \in \text{Coh}_{>v}(X)$ and $G \in \text{Coh}_{\leqslant v}(X)$. For a spherical sheaf E of class $(r,d) \in K_0'(X)$ and of slope $\mu = \frac{d}{r}$, the autoequivalence T_E establishes an autoequivalence between $\text{Coh}\,(X)$ and $\text{Coh}_v(X)$ where $v = -\infty$ for $\mu = \infty$ and $v = \mu - \frac{1}{r^2}$ otherwise.

We have mentioned earlier that the most naive approach from a geometrical point of view for doubling the Hall algebra in order to pass from the positive part of a quantum group to the full quantum group is to look for a double version (in some way) of the abelian category $\mathcal{A}$. For this purpose, define the root category $\mathbf{R}_X$ as the orbit category $\mathbf{R}_X = D^b(X)/[2]$; $\mathbf{R}_X$ has a triangulated structure [4]. One can think of $\mathbf{R}_X$ as the category of $2-$periodic complexes of coherent sheaves on X. For any object $A \in \text{Coh}\,(X)$, we denote by A^+ the image of A in $\mathbf{R}_X$ and by A^- the image of $A[-1]$ in $\mathbf{R}_X$. We define the semistable objects of $\mathbf{R}_X$ to be the objects $A^{\pm}$, where A is semistable in $\text{Coh}\,(X)$. The action of the group $\widetilde{SL(2,\mathbb{Z})}$ breaks to an action of $SL(2,\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ in $\mathbf{R}_X$ because $[1]^2 = \text{id}$ in $\mathbf{R}_X$. From now on, whenever we refer to the action of $SL(2,\mathbb{Z})$ on $\mathbf{R}_X$, we refer to this particular action. Using tilted hearts, one can prove that the set of semistable objects of the root category $\mathbf{R}_X$ is invariant under the action of $SL(2,\mathbb{Z})$.

3.2. **A PBW theorem for the full Hall algebra.** The algebra we are interested in, the elliptic Hall algebra, will be defined as a subalgebra of the Hall algebra of X. One of the main theorems we will prove about the EHA is a PBW theorem. Before doing it, we need a PBW theorem for $\mathbb{H}_X$ which is significantly easier to prove.

First, we need to fix some notation. Let $\mathbb{H}_X^a \subset \mathbb{H}_X$ be the subspace spanned by classes of sheaves $F \in C_a$. The category C_a is stable under extensions, thus $\mathbb{H}_X^a$ is a subalgebra of the Hall algebra. Further, Atiyah's theorem provides algebra isomorphisms

$$\varepsilon_{a,b} : \mathbb{H}_X^b \to \mathbb{H}_X^a.$$

Proposition 3.2. *(PBW theorem for the Hall algebra) The multiplication map*

$$m : \otimes_a \mathbb{H}_X^a \to \mathbb{H}_X$$

is an isomorphism.

Proof. We have that $\mathrm{Hom}\,(G,F) = \mathrm{Ext}\,(F,G) = 0$ for $F \in C_a$ and $G \in C_b$, where $a < b$. Thus, up to a power of v, we have that

$$[F_1]\cdots[F_r] = [F_1 \oplus \cdots \oplus F_r],$$

where $F_i \in C_{a_i}$ and $a_1 < \cdots < a_r$. Any sheaf can be decomposed in such a direct sum of semistable sheaves by Theorem 3.1.1, so the multiplication map is surjective. To show injectivity, pick a finite sum S with the minimal number of terms possible which sums up to zero $\sum_{\mu(F_1)<\cdots<\mu(F_a)} c[F_1]\cdots[F_a] = 0$, where all the sheaves F_i appearing in the sum are semistable; then $\sum c'[F_1 \sum \cdots \sum F_a] = 0 \in \mathbb{H}_X$. Assume that the not all coefficients are zero. Let F be a sheaf with maximal slope appearing in the sum, and which is not contained in any other sheaf G appearing in the sum. All the terms $[F_1 \cdots \mathcal{F}_a]$ that are equal, up to a constant, to a term that contains the sheaf F must have the last term $F_a = F$. Then $\sum_{F_a=F} c[F_1]\cdots[F_{a-1}] = 0$, which by the minimality of the chosen sum implies that all the coefficients in the sum $\sum_{F_a=F} c[F_1]\cdots[F_{a-1}] = 0$ are zero, and thus that $[F]$ appears with coefficient zero in the sum S, which contradicts our assumption on $[F]$. Thus m is injective and thus an isomorphism. $\square$

3.3. **Drinfeld double of the Hall algebra.** The Drinfeld double of the Hall algebra carries more symmetries than the Hall algebra alone− for example, the derived equivalences of $D(X)$ give algebra automorphisms of the Drinfeld double $\mathbb{DH}_X$. As we have already seen in the previous talks, this symmetry can be used reduce general statements to simpler ones, see the proof of Theorem 3.10.

Before we start proving that derived equivalences give algebra automorphisms of $\mathbb{DH}_X$, we want to find generators and relations for $\mathbb{DH}_X$. We can actually phrase all the relations in terms of semistable sheaves only.

Proposition 3.3. *The Hall algebra $\mathbb{H}_X$ is isomorphic to the $K-$algebra generated by $\{x_F | F$ semistable$\}$ subject to the relations $P([F],[G])$:*

$$x_F x_G = v^{-\chi(F,G)} \sum_H c^H_{F,G} \underline{x}_H,$$

where F and G are semistable, and $\underline{x}_H := v^{\sum_{i<j}\langle H_i,H_j\rangle} x_{H_1} \cdots x_{H_r}$, where $H = \oplus H_i$ is the Harder Narasimhan decomposition of H.

We can also rephrase the Drinfeld double relations $R([F],[G])$ in fairly explicit terms. We will not write all these relations, see [1], but here is an example: suppose F is $a-$semistable and G is $b-$semistable and $a < b$. The relation $R([F],[G])$ becomes

$$[F]^-[G]^+ = v^{\langle F,G\rangle} \sum_{B,C} v^{-\langle C,B\rangle} c^{B[1]\oplus C}_{F[1],G} [C]^+[B]^-,$$

where $c^{B[1]\oplus C}_{F[1],G}$ is the number of distinguished triangles

$$G \to B[1] \oplus C \to F[1].$$

In our case, the relation for $a = b$ says that the two halves commute, and this will be used in the proof of Theorem 3.10

Theorem 3.4. *Let Φ be an autoequivalence of $D(X)$ in the group $\widetilde{SL(2,\mathbb{Z})}$. The assignment $[F] \to [\Phi F]$ for F semistable object of $\mathbf{R}_X$ extends to an algebra automorphism of $\mathbb{DH}_X$.*

Proof. The algebra $\mathbb{DH}_X$ is isomorphic to the $K-$algebra generated by $[F]^+$ and $[F]^-$, where F is a semistable sheaf, subject to:

(i) the Hall algebra relations $P([F]^+,[G]^+)$,

(ii) the Drinfeld double relations $R([F],[G])$.

Le F be $a-$semistable and G be $b-$semistable, and Φ a derived equivalence in $SL(2,\mathbb{Z})$. Denote by $\tilde{F}[i]$ and $\tilde{G}[j]$ their images under Φ, and we can assume that i and j are both 0 or -1. The sheaves $\tilde{F}$ and $\tilde{G}$ are semistable, of slopes $\tilde{a}$ and $\tilde{b}$, respectively. We need to check that the relations $P([\tilde{F}],[\tilde{G}])$ and $R([\tilde{F}],[\tilde{G}])$ hold in $\mathbb{DH}_X$. It is clear that after applying the autoequivalence [1] the sheaves will continue to satisfy the two relations.

(1) Assume that $i = j$. Assume that $a > b$; then $\tilde{a} > \tilde{b}$. We thus get an isomorphism of Hall algebras for exact categories

$$\Phi : C[b, a] \to C[\tilde{b}, \tilde{a}]$$

preserving all the Hall algebra constants, and thus the relations $P([F], [G])$ are mapped to $P([\tilde{F}], [\tilde{G}])$.

(2) Assume that i and j are different, and say for simplicity that i is odd and j is even. Also assume that $a > b$. This case can only happen for $E = \mathcal{O}_x$. Once again, an easy computation shows that $\tilde{a} < \tilde{b}$. By the description of the derived equivalences via tilted hearts, there exists an integer $b \leqslant k < a$ such that

$$\Phi(C_\phi) \in \mathrm{Coh}\,(X)[i]$$

for $k < \phi \leqslant a$ and

$$\Phi(C_\phi) \in \mathrm{Coh}\,(X)[i-1]$$

for $b \leqslant \phi \leqslant k$.

Split any extension H of G by F as $H = H_0 \oplus H_1$, where H_0 has all semistable factors with slopes $\leqslant k$, and H_1 has all direct summands with slopes $> k$. Relation $P([F], [G])$ becomes

$$[F][G] = v^{-\langle F,G\rangle} \sum_{H_0,H_1} c_{F,G}^{H_0\oplus H_1} v^{\langle H_0,H_1\rangle}[H_0][H_1].$$

Now, $\Phi(F) = \tilde{F}[i], \Phi(H_1) = \tilde{H}_1[i-1], \Phi(G) = \tilde{G}[i]$, and $\Phi(H_0) = \tilde{H}_0[i]$. We also have $\langle F, G\rangle = -\langle \tilde{F}, \tilde{G}\rangle$ and $\langle H_0, H_1\rangle = -\langle \tilde{H}_0, \tilde{H}_1\rangle$.

The relation $P(\Phi(F), \Phi(G))$ we need to prove is actually

$$[\tilde{F}]^-[\tilde{G}]^+ = v^{\langle \tilde{F},\tilde{G}\rangle} \sum_{\tilde{H}_0,\tilde{H}_1} v^{-\langle \tilde{H}_0,\tilde{H}_1\rangle} c_{\tilde{F}[1],\tilde{G}}^{\tilde{H}_0\oplus \tilde{H}_1[1]}[\tilde{H}_0]^+[\tilde{H}_1]^-.$$

The equality to be proven is simply the Drinfeld double relation for $\tilde{F}$ and $\tilde{G}$.

The relations $R([\tilde{F}], [\tilde{G}])$ can be checked in a similar manner. □

Corollary 3.5. *The universal cover $\widetilde{SL}(2,\mathbb{Z})$ acts by algebra automorphisms on $\mathbb{DH}_X$.*

3.4. **The Elliptic Hall Algebra.** Recall the elements $T_r^{(\infty)} = \sum T_{r,x}^{(\infty)}$ defined in section 2.3, $T_r^{(\infty)} \in \mathbb{H}_X^{(\infty)}$. For arbitrary $\mu \in \mathbb{Q}$, define

$$T_r^{(\mu)} = \varepsilon_{\mu,\infty}(T_r^{(\infty)}).$$

Observe that $\varepsilon_{a,b}(T_r^{(b)}) = T_r^{(a)}$, for any slopes a and b.

Definition 3.6. Let $\mathbb{U}_X^+ \subset \mathbb{H}_X^+$ be the subalgebra generated by all elements $T_r^{(\mu)}$, for $r \geqslant 1$ and $\mu \in \mathbb{Q} \cup \{\infty\}$. Define $\mathbb{U}_X^- \subset \mathbb{H}_X^-$ similarly and let $\mathbb{U}_X \subset \mathbb{DH}_X$ be the subalgebra generated by $\mathbb{U}_X^+$ and $\mathbb{U}_X^-$.

We will introduce different notation for the generators of $\mathbb{U}_X$: for $\mu = \frac{l}{n}$ with l and n relatively prime, $n \geqslant 1$, write $T_{(\pm rn, \pm rl)} = (T_r^{(\mu)})^{\pm} \in \mathbb{U}_X^+$, $T_{(0,r)} = (T_r^{(\infty)})^+, T_{(0,0)} = 1$. Also define $(\mathbb{Z}^2)^{\pm} = \{(q,p) \in \mathbb{Z}^2 | \pm q > 0 \text{ or } \pm q = 0, p > 0\}$ and similarly for the minus half. Then $\mathbb{U}_X^{\pm}$ is the subalgebra of $\mathbb{DH}_X$ generated by $T_{(q,p)}$ for $(q,p) \in (\mathbb{Z}^2)^{\pm}$.

The $\widetilde{SL}(2,\mathbb{Z})$ action on $\mathbb{DH}_X$ preserves $\mathbb{U}_X$. This action factors through $SL(2,\mathbb{Z})$, as mentioned earlier, and for $\gamma \in SL(2,\mathbb{Z})$ we have $\gamma T_{(p,q)} = T_{\gamma(p,q)}$.

Finally, in some situations it will be more convenient to use another set of generators for the algebra $\mathbb{U}_X$. For $a \in (\mathbb{Z}^2)^+$ define

$$1_a^{ss} = \sum_{H \text{sstable slope } a} [H] \in \mathbb{H}_X^+[a].$$

For $a = (q,p)$ with q and p relatively prime, a computation using torsion sheaves and Macdonald's theorem shows that

$$1 + \sum_{r \geqslant 1} 1_{ra}^{ss} s^r = \exp(\sum_{r \geqslant 1} \frac{T^{ra}}{[r]} s^r).$$

This makes transparent that the elements 1_a^{ss} with $a \in (\mathbb{Z}^2)^{\pm}$ generate $\mathbb{U}_X^{\pm}$. In the next part, we list some results about the generators of $\mathbb{U}_X$ that will be used later in the talk. We do not provide (complete) proofs, which can be found in [1]:

(1) If $a = (p,q)$ with p and q coprime, then $T_a = 1_a^{ss}$. This is immediate from the above identity.
(2) Macdonald's theorem implies that $\Delta(T_{(0,n)}) = T_{(0,n)} \otimes 1 + 1 \otimes T_{(0,n)}$.
(3) If we define $1_a = \sum_{F \text{ of class } a} [F] \in \mathbb{H}_X$, we have that $1_{(0,l)} = 1_{(0,l)}^{ss}$ and $1_{(1,l)} = \sum_{F=(1,l)} [F] = 1_{(1,l)}^{ss} + \sum_{d>0} v^d 1_{(1,l-d)}^{ss} 1_{(0,d)}$ as any coherent sheaf on X of rank one splits uniquely as the sum of a line bundle and of a torsion sheaf.
(4) $\Delta_{a,b}(1_{a+b}) = v^{\langle a,b \rangle} 1_a \otimes 1_b$. This is a general result for Hall algebras of abelian categories of gldim $\leqslant 1$. Here, $\Delta_{a,b} : \mathbb{U}_X[a+b] \to \mathbb{U}_X[a] \times \mathbb{U}_X[b]$ is the (a,b) component of the coproduct Δ.

The next two results will be used in the proof of Theorem 3.10 which gives generators and relations for the EHA. We denote by $c_i(X) = \frac{|X(\mathbb{F}_{q^i})| v^i [i]}{i}$.

(5) Let $x = (q,p) \in (\mathbb{Z}^2)^+$, and define $r = \gcd(p,q)$. Then

$$(T_x, T_x) = \frac{c_r(X)}{v^{-1} - v}.$$

One can use the $SL(2,\mathbb{Z})$ action and reduce the computation to the case $x = (0,r)$. By Macdonald's theorem, one can compute explicitly $(T_{r,x}^{(\infty)}, T_{r,x}^{(\infty)}) = \frac{v^r[r]d}{r(v^{-1}-v)}$. Also, recall that Macdonalds theorem also says that $T_{r,x}^{(\infty)}$ are orthogonal to each other.

This computation is used in proving the next result:

(6) For any $n \geqslant 0$ and any $a = (r,d) \in (\mathbb{Z}^2)^+$ we have

$$[T_{(0,n)}, 1_a] = c_n(X) \frac{v^{rn} - v^{-rn}}{v - v^{-1}} 1_{a+(0,n)}.$$

We will only say a few words about the argument, a full proof can be found in [1][Lemma 4.11]. One introduces the elements $1^{vec} = \sum_{F \text{ v.b. class } a} [F] \in \mathbb{U}_X^+$. Because 1_a can be written explicitly in function of 1_a^{vec}, using identities similar to the ones in item (3), one reduces the above statement to the one where 1_a is replaced by 1_a^{vec}. The first part of the proof is showing that $[T_{(0,n)}, 1_a] \in \mathbb{H}_X^{vec}$.

Next, one computes directly the coefficient of a vector bundle $[V]$ in the commutator $[[T], 1_a^{vec}]$, where T is a torsion sheaf. The answer ends up depending on T and the rank of V only. This implies that $[T_{(0,n)}, 1_a^{vec}] = u_r 1_a^{vec}$, where u_r is a constant depending on r only.

The general case can be reduced to the rank one case. In the rank one case, by the $SL(2,\mathbb{Z})$ action we can assume that $a = (1,0)$. Then u_1 is computed by expressing the scalar product $(T_{(0,n)}T_{(1,0)}, 1_{(1,n)})$ in two different ways, one using the Hopf pairing and item (4), and one using item (3).

3.5. **A PBW theorem for the EHA.** Recall the PBW decomposition we have obtained for the Hall algebra in Section 3.2: the multiplication map induces isomorphisms $\otimes_a \mathbb{H}_X^{a,\pm} \to \mathbb{H}_X^{\pm}$ and $\otimes_a \mathbb{H}_X^{a,+} \otimes \otimes_a \mathbb{H}_X^{a,-} \to \mathbb{H}_X$. In this section, we prove the analogous theorem for the EHA.

Theorem 3.7. *(Burban-Schiffmann) The multiplication map induces isomorphisms of $K-$vector spaces $\otimes_a \mathbb{U}_X^{a,\pm} \to \mathbb{U}_X^{\pm}$ and $\otimes_a \mathbb{U}_X^{a,+} \otimes \otimes_a \mathbb{U}_X^{a,-} \to \mathbb{U}_X$.*

Proof. Let $\mathbb{H}_X^{vec} = m(\otimes_{a<\infty} \mathbb{H}_X^{a,+})$ be the subspace generated by classes of vector bundles.

Claim: $\mathbb{U}_X^+ \subset \mathbb{H}_X^{vec} \otimes \mathbb{U}_X^{\infty}$.

Let's assume the claim for the moment. For any slope a, there exists $c \in SL(2,\mathbb{Z})$ such that $c(a) = \infty$. Recall that $SL(2,\mathbb{Z})$ acts on $\mathbb{DH}_X$, preserves the subalgebra $\mathbb{U}_X$, and permutes the factors $\mathbb{H}_X^{a,+}$. Using the claim, we obtain:

$$c(\mathbb{U}_X^+) \subset \mathbb{U}_X^+ \otimes \mathbb{U}_X^- \subset (\otimes_{a<\infty}\mathbb{H}_X^{a,+} \otimes \mathbb{U}_X^{\infty,+}) \otimes (\otimes_{a<\infty}\mathbb{H}_X^{a,-} \otimes \mathbb{U}_X^{\infty,-}),$$

from which we deduce, after applying c^{-1}, that

$$\mathbb{U}_X^+ \subset \otimes_{\mu<a}\mathbb{H}_X^{\mu,+} \otimes \mathbb{U}_X^{a,+} \otimes \otimes_{\mu>a}\mathbb{U}_X^{a,+}.$$

This is true for all slopes a, and thus

$$\mathbb{U}_X^+ \subset \otimes_a \mathbb{U}_X^{a,+},$$

as desired.

We still need to establish the above claim in order to finish the proof. Consider an element $u \in \mathbb{U}_X^+[a]$, and expand it as $u = \sum_l u_l$, where $u_l = \sum_i u'_{l,i}u''_{l,i}$, where $u'_{l,i} \in \mathbb{H}_X^{vec}$ and $u''_{l,i} \in \mathbb{H}_X^{(\infty)}[(0,l)]$ for all i and l. Denote by $\pi : \mathbb{H}_X \to \mathbb{H}_X^{vec}$. We can compute that

$$(\pi \otimes 1)\Delta_{a-(0,l),(0,l)}(u) = v^{\langle a-(0,l),(0,l)\rangle} u_l.$$

On the other hand, the fact above tells us that

$$\Delta_{a-(0,l),(0,l)}(u) \in \mathbb{U}_X^+[a-(0,l)] \otimes \mathbb{U}_X^+[(0,l)].$$

From these two relations we deduce that $u_l \in \mathbb{H}_X^{vec} \otimes \mathbb{U}_X^+[(0,l)]$, and thus that u has the claimed property.

□

3.6. **EHA via generators and relations.** Once we have proven the PBW theorem, we can start identifying the algebra $\mathbb{U}_X$ defined in these notes with the EHA defined by generators and relations as defined in [2]. To differentiate between the two algebras, we will call the latter $\mathcal{E}$. Let's review the definition of $\mathcal{E}$.

Let o be the origin in $\mathbb{Z}^2$, and let Conv$'$ be the set of all convex paths $p = (x_1, \cdots, x_r)$ satisfying $\angle x_1 L_0 \geqslant \cdots \geqslant \angle x_r L_o \geqslant 0$; here, the notation $\angle x_1 L_0$ means the angle between the vector ox_1 and the vector L_0 which joins the origin o and the point $(0,-1)$. Two convex paths p and q are equivalent if they are obtained from each other by a permutation of their edges. Let Conv be the set of equivalence classes of convex paths Conv$'$, Conv$^+$ its subset of convex paths with all angles $\geqslant \pi$, and Conv$^-$ its subset with all paths $< \pi$. Concatenation gives an identification

$$\mathrm{Conv} = \mathrm{Conv}^+ \times \mathrm{Conv}^-.$$

The PBW theorem proved in the previous section says that the elements $\{T_p | p \in \mathrm{Conv}^\pm\}$ are a $K-$basis of $\mathbb{U}_X^\pm$, where to a path $p = (x_1, \cdots, x_r)$ we associate the element $T_p := T_{x_1} \cdots T_{x_r} \in \mathbb{U}_X$. For $x \in \mathbb{Z}^2 - o$, define $\deg(x) := \gcd(p,q) \in \mathbb{N}$. Also, for $x, y \in \mathbb{Z}^2 - o$ noncollinear, denote by $\varepsilon(x,y) = \mathrm{sign}\,(\det(x,y)) \in \{-1, 1\}$.

Definition 3.8. Fix $\sigma, \bar{\sigma} \in \mathbb{C} - \{0, -1, 1\}$, and let $v := (\sigma\bar{\sigma})^{-1/2}$ and $c_i(\sigma, \bar{\sigma}) = (\sigma^{i/2} - \sigma^{-i/2})(\bar{\sigma}^{i/2} - \bar{\sigma}^{-i/2})\frac{[i]_v}{i}$.

Let $\mathcal{E}_{\sigma,\bar{\sigma}}$ be the $\mathbb{C}-$algebra generated by $\{t_x | x \in \mathbb{Z}^2 - o\}$ modulo the following relations:

(1) if x and y belong to the same line in $\mathbb{Z}^2$, then $[t_x, t_y] = 0$,

(2) if x is of degree one and y is another nonzero lattice point such that $\Delta(x,y)$ has no interior lattice points, then

$$[t_y, t_x] = \varepsilon(x,y) c_{\deg(y)}(\sigma, \bar{\sigma}) \frac{\theta_{x+y}}{v^{-1} - v},$$

where the elements θ_z, for $z \in \mathbb{Z}^2 - o$, are defined as follows:

$$\sum_i \theta_{ix_0} s^i = \exp((v^{-1} - v) \sum_{t \geqslant 1} t_{rx_0} s^r),$$

for any degree one element $x_0 \in \mathbb{Z}^2 - o$.

We also denote by $\mathcal{E}^{\pm}_{\sigma,\bar{\sigma}}$ the subalgebras generated by t_x with $x \in (\mathbb{Z}^2)^{\pm}$.

Let X be an elliptic curve over $\mathbb{F}_{q^r}$. Then by Hasse's theorem there exist complex numbers σ and $\bar{\sigma}$ such that $\sigma\bar{\sigma} = q$, and

$$|X(\mathbb{F}_{q^r})| = q^r + 1 - (\sigma^r + \bar{\sigma}^r).$$

Observe that $c_i(\sigma, \bar{\sigma}) = \frac{v^i [i] |X(\mathbb{F}_{q^i})|}{i} = c_i(X)$.

Theorem 3.9. *(Burban-Schiffmann)*

The assignment $\Omega : t_x \to T_x$ for all $x \in \mathbb{Z}^2 - o$ extends to an isomorphism

$$\Omega : \mathcal{E}_{\sigma,\bar{\sigma}} \to \mathbb{U}_X \otimes_K \mathbb{C}.$$

Proof. The proof consists of three parts. First, we need to check that Ω is well defined, that is, we need to check that the generators T_x of the Hall algebra $\mathbb{U}_X$ satisfy relations (1) and (2) from definition 3.9. The second part is a counting dimensions argument which will show that Ω restricts to isomorphisms to both halves. One can use their inverses to construct a map of vector spaces

$$\Omega^{-1} : \mathbb{U}_X \otimes_K \mathbb{C} \to \mathcal{E}_{\sigma,\bar{\sigma}},$$

which restricts to algebra isomorphisms on both halves. The third part is checking that Ω^{-1} is well-defined, that is, that the Drinfeld double relations hold in $\mathcal{E}_{\sigma,\bar{\sigma}}$.

The first step is checking that relations (1) and (2) in definition 3.9 hold for $\mathbb{U}_X$. For (1), by the $SL(2, \mathbb{Z})$ invariance of both algebras, we can assume that $x = (0, r)$

and $y = (0,s)$, Then the relation follows because $\mathbb{H}_X^{(\infty)}$ is commutative [1]. For relation (2), since $\deg(x) = 1$, we cannot have both $\deg(y)$ and $\deg(x+y)$ equal to 2 by arithmetic reasons, or one of them $\geqslant 3$ and the other $\geqslant 2$ by an application of Pick's theorem. Thus we have either $\deg(x+y) = 1$ or $\deg(y) = 1$. We only discuss the first possibility. Using the $SL(2,\mathbb{Z})$ action, we can assume $x = (1,0)$, and if $\det(x,y) = r$, we can further assume $y = (s,r)$ with $0 \leqslant s < r$. Because there are no lattice points in $\Delta(x,y)$, we deduce that $y = (0,r)$. We thus need to check (2) for $T_{(1,0)}$ and $T_{(0,r)}$. The relation we need to prove is

$$[T_{(0,r)}, T_{(1,0)}] = c_r(\sigma,\bar{\sigma})\frac{\theta_{(1,r)}}{v^{-1}-v}.$$

Observe that $\theta_{(1,r)} = (v^{-1}-v)t_{(1,r)}$, and thus the relation we need to prove becomes

$$[T_{(0,r)}, T_{(1,0)}] = c_r T_{(1,r)},$$

which is exactly relation (6) in section 3.4. This implies that Ω is extends to a surjective $SL(2,\mathbb{Z})$ equivariant algebra morphism.

By the PBW theorem for the algebras $\mathcal{E}^{+-}_{\sigma,\bar{\sigma}}$, we deduce that Ω restricts to isomorphisms on both halves of these two algebras. Denote their inverses by

$$\Omega_{\pm}^{-1} : \mathbb{U}_X^{\pm} \otimes_K \mathbb{C} \to \mathcal{E}^{\pm}_{\sigma,\bar{\sigma}}.$$

Recall that the algebra $\mathbb{U}_X$ is generated by the two halves $\mathbb{U}_X^{\pm}$ modulo the Drinfeld double relations $R(h,g)$, for $h,g \in \mathbb{U}_X^{+}$ both classes of semistable sheaves. We need to check that these relations hold in $\mathcal{E}_{\sigma,\bar{\sigma}}$ in order to conclude that Ω^{-1} is an algebra morphism.

There are two cases to consider. The first one is when the slopes of g and h are equal to μ. With some work, one can see that the relation $R(h,g)$ says that the algebras $\mathbb{U}_X^{+,\mu}$ and $\mathbb{U}_X^{-,\mu}$ commute, which is true in $\mathcal{E}_{\sigma,\bar{\sigma}}$ by relation (1). The second case is when the slopes are different. There exists an element $c \in SL(2,\mathbb{Z})$, such that $c(g) \in \mathbb{U}_X^{+}$ and $c(h) \in \mathbb{U}_X^{-}$. Once again, we want to show that the relation $\Omega^{-1}(R(h,g))$ holds in $\mathcal{E}$. The relation $R(h,g)$ holds in $\mathbb{U}_X$; applying c we obtain that the relation $cR(h,g)$ in $\mathbb{U}_X^{+}$. Because Ω is an algebra isomorphism on the two halves, we obtain the relation $\Omega^{-1}(cR(h,g))$ in $\mathcal{E}^{+}$. By the $SL(2,\mathbb{Z})-$equivariance property of $\mathcal{E}$, we obtain the relation $c\Omega^{-1}(R(h,g))$ in $\mathcal{E}^{+}$, and, consequently, the relation $\Omega^{-1}(R(h,g))$ in $\mathcal{E}$, which is what we wanted to prove.

$\square$

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