Free Field Realization

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Abstract

The notes are prepared for the seminar *Representations of affine Kac-Moody algebras at the critical level* at MIT in Spring 2024.

1 Overview

We are in the process of proving an isomorphism between $\mathfrak{z}(V_{\kappa_c}(\mathfrak{g}))$ and $\mathbb{C}[\mathrm{Op}_{^LG}(D)]$, which further implies an isomorphism $Z(\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})) \cong \mathbb{C}[\mathrm{Op}_{^LG}(D^\times)]$. The strategy for the proof is to embed both algebras inside $\mathbb{C}[\mathfrak{h}^*[[t]]dt]$.

We begin by reviewing the counterpart of the embedding $\mathfrak{z}(V_{\kappa_c}(\mathfrak{g})) \to \mathbb{C}[\mathfrak{h}^*[[t]]dt]$ in the finite-dimensional case, which coincides with the map used in the Harish-Chandra isomorphism, as well as constructions that are useful in the affine case.

2 The finite-dimensional case

2.1 Recollection about vector fields

Recall that if we have a group action $\alpha : G \times X \to G$, with G a Lie group, there is an induced Lie algebra homomorphism

$$\alpha_* : \mathfrak{g} \longrightarrow \mathrm{Vect}(X)$$

sending $z \in \mathfrak{g}$ to the vector field $\alpha_*(z)$ on X defined by

$$(\alpha_*(z)f)(x) \coloneqq \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f(\exp(-\epsilon z)x).$$

Let us briefly review some properties of vector fields. For a smooth affine scheme X, the vector fields on X are precisely the derivations on the ring of functions $\mathbb{C}[X]$. For smooth affine schemes X and Y, we have a map $\mathrm{Vect}(X \times Y) \to \mathbb{C}[X] \otimes \mathrm{Vect}(Y) \oplus \mathbb{C}[Y] \otimes \mathrm{Vect}(X)$, or equivalently,

$$\mathrm{Der}(\mathbb{C}[X] \otimes \mathbb{C}[Y]) \longrightarrow (\mathbb{C}[X] \otimes \mathrm{Der}(\mathbb{C}[Y])) \oplus (\mathbb{C}[Y] \otimes \mathrm{Der}(\mathbb{C}[X])) \tag{1}$$

which sends $\varphi \longmapsto (\varphi|_{1\otimes\mathbb{C}[Y]}, \varphi|_{\mathbb{C}[X]\otimes 1})$. It is an isomorphism of Lie algebras with respect to the Lie bracket on RHS defined by $[f\varphi, g\psi] = f\varphi(g)\psi - g\psi(f)\varphi$ for $f \in \mathbb{C}[X]$, $\varphi \in \mathrm{Vect}(Y)$, $g \in \mathbb{C}[Y]$ and $\psi \in \mathrm{Vect}(X)$. Consequently, letting $D(X)$ be the algebra of differential operators on X, we can check that

$$D(X \times Y) = D(X) \otimes D(Y)$$

is an isomorphism.

Let G be an algebraic group G The Lie algebra of left-invariant (resp. right-invariant) vector fields $\mathfrak{g}_l$ (resp. $\mathfrak{g}_r$) are identified with $\mathfrak{g}$.

The action of G on itself induces a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathrm{Vect}(G)$ which is equivariant for the adjoint action of G on $\mathfrak{g}$ and the left G-action on $\mathrm{Vect}(G)$. It factors through an isomorphism

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}_r \subset \mathrm{Vect}(G)$$

mapping $x \in \mathfrak{g}$ to $-x_r$ where x_r is the corresponding right G-equivariant vector field.

Remark 2.1. If G is abelian, then we can identify x_l and x_r. In general, letting $\iota : G \to G$ be the inversion, $d\iota$ gives an isomorphism between $\mathfrak{g}_l$ and $\mathfrak{g}_r$.

Remark 2.2. The inclusion $\mathfrak{g}_l \subset D(G)^{G_l}$ lifts to an isomorphism $U(\mathfrak{g}_l) \cong D(G)^{G_l}$. Similarly, $D(G)^{G_r} \cong U(G_r)$.

2.2 The homomorphism $\mathfrak{g} \to \mathrm{Vect}(N_+) \oplus (\mathbb{C}[N_+] \otimes \mathfrak{h})$.

Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ with Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and let $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ be Borel subalgebras.

Let G be the connected simply connected algebraic group corresponding to $\mathfrak{g}$, and let $N_\pm$ and $B_\pm$ be the unipotent and Borel subgroups corresponding to $\mathfrak{n}_\pm$ and $\mathfrak{b}_\pm$ respectively. There is an isomorphism of varieties $N_+ \times H \cong B_+$ sending $(n,t) \mapsto nt$. Note that $N_+ \times H$ admits the following actions:

- An N_+-action from the left: $(n', (n,t)) \mapsto (n'n, t)$
- An H-action from the left: $(t', (n,t)) \mapsto (t'nt'^{-1}, t't)$
- An H-action from the right: $((n,t), t'') \mapsto (n, tt'')$

Now the isomorphism 1 becomes

$$\mathrm{Vect}(N_+ \times H) \cong (\mathrm{Vect}(N_+) \otimes \mathbb{C}[H]) \oplus (\mathrm{Vect}(H) \otimes \mathbb{C}[N_+])) \otimes \mathbb{C}[N_+]).$$

Noting that right H-action is trivial on $\mathfrak{h}$ and $\mathrm{Vect}(N_+)$, while standard on $\mathbb{C}[H]$, it follows that

$$\mathrm{Vect}(N_+ \times H)^{H_r} \cong \mathrm{Vect}(N_+) \oplus (\mathbb{C}[N_+] \otimes \mathfrak{h}). \tag{2}$$

Remark 2.3. We can upgrade the isomorphism 2 to the level of algebra

$$D(N_+ \times H)^{H_r} = (D(N_+) \times D(H))^{H_r} = D(N_+) \otimes U(\mathfrak{h}).$$

Now consider the homogeneous space G/N_- with left G-action and right H-action, noting that these actions commute. There is an induced map of Lie algebras

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(G/N_-)^{H_r}.$$

By considering the restriction to the open B_+-orbit $B_+[1] \subset G/N_-$, it induces a map of Lie algebras

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^H = \mathrm{Vect}(N_+) \oplus (\mathbb{C}[N_+] \otimes \mathfrak{h}). \tag{3}$$

Remark 2.4. We first note that the map $U(\mathfrak{g}) \to D(B_+)$ induced by $\mathfrak{g} \to \mathrm{Vect}(B_+)$ preserves the filtrations with respect to the PBW filtration on $U(\mathfrak{g})$ and the order of differential operators on $D[B_+]$. Then, the associated graded $S(\mathfrak{g}) \longrightarrow \mathbb{C}[T^*B_+]$, where T^*B_+ denotes the cotangent bundle, can be described as the composition of the following maps:

1. The classical comoment map $S(\mathfrak{g}) \longrightarrow \mathbb{C}[T^*(G/N_-)]$
2. The restriction $\mathbb{C}[T^*(G/N_-)]] \longrightarrow \mathbb{C}[T^*B_+]$

The first map is injective because $T^*(G/N_-) \to \mathfrak{g}^*$ is dominant, while the second map is clearly injective.

2.3 Geometric realization of dual Verma modules

Definition 2.5. Let $\chi \in \mathfrak{h}^*$. Consider the one-dimensional representation $\mathbb{C}_\chi$ of $\mathfrak{b}_+$ on which $\mathfrak{h}$ acts by χ and $\mathfrak{n}_+$ acts by zero. The *Verma module* with highest weight $\chi \in \mathfrak{h}^*$ is the $\mathfrak{g}$-module defined by

$$M_\chi := \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\chi.$$

Remark 2.6. The underlying $\mathfrak{n}_-$-module of M_χ is always isomorphic to $U(\mathfrak{n}_-)$, while its the $\mathfrak{h}$-module structure is the tensor product $U(\mathfrak{n}_-) \otimes \mathbb{C}_\chi$.

Remark 2.7. Noting that we have the weight decomposition $M_\chi = \bigoplus_{\mu \in \chi - Q_+} M_\chi[\mu]$, where Q_+ is the positive part of the root lattice of $\mathfrak{g}$, i.e., $Q_+ = \{\sum_i n_i \alpha_i : n_i \geq 0\}$. The *dual* $\mathfrak{g}$-*module* M_χ^* is the $\mathfrak{g}$-module

$$M_\chi^* := \bigoplus_{\mu \in \chi - Q_+} M_\chi[\mu]^\vee$$

with the $\mathfrak{g}$-action defined by

$$(x \cdot \varphi)(m) = \varphi(-\tau(x) \cdot m)$$

for $x \in \mathfrak{g}$, $\varphi \in M_\chi^*$ and $m \in M_\chi$, where τ is the involutive automorpshim on $\mathfrak{g}$ such that $\tau(h_i) = -h_i$, $\tau(e_i) = f_i$, $\tau(f_i) = e_i$.

Exercise 2.8. The duality functor is exact and contravariant, and its square is the identity functor. Moreover, the duality functor preserves the formal character.

We now define a modified $\mathfrak{g}$-module structure on $\mathbb{C}[N_+]$ which depends on $\chi \in \mathfrak{h}^*$.

Definition 2.9. For $\chi \in \mathfrak{h}^*$, let us write $\mathrm{ev}_\chi : U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}$. The modified $\mathfrak{g}$-module structure on $\mathbb{C}[N_+]$ is defined by the composition

$$U(\mathfrak{g}) \longrightarrow D(N_+) \otimes U\mathfrak{h} \xrightarrow{-\otimes \mathrm{ev}_\chi} D(N_+),$$

noting that $\mathbb{C}[N_+]$ is naturally a $D(N_+)$-module. The resulting $\mathfrak{g}$-module is denoted by $\mathbb{C}[N_+]_\chi$.

Theorem 2.10. *There is an isomorphism between* $\mathfrak{g}$*-modules*

$$\mathbb{C}[N_+]_\chi \cong M_\chi^*.$$

Proof. We will prove the dual statement $\mathbb{C}[N_+]_\chi^* \cong M_\chi$. We first note that the Killing form identifies $\mathfrak{n}_+^*$ with $\mathfrak{n}_-$. The exponential map identified N_+ with $\mathfrak{n}_+$, and it is H-equivariant. It follows that the character of the dual $\mathfrak{g}$-module $\mathbb{C}[N_+]_\chi^*$ can be computed as

$$\begin{aligned} \mathrm{char}\, \mathbb{C}[N_+]_\chi^* = \mathrm{char}\, \mathbb{C}[N_+]_\chi &= e^\chi \sum_\lambda \mathbb{C}[N_+]_\lambda e^\lambda \\ &= e^\chi \sum_\lambda S(\mathfrak{n}_+^*)_\lambda e^\lambda \\ &= e^\chi \sum_\lambda S(\mathfrak{n}_-)_\lambda e^\lambda \\ &= \mathrm{char}\, M_\chi. \end{aligned}$$

Note that there is a pairing $\langle -, - \rangle : U(\mathfrak{n}_+) \times \mathbb{C}[N_+] \to \mathbb{C}$ given by

$$\langle \alpha, f \rangle \coloneqq (\alpha \cdot f)(1)$$

for $\alpha \in U(\mathfrak{n}_+)$ and $f \in \mathbb{C}[N_+]$, where we view α as a left-invariant differential operator on N_+ via the identification $U(\mathfrak{n}_+) \cong D(N_+)^{N_+}$. Note the following properties of the pairing:

- The pairing is non-degenerate in th first argument. Indeed, if we have $(\alpha \cdot f)(1) = 0$ for all $f \in \mathbb{C}[N_+]$, then by the left-invariance α it follows that $(\alpha \cdot f)(n) = 0$ for all $f \in \mathbb{C}[N_+]$ and $n \in N_+$, hence $\alpha = 0$.
- The pairing is H-equivariant, so we can split it into pairings between the individual weight spaces. It follows that the induced pairing between the individual weight spaces is perfect.
- The pairing is $U(\mathfrak{n}_+)$-equivariant, i.e., $\langle \alpha\alpha', f \rangle = \langle \alpha', \alpha' f \rangle$ for $\alpha, \alpha' \in U(\mathfrak{n}_+)$ and all $f \in \mathbb{C}[N_+]$.

As a consequence, there is an isomorphism $U(\mathfrak{n}_+) \cong \mathbb{C}[N_+]^*$ as right $U(\mathfrak{n}_+)$-modules, and so there is an isomorphism $\mathbb{C}[N_+]^* \cong U(\mathfrak{n}_-)$ as left $U(\mathfrak{n}_-)$-modules via the anti-isomorphism. We can see that $\mathbb{C}[N_+]_\chi^*$ is a free $U(\mathfrak{n}_-)$-module generated by its highest weight vector. We conclude that $\mathbb{C}[N_+]_\chi^* \cong M_\chi$ as $\mathfrak{g}$-modules. □

Exercise 2.11.

1. The algebra homomorphism $U(\mathfrak{g}) \to D(B_+)^{H_r} = D(N_+) \otimes U(\mathfrak{h})$ restricts to an embedding $\iota : U(\mathfrak{g})^G \to U(\mathfrak{h})$.

   Hint. The left-B_+-invariant part $(D(B_+)^{H_r})^{B_+}$ is precisely the left H-invariant part of $(D(B_+)^{H_r})^{N_+}$. The latter invariants coincide with $U(\mathfrak{n}_+) \otimes U(\mathfrak{h})$.

2. From Theorem 2.10, conclude that an element z of $U(\mathfrak{g})^G$ acts on M_χ^* by the scalar $\mathrm{ev}_\chi(\iota(z))$.

3. Conclude that ι coincides with the embedding used to construct the Harish-Chandra isomorphism.

   Hint. The center $Z(U(\mathfrak{g}))$ acts on M_χ and M_χ^* by the same scalars. To see this, note that the action is by scalars on both $\mathfrak{g}$-modules and recall that they have the same irreducible constituents.

3 Formulas for the action on vector fields

Let us now discuss an explicit formula to compute the Lie algebra homomorphim (3), $\rho : \mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^{H_r}$. For this, let $x \in \mathfrak{g}$ and $y \in B_+$. Since $G^\circ := B_+N_- \subset G$ is open and dense, we can write $\exp(-tx)y \in G^\circ$ for sufficiently small t. In the open dense subset G° every element z can be uniquely expressed as the product $z = z_+z_-$ with $z_+ \in B_+$ and $z_- \in N_-$. Then, for $x \in \mathfrak{g}$, its image $\rho(x) \in \mathrm{Vect}(B_+)$ satisfies $(\rho(x)\cdot f)(y) = \frac{d}{d\epsilon}|_{\epsilon=0} f((\exp(-\epsilon x)y)_+)$.

3.1 The case of $\mathfrak{sl}_2$

Let us first work out the case $G = \mathrm{SL}_2$ explicitly. Let y be a coordinate on $\mathfrak{n}_+$.

Image of e. By the choice of coordinate y, we have

$$e \longmapsto \frac{\partial}{\partial y}.$$

Image of h. We can compute the image of h by considering the left H-action on $B_+ \cong N_+ \times H$ given by $(t', (n,t)) \longmapsto (t'nt'^{-1}, t't)$. Then we can see that

$$h \longmapsto -2y\frac{\partial}{\partial y} + h \in \mathrm{Vect}(\mathfrak{n}_+) \oplus (\mathbb{C}[\mathfrak{n}_+] \otimes \mathfrak{h}).$$

Image of f. Since f has weight -2, its image $\rho(f)$ lies in $(\mathrm{Vect}(\mathfrak{n}_+) \oplus (\mathbb{C}[\mathfrak{n}_+] \otimes \mathfrak{h}))_{-2}$ so it can be written in the form $\rho(f) = \alpha y^2 \frac{\partial}{\partial y} + \beta y h$ for some α, β. The commutator relation $[e, f] = h$ implies

$$[\frac{\partial}{\partial y}, \alpha y^2 \frac{\partial}{\partial y} + \beta y h] = 2t\alpha \frac{\partial}{\partial y} + \beta h,$$

so we find that $\alpha = -1, \beta = 1$. We conclude that

$$f \longmapsto -y^2\frac{\partial}{\partial y} + yh.$$

3.2 The general case

Let $\{y_\alpha\}_{\alpha\in\Delta_+}$ be the coordinates on N_+ so that $\mathbb{C}[N_+] = \mathbb{C}[y_\alpha]$ holds and the weight of y_α is α, which is possible by considering an H-equivariant isomorphism between $\mathfrak{n}_+$ and N_+, say the exponential map $\exp : \mathfrak{n}_+ \to N_+$. Note that with this choice of coordinates, the variables y_α correspond to coordinate functions g_α on the root subspace $\mathfrak{n}_+$.

Proposition 3.1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let e_i, f_i, g_i be the generators of $\mathfrak{g}$. Then we have*

$$\rho(e_i) = \frac{\partial}{\partial y_{\alpha_i}} + \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} P^i_\beta(y_\alpha)\frac{\partial}{\partial y_\beta}$$

$$\rho(h_i) = -\sum_{\beta\in\Delta_+} \beta(h_i) y_\beta \frac{\partial}{\partial y_\beta} + h_i$$

$$\rho(f_i) = \sum_{\beta\in\Delta_+} Q^i_\beta(y_\alpha)\frac{\partial}{\partial y_\beta} + h_i y_{\alpha_i}$$

where $P^i_\beta(y_\alpha)$ and $Q^i_\beta(y_\alpha)$ are polynomials in $y_\alpha, \alpha \in \Delta_+$, of degree $\alpha_i - \beta$ and $-\alpha_i - \beta$ respectively.

Example 3.2. Let $e_{\alpha+\beta} = E_{13}$, $e_\alpha = E_{12}$, $e_\beta = E_{23}$. For $\mathfrak{n}_+ \subset \mathfrak{sl}_3$, the Baker–Campbell–Hausdorff formula simplifies to $x * y = x + y - \frac{1}{2}[x,y]$. It follows that

$$\rho(e_{\alpha+\beta}) = \frac{\partial}{\partial y_{\alpha+\beta}}$$

$$\rho(e_\alpha) = \frac{\partial}{\partial y_\alpha} - \frac{1}{2} y_\beta \frac{\partial}{\partial y_{\alpha+\beta}}$$

$$\rho(e_\beta) = \frac{\partial}{\partial y_\beta} + \frac{1}{2} y_\alpha \frac{\partial}{\partial y_{\alpha+\beta}}.$$

3.3 The homomorphism $\rho_0 : \mathfrak{g} \to \mathbf{Vect}(N_+)$

By passing (3) to the first direct summand, we obtain a Lie algebra homomorphism $\rho_0 : \mathfrak{g} \to \mathrm{Vect}(N_+)$. Following the approach in [2], we work with the flag variety G/B_- and consider the open dense orbit $U \coloneqq N_+[1] \subset G/B_-$. Choose a faithful representation V of $\mathfrak{g}$, say the adjoint representation.

Proposition 3.3. *For $a \in \mathfrak{g}$ and $z \in N_+$, we have*

$$\rho_0(a) \cdot z = -z(z^{-1}az)_+ \tag{4}$$

where b_+ denotes the projection of an element $b \in \mathfrak{g}$ onto $\mathfrak{n}_+$ along $\mathfrak{b}_-$.

Proof. For sufficiently small ϵ, we can write $\exp(1 - \epsilon a)z = Z_+(\epsilon)Z_-(\epsilon)$ with $Z_+(\epsilon) \in N_+$ and $Z_-(\epsilon) \in B_-$ By the definition of the action we have

$$(\rho_0(a) \cdot f)(y) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f(Z_+(\epsilon)).$$

For the rest of the proof, we work with the dual number $\epsilon \in \mathbb{C}[\epsilon]/\epsilon^2$. Choosing a faithful representation V, we have an embedding $G(\mathbb{C}[\epsilon]/\epsilon^2) \hookrightarrow \mathrm{GL}(V)(\mathbb{C}[\epsilon]/\epsilon^2)$. Noting that $a \in \mathfrak{g} \subset \mathrm{End}\,(V)$ and $z \in N_+ \subset \mathrm{GL}(V)$, we can write

$$(1 - \epsilon a)z = Z_+(\epsilon)Z_-(\epsilon) \in G(\mathbb{C}[\epsilon]/\epsilon^2). \tag{5}$$

Let $Z_+(\epsilon) = z + \epsilon Z_+^{(1)}$ and $Z_-(\epsilon) = 1 + \epsilon Z_-^{(1)}$ for some $Z_+^{(1)} \in \mathfrak{n}_+$ and $Z_-^{(1)} \in \mathfrak{b}_-$. Note that $Z_+^{(1)}$ coincides with $\rho_0(a) \cdot z$. By comparing the coefficient of ϵ in (5), we obtain $-az = Z_+^{(1)} + zZ_-^{(1)}$. It follows that

$$-z^{-1}az = z^{-1}Z_+^{(1)} + Z_-^{(1)}.$$

We can ensure that $\mathfrak{h}_+ \subset \{\text{strictly upper triangular matrices}\}$, $\mathfrak{b}_- \subset \{\text{lower triangular matrices}\}$ and $N_+ \subset \{\text{upper unitriangular matrices}\}$. For such a presentation, choose a weight basis $v_1, \cdots, v_n$ of V and order it so that $\mathrm{wt}(v_i) > \mathrm{wt}(v_j)$ implies $i < j$. Then the term $z^{-1}Z_+^{(1)}$ in (5) is a strictly upper triangular matrix and is also an element of $\mathfrak{g}$ because the remaining two terms are. It follows that $z^{-1}Z_+^{(1)} \in \mathfrak{n}_+$. Since $-z^{-1}az$ lies in $\mathfrak{g}$ and $Z_-^{(1)}$ lies in $\mathfrak{b}_-$, we obtain $-(z^{-1}az)_+ = z^{-1}Z_+^{(1)}$. The statement now follows by rewriting $Z_+^{(1)} = -z(z^{-1}az)_+$. □

4 The Lie algebra homomorphism $\mathcal{L}\mathfrak{g} \to \mathbf{Vect}(\mathcal{L}U)$

In the finite-dimensional case, we obtained a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Vect}(U)$ by considering the left G-action on the flag variety G/B_- and restricting the induced vector field on the open dense orbit $U = N_+[1] \subset G/B_-$. Our next goal is to obtain an affine analog of this.

4.1 The ind-scheme $\mathcal{L}U$

For $N \in \mathbb{Z}$, we consider the scheme $t^N\mathbb{C}[[t]]$ whose R-points are given by

$$t^N\mathbb{C}[[t]](R) = \{\sum_{i\geq N} a_i t^i : a_i \in R\} = \prod_{i\geq N} \mathbb{A}^1(R).$$

Equivalently, it is the affine scheme

$$t^N\mathbb{C}[[t]] = \text{Spec } \mathbb{C}[x_i]_{i\geq N}.$$

For each N, there is a closed embedding $t^N\mathbb{C}[[t]] \to t^{N-1}\mathbb{C}[[t]]$ coming from the surjective homomorphism $\mathbb{C}[x_i : i \geq N-1] \to \mathbb{C}[x_i : i \geq N]$ sending $x_i \mapsto x_i$ for all $i \geq N$ and $x_{N-1} \mapsto 0$.

Definition 4.1. We define the *loop space* $\mathcal{L}U$ of U as the following ind-affine ind-scheme

$$\mathcal{L}U := \varinjlim_{N<0} U \times t^N\mathbb{C}[[t]].$$

For the basics of ind-affine ind-schemes, we refer the reader to [1, Appendix].

Since we can identify the orbit $U = N_+[1]$ with N_+ and further with $\mathfrak{n}_+$, say via the exponential map, we can write $U = \text{Spec } \mathbb{C}[y_\alpha]_{\alpha\in\Delta_+}$. For each $N < 0$, we can identify $U \times t^N\mathbb{C}[[t]]$ as the affine scheme $\text{Spec } \mathbb{C}[y_{\alpha,n}]_{\alpha\in\Delta_+, n\geq N}$, and we can consider the canonical restriction map

$$\begin{aligned} \mathbb{C}[y_{\alpha,n}]_{n\geq N} &\longrightarrow \mathbb{C}[y_{\alpha,n}]_{n\geq N'} \\ y_{\alpha,n} &\longmapsto \begin{cases} y_{\alpha,n} & n \geq N' \\ 0 & n < N' \end{cases} \end{aligned}$$

We define the algebra of functions on $\mathcal{L}U$ to be the invserse limit $\varprojlim_{N<0} \mathbb{C}[y_{\alpha,n}]_{n\geq N}$. More concretely, any element of $\mathbb{C}[\mathcal{L}U]$ can be uniquely represented as a possibly infinite series

$$P_0 + \sum_{N<0} \sum_{\alpha\in\Delta_+} P_{\alpha,N} y_{\alpha,N}$$

where $P_0 \in \mathbb{C}[y_{\alpha,n}]_{n\geq 0}$ and $P_{\alpha,N} \in \mathbb{C}[y_{\alpha,n}]_{n\geq N}$ for each N.

Remark 4.2. Note that $\mathbb{C}[\mathcal{L}U]$ is a topological algebra with the natural topology of an inverse limit. The basis of open neighborhoods of 0 in $\mathbb{C}[\mathcal{L}U]$ is given by the ideals $I_N = (y_{\alpha,n})_{n\leq N}$ for $N \leq 0$.

4.2 Vector fields on $\mathcal{L}U$

Definition 4.3. The *vector fields* on $\mathcal{L}U$ are the continuous derivations of the topological algebra $\mathbb{C}[\mathcal{L}U]$.

More explicitly, using the topology on $\mathbb{C}[\mathcal{L}U]$, we can write

$$\text{Vect}(\mathcal{L}U) = \left\{ \sum_\alpha \sum_{n\in\mathbb{Z}} P_{\alpha,n} \frac{\partial}{\partial y_{\alpha,n}} : P_{\alpha,n} \in \mathbb{C}[\mathcal{L}U] \text{ such that } \lim_{n\to-\infty} P_{n,\alpha} = 0 \right\}.$$

First, every vector field on $\mathcal{L}U$ can be expressed in this form since it is uniquely determined by the images of the topological basis $y_{\alpha,n}$ and it has to send sequences converging to 0 to sequences converging to 0. To see that such an expression defines a continuous derivation, note that every infinite sum with $P_{\alpha,n} \in \mathbb{C}[\mathcal{L}U]$ gives a derivation $\mathbb{C}[y_{\alpha,n}] \to \mathbb{C}[\mathcal{L}U]$.

Exercise 4.4. Show that this linear map extends to $\mathbb{C}[\mathcal{L}U]$ by continuity if and only if $\lim_{n\to-\infty} P_{\alpha,n} = 0$. In this case, the extension is automatically a derivation.

Remark 4.5. $\mathrm{Vect}(\mathcal{L}U)$ is a topological space where the basis of open neighborhoods of 0 is given by

$$I_{N,M} = \left\{ \sum_{\alpha} \sum_{n\in\mathbb{Z}} P_{\alpha,n} \frac{\partial}{\partial y_{\alpha,n}} : P_{\alpha,n} \in I_N \text{ for all } n < M \right\}$$

for $N \le 0$ and $M \ge 0$. Equivalently, $I_{N,M}$ is the subspace generated by $y_{\alpha,n}, n \le N$ and $\frac{\partial}{\partial y_{\alpha,m}}, m \ge M$. The commutator of derivations turns $\mathrm{Vect}(\mathcal{L}U)$ into a topological Lie algebra.

4.3 The homomorphism $\mathcal{L}\mathfrak{g} \to \mathrm{Vect}(\mathcal{L}U)$ when $\mathfrak{g} = \mathfrak{sl}_2$

We want to construct a homomorphism of Lie algebras

$$\rho : \mathcal{L}\mathfrak{g} \longrightarrow \mathrm{Vect}(\mathcal{L}U).$$

We describe the homomorphism explicitly in the case of $\mathfrak{g} = \mathfrak{sl}_2$. Note that in this case the flag variety G/B is identified with the projective line $\mathbb{P}^1$. Let us consider the subspace $\mathcal{L}\mathbb{A}^1 = \left\{ \begin{pmatrix} x(t) \\ -1 \end{pmatrix} \right\} \subset \mathcal{L}(G/B) = \mathcal{L}\mathbb{P}^1$ where $x(t) = \sum_{m\in\mathbb{Z}} x_m t^m$. Note that the target is not an ind-scheme. For the computation, let us choose the topological basis for $\mathcal{L}\mathfrak{sl}_2$: $e_n = e \otimes t^n = \begin{pmatrix} 0 & t^n \\ 0 & 0 \end{pmatrix}$, $h_n = h \otimes t^n = \begin{pmatrix} t^n & 0 \\ 0 & -t^n \end{pmatrix}$ and $f_n = f \otimes t^n = \begin{pmatrix} 0 & 0 \\ t^n & 0 \end{pmatrix}$.

Lemma 4.6. *The map* $\rho : \mathcal{L}\mathfrak{sl}_2 \to \mathrm{Vect}(\mathcal{L}U)$ *sends*

$$\begin{aligned} e_n &\longmapsto \frac{\partial}{\partial x_n} \\ h_n &\longmapsto -2 \sum_{i+j=n} x_{-i} \frac{\partial}{\partial x_j} \\ f_n &\longmapsto - \sum_{i+j+k=n} x_{-i} x_{-j} \frac{\partial}{\partial x_k} \end{aligned}$$

Proof. The computation is similar for each case of e, h and f, but we discuss all cases for completeness. We may work with the dual number $\epsilon \in \mathbb{C}[\epsilon]/\epsilon^2$.

1. We first consider $e_n = \begin{pmatrix} 0 & t^n \\ 0 & 0 \end{pmatrix} \in \mathcal{L}\mathfrak{sl}_2$.

$$\begin{aligned} (\rho(e_n)\varphi) \begin{pmatrix} x(t) \\ -1 \end{pmatrix} &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi \left(\exp(-\epsilon e_n) \begin{pmatrix} x(t) \\ -1 \end{pmatrix} \right) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi \left(\begin{pmatrix} 1 & -\epsilon t^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ -1 \end{pmatrix} \right) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi \begin{pmatrix} \sum_m x_m t^m + \epsilon t^n \\ -1 \end{pmatrix} \end{aligned}$$

so it implies that $\rho(e_n) = \frac{\partial}{\partial x_n}$.

2. Next, we consider $f_n = \begin{pmatrix} 0 & 0 \\ t^n & 0 \end{pmatrix}$.

$$\begin{aligned}(\rho(f_n)\varphi)\begin{pmatrix} x(t) \\ -1 \end{pmatrix} &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\left(\exp(-\epsilon f_n)\begin{pmatrix} x(t) \\ -1 \end{pmatrix}\right) \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\left(\begin{pmatrix} 1 & 0 \\ -\epsilon t^n & 1 \end{pmatrix}\begin{pmatrix} x(t) \\ -1 \end{pmatrix}\right) \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\begin{pmatrix} x(t) \\ -1 - \epsilon t^n x(t) \end{pmatrix}.\end{aligned}$$

Notice that the second entry is invertible so in the projective space we can write the result as follows

$$\begin{aligned}(\rho(f_n)\varphi)\begin{pmatrix} x(t) \\ -1 \end{pmatrix} &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\begin{pmatrix} (1 - \epsilon t^n x(t))x(t) \\ -1 \end{pmatrix} \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\begin{pmatrix} x(t) - \epsilon \sum_{m\in\mathbb{Z}} \sum_{i+j=m} x_i x_j t^{m+n} \\ -1 \end{pmatrix}\end{aligned}$$

From this computation, it follows that

$$\rho(f_n) = \sum_{m\in\mathbb{Z}} \sum_{i+j=m} x_i x_j \frac{\partial}{\partial x_{m+n}} = \sum_{i+j+k=n} x_{-i} x_{-j} \frac{\partial}{\partial x_k}.$$

3. Finally, we consider $h_n = \begin{pmatrix} t^n & 0 \\ 0 & -t^n \end{pmatrix}$.

$$\begin{aligned}(\rho(h_n)\varphi)\begin{pmatrix} x(t) \\ -1 \end{pmatrix} &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\left(\exp(-\epsilon h_n)\begin{pmatrix} x(t) \\ -1 \end{pmatrix}\right) \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\begin{pmatrix} e^{-2\epsilon t^n} x(t) \\ -1 \end{pmatrix} \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi\begin{pmatrix} x(t) - 2\epsilon \sum_m x_{m-n} t^m \\ -1 \end{pmatrix}\end{aligned}$$

From this computation, we obtain

$$\rho(h_n) = -2 \sum_{m\in\mathbb{Z}} x_{m-n} \frac{\partial}{\partial x_m} = -2 \sum_{i+j=n} x_{-i} \frac{\partial}{\partial x_j}.$$

□

Lemma 4.7. *The map $\rho : \mathcal{L}\mathfrak{sl}_2 \to \mathrm{Vect}(\mathcal{L}U)$ is a Lie algebra homomorphism.*

Proof. Note that for $x, y \in \mathfrak{sl}_2$ we have $[x_n, y_m] = [xt^n, yt^m] = [x, y]t^{n+m}$. For example, $[e_n, h_m] = -2e_{n+m}$. It is easy to check the commutator relation directly. For example, we can see that

$$[\rho(e_n), \rho(h_m)] = [\frac{\partial}{\partial x_n}, -2 \sum_{i+j=n} x_{-i} \frac{\partial}{\partial x_j}] = -2 \frac{\partial}{\partial x_{n+m}} = -2\rho(e_{n+m}).$$

It implies that $\rho([e_n, h_m]) = \rho(-2e_{n+m})$ coincides with $[\rho(e_n), \rho(h_m)]$. □

Remark 4.8. We can assemble these formulas in a simple way by introducing the generating function $e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1}$ and similarly for h and f. For convenience, we write $a_n = \frac{\partial}{\partial x_n}$ and $a_n^* = x_{-n}$, and consider the generating functions

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} = \sum_{n\in\mathbb{Z}} \frac{\partial}{\partial x_n} z^{-n-1}$$

$$a^*(z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n} = \sum_{n\in\mathbb{Z}} x_{-n} z^{-n}.$$

Then we can simplify the formulas in Lemma 4.6 as follows

$$\begin{aligned} e(z) &\longmapsto a(z) \\ h(z) &\longmapsto -2a^*(z)a(z) \\ f(z) &\longmapsto -a^*(z)^2 a(z) \end{aligned}$$

4.4 The general case

Now let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. In this general situation, we can construct a homomorphism of Lie algebras

$$\widehat{\rho}\colon \mathcal{L}\mathfrak{g} \longrightarrow \mathrm{Vect}(\mathcal{L}U).$$

For this, we choose a faithful representation V of $\mathfrak{g}$, again say the adjoint representation, and use the following formula analogous to (4)

$$\widehat{\rho}(a \otimes t^m) \cdot x(t) = -x(t)\left(x(t)^{-1}(a\otimes t^m)x(t)\right)_+$$

for $a \in \mathfrak{g}$ and $x(t) \in N_+((t))$, where z_+ denotes the projection of an element $z \in \mathfrak{g}((t))$ onto $\mathfrak{n}_+((t))$ along $\mathfrak{b}_-((t))$.

There are several ways to check that $\widehat{\rho}$ is a Lie algebra homomorphism. For the approach that uses formal loops, we refer to [3].

Moreover, we can establish an affine-analog of Proposition 3.1 by replacing each element $a \in \mathfrak{b}$ by its generating function $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-m-1}$.

Proposition 4.9. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let e_i, f_i, g_i be the generators of $\mathfrak{g}$. Then we have*

$$\widehat{\rho}(e_i(z)) = a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} P_\beta^i(a_\alpha^*(z))a_\beta(z)$$

$$\widehat{\rho}(h_i(z)) = -\sum_{\beta\in\Delta_+} \beta(h_i)a_\beta^*(z)a_\beta(z)$$

$$\widehat{\rho}(f_i(z)) = \sum_{\beta\in\Delta_+} Q_\beta^i(a_\alpha^*(z))a_\beta(z)$$

where $P_\beta^i(y_\alpha)$ and $Q_\beta^i(y_\alpha)$ are polynomials in $y_\alpha, \alpha \in \Delta_+$, of degree $\alpha_i - \beta$ and $-\alpha_i - \beta$ respectively.

5 The completed Weyl algebra

5.1 Definition

As before, let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let Δ_+ be the set of positive roots of $\mathfrak{g}$. We first consider the Weyl algebra $\mathcal{A}$ generated by $a_{\alpha,n} = \frac{\partial}{\partial y_{\alpha,n}}$ and $a_{\alpha,n}^* = y_{\alpha,-n}$ for $\alpha \in \Delta_+$ and $n \in \mathbb{Z}$ with the commutator relations $[a_{\alpha,n}, a_{\beta,m}^*] = \delta_{\alpha,\beta}\delta_{n,-m}$ (and all other commutators vanish).

Definition 5.1. The *completed Weyl algebra* is the following completion of $\mathcal{A}$:

$$\widehat{\mathcal{A}} = \varprojlim_{N\to\infty} \mathcal{A}/\mathcal{A}(a_{\alpha,n}, a^*_{\alpha,m})_{n,m\geq N}.$$

Exercise 5.2. Show that the completed Weyl algebra $\widehat{\mathcal{A}}$ has a well-defined product.

More explicitly, $\widehat{\mathcal{A}}$ consists of power series of the form

$$\sum_{n\geq N_0} (P_{\alpha,n} a_{\alpha,n} + Q_{\alpha,n} a^*_{\alpha,n})$$

where $P_{\alpha,n}, Q_{\alpha,n} \in \mathcal{A}$ and $N_0 \in \mathbb{Z}$. Note that such an expression is not unique.

5.2 A filtration on $\widehat{\mathcal{A}}$ and a short exact sequence

Consider a filtration of $\mathcal{A}$ whose n-th piece $\mathcal{A}_{\leq n}$ consists of sums of monomials each containing at most n variables of the form $a_{\alpha,n}$. We can similarly define a filtration on $\widehat{\mathcal{A}}$, which is not exhaustive.

Explicitly, the zeroth piece is given by

$$\widehat{\mathcal{A}}_0 = \{ \sum_{n\geq N_0} Q_{\alpha,n} a^*_{\alpha,n} : Q_{\alpha,n} \in \mathbb{C}[a^*_{\alpha,n}]_{n\in\mathbb{Z}} \}.$$

Note that $\widehat{\mathcal{A}}_0$ is a commutative topological algebra identified with $\mathbb{C}[\mathcal{L}U]$.

Similarly, we can describe the first piece of the filtrarion as follows

$$\widehat{\mathcal{A}}_{\leq 1} = \{ \sum_{n\geq N_0} (P_{\alpha,n} a_{\alpha,n} + Q_{\alpha,n} a^*_{\alpha,n}) : P_{\alpha,n} \in \mathbb{C}[a^*_{\alpha,n}]_{n\in\mathbb{Z}}, Q_{\alpha,n} \in \mathcal{A}_{\leq 1} \}.$$

Exercise 5.3.

1. Prove that $[\mathcal{A}_{\leq i}, \mathcal{A}_{\leq j}] \subset \mathcal{A}_{i+j-1}$ and use the continuity argument to deduce that $[\widehat{\mathcal{A}}_{\leq i}, \widehat{\mathcal{A}}_{\leq j}] \subset \widehat{\mathcal{A}}_{i+j-1}$.
2. Conclude that $\widehat{\mathcal{A}}_{\leq 1}$ admits a Lie algebra structure.

Proposition 5.4. *There is a short exact sequence of Lie algebras (where $\mathbb{C}[\mathcal{L}U]$ is an ideal in $\widehat{\mathcal{A}}_{\leq 1}$)*

$$0 \longrightarrow \mathbb{C}[\mathcal{L}U] \longrightarrow \widehat{\mathcal{A}}_{\leq 1} \longrightarrow \mathrm{Vect}[\mathcal{L}U] \longrightarrow 0.$$

Proof. In order to define the map $\varphi : \widehat{\mathcal{A}}_{\leq 1} \longrightarrow \mathrm{Vect}[\mathcal{L}U]$, we need to associate to each element of $\widehat{\mathcal{A}}_{\leq 1}$ some continuous derivation on $\mathcal{L}U$. For this, letting $\alpha \in \widehat{\mathcal{A}}_{\leq 1}$, we consider the endomorphism on $\mathbb{C}[\mathcal{L}U]$ by

$$\alpha \cdot f = [\alpha, f]$$

for $f \in \mathbb{C}[\mathcal{L}U]$ via the identification $\mathbb{C}[\mathcal{L}U] = \widehat{\mathcal{A}}_0$.

Exercise 5.5. This is a continuous derivation of $\mathbb{C}[\mathcal{L}U]$.

Since $\widehat{\mathcal{A}}_0$ is commutative with respect to the commutator, it is clear that $\varphi : \widehat{\mathcal{A}}_{\leq 1} \longrightarrow \mathrm{Vect}[\mathcal{L}U]$ factors through the quotient,

$$\overline{\varphi} : \widehat{\mathcal{A}}_{\leq 1}/\widehat{\mathcal{A}}_0 \longrightarrow \mathrm{Vect}[\mathcal{L}U].$$

We want to show that $\overline{\varphi}$ is an isomorphism of Lie algebras.

Idea for the proof. Before going into the detail, let us explain the idea for the proof. For simplicity, suppose that there is only one variable y, i.e., Δ_+ is a singleton. We can construct the inverse map of $\overline{\varphi}$ by sending a vector field $\sum_n P_n \frac{\partial}{\partial y_n}$ to the expression $\sum_{n\geq 0} P_n a_n + + \sum_{n<0} a_n P_n$.

Exercise 5.6. The expression is indeed an element of $\widehat{\mathcal{A}}_{\leq 1}$ and this construction gives a two-sided inverse of $\overline{\varphi}$.

Let us give the proof in general. We note that an arbitrary element of $\widehat{\mathcal{A}}_{\leq 1}$ can be written in the form

$$\sum_{n\geq N_0} (P_{\alpha,n} a_{\alpha,n} + Q_{\alpha,n} a^*_{\alpha,n})$$

where $P_{\alpha,n} \in \mathcal{A}_0 = \mathbb{C}[a^*_{\alpha,n}]_{n\in\mathbb{Z}}, Q_{\alpha,n} \in \mathcal{A}_{\leq 1}$. Moreover, we can express each $Q_{\alpha,n}$ as $Q^0_{\alpha,n} + \sum_{m\in K_n} R^{\beta,m}_{\alpha,n} a_{\beta,m}$ where $Q^0_{\alpha,n}, R^{\beta,m}_{\alpha,n} \in \mathcal{A}_0$ and K_n is a finite set. Then φ maps the n-th term $P_{\alpha,n} a_{\alpha,n} + Q_{\alpha,n} a^*_{\alpha,n}$ to

$$\begin{aligned}
\varphi(P_{\alpha,n} a_{\alpha,n} + Q_{\alpha,n} a^*_{\alpha,n}) &= \varphi\left(P_{\alpha,n} a_{\alpha,n} + (Q^0_{\alpha,n} + \sum_{m\in K_n} R^{\beta,m}_{\alpha,n} a_{\beta,m}) a^*_{\alpha,n}\right) \\
&= \varphi\left(P_{\alpha,n} a_{\alpha,n} + Q^0_{\alpha,n} a^*_{\alpha,n} + \sum_{m\in K_n} R^{\beta,m}_{\alpha,n} a^*_{\alpha,n} a_{\beta,m} + \delta_{\alpha,\beta}\delta_{n,-m} R^{\beta,m}_{\alpha,n}\right) \\
&= \varphi\left(P_{\alpha,n} a_{\alpha,n} + \sum_{m\in K_n} a^*_{\alpha,n} R^{\beta,m}_{\alpha,n} a_{\beta,m}\right) \\
&= P_{\alpha,n} \frac{\partial}{\partial y_{\alpha,n}} + y_{\alpha,-n} \sum_{m\in K_n} R^{\beta,m}_{\alpha,n} \frac{\partial}{\partial y_{\beta,m}},
\end{aligned}$$

where we used the fact that φ annihilates $\widehat{\mathcal{A}}_0$. The image clearly lies in the ideal $I_{n,n} \subset \mathrm{Vect}(\mathcal{L}U)$. It follows that φ matches the bases of open neighborhoods of 0.

From the explicit description, it is clear that the kernel of φ consists of the power series with all coefficients $P_{\alpha,n}$ and $R^{\beta,m}_{\alpha,n}$ zero, i.e., $\ker\varphi = \widehat{\mathcal{A}}_0$. By setting $Q^0_{\alpha,n}$ to be zero, we can see that φ is surjective. □

5.3 Non-splitting

Theorem 5.7. *The short exact sequence in Proposition 5.4 does not split.*

Instead of giving a proof, we explain an indication of non-splitting by contrasting the situation to the finite-dimensional case when the analogous short exact sequence splits.

Let X be a smooth affine scheme. Then there is a short exact sequence

$$0 \longrightarrow \mathbb{C}[X] \longrightarrow D(X)_{\leq 1} \longrightarrow \mathrm{Vect}(X) \longrightarrow 0.$$

For the splitting, we construct a map $\mathrm{Vect}(X) \to D(X)_{\leq 1}$ by sending a vector field ξ to the unique first-order differential operator D_ξ which annihilates the constant function 1. Here, the construction crucially relies on the fact that the algebra $D(X)$ of differential operators acts on $\mathbb{C}[X]$.

In contrast, the completed Weyl algebra $\widehat{\mathcal{A}}$ does not act on $\mathbb{C}[\mathcal{L}U]$. For example, if we consider the element

$$:\sum_{n\in\mathbb{Z}} a^*_{\alpha,-n} a_{\alpha,n} := \sum_{n<0} a_{\alpha,n} a^*_{\alpha,-n} + \sum_{n\geq 0} a^*_{\alpha,-n} a_{\alpha,n} \in \widehat{\mathcal{A}}_{\leq 1},$$

its naive action on the constant function $1 \in \mathbb{C}[\mathcal{L}U]$ diverges. Note that the image of $:\sum_{n\in\mathbb{Z}} a^*_{\alpha,-n} a_{\alpha,n}:$ under φ is exactly the Euler vector field $\sum_{n\in\mathbb{Z}} y_{\alpha,n} \frac{\partial}{\partial y_{\alpha,n}} \in \mathrm{Vect}(\mathcal{L}U)$.

Instead, the completed Weyl algebra $\widehat{\mathcal{A}}$ acts on its Fock representation $M_{\mathfrak{g}} = \mathbb{C}[a_{\alpha,n}, a^*_{\alpha,m}]_{n<0,m\leq 0}$, which is the quotient of $\widehat{\mathcal{A}}$ by the left ideal generated by $a_{\alpha,n}, n \geq 0$ and $a^*_{\alpha,m}, m > 0$.

5.4 What comes next

The non-splitting of the short exact sequence means that we cannot canonically lift $\mathcal{L}\mathfrak{g} \longrightarrow \mathrm{Vect}(\mathcal{L}U)$ to $\mathcal{L}\mathfrak{g} \longrightarrow \widehat{\mathcal{A}}_{\leq 1}$. In fact, there is no such lift. However, it turns out that we can lift it to $\widehat{\mathfrak{g}}_{\kappa_c} \longrightarrow \widehat{\mathcal{A}}_{\leq 1}$ where $\widehat{\mathfrak{g}}_{\kappa_c}$ is the central extension at the critical level. Since the construction is technical and requires effort, we refer the reader to [2, Sections 5.5, 5.6] for the proof.

Moreover, by considering deformations, we can construct a homomorphism

$$\widehat{\mathfrak{g}}_{\kappa_c} \longrightarrow \widetilde{D}(\mathcal{L}N_+) \widehat{\otimes} \mathbb{C}[\mathfrak{h}((t))]$$

where $\widetilde{D}(\mathcal{L}N_+)$ denotes the completed Weyl algebra $\widehat{\mathcal{A}}$. We can show that it gives rise to an embedding $Z(\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})) \hookrightarrow \mathbb{C}[\mathfrak{h}((t))]$, which is exactly what we need.

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