Intertwining operators for $\mathfrak{sl}_2$.

Daniil Klyuev

1 Plan and first steps

Let $\mathfrak{g}$ be a simple Lie algebra of rank r, $\mathfrak{b} = \mathfrak{b}_+$ be a Borel subalgebra, $\mathfrak{h}$ be a Cartan subalgebra,

We want to prove that the center $\mathfrak{z}(\hat{\mathfrak{g}})$ of vertex algebra $V_{\kappa_c}(\mathfrak{g})$ at the critical level is isomorphic to $\operatorname{Fun}\operatorname{Op}_{G^\vee}(D)$.

In order to do this we will use the homomorphism of vertex algebras

$$\omega_{\kappa_c} \colon V_{\kappa_c}(\mathfrak{g}) \to W_{0,\kappa_c} = M_{\mathfrak{g}} \otimes V_0(\mathfrak{h}),$$

constructed in Section 4 of [W] where the notation is as follows: $M_{\mathfrak{g}}$ is the Weyl vertex algebra whose underlying vector space of states is the Fock representation of the Weyl algebra $\mathcal{A}^{\mathfrak{g}}$ and $V_0(\mathfrak{h}) = \pi_0$ is the commutative vertex algebra associated to $L\mathfrak{h}$.

The plan is as follows.

1. Show that ω_{κ_c} is injective.
2. Show that $\omega_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}})) \subset \pi_0$. Hence we need to describe the image of $\mathfrak{z}(\hat{\mathfrak{g}})$ in π_0.
3. Construct *screening operators* $\overline{S}_i, i = 1, \ldots, r$, where r is the rank of $\mathfrak{g}$, $\hat{\mathfrak{g}}_{\kappa_c}$-linear maps from W_{0,κ_c} to other modules.
4. Show that $\omega_{\kappa_c}(V_{\kappa_c}(\mathfrak{g})) \subset \ker \overline{S_i}$ for all i. Hence the image of $\mathfrak{z}(\hat{\mathfrak{g}})$ is contained in $\bigcap\limits_{i=1}^{l} \ker \overline{V_i}[1]$, where $\overline{V_i}[1]$ is the restriction of $\overline{S_i}$ to π_0.
5. Using the isomorphism between the Wakimoto module $W^+_{\kappa_c}$ and the Verma module $\mathbf{M}_{0,\kappa_c}$ constructed in Kenta's lecture we will compute the

graded character of $\mathfrak{z}(\hat{\mathfrak{g}})$. We will show that it is equal to the character of $\bigcap_{i=1}^{l} \ker \overline{V_i}[1]$. It follows that

$$\mathfrak{z}(\hat{\mathfrak{g}}) = \bigcap_{i=1}^{l} \ker \overline{V_i}[1].$$

6. By using *Miura opers* constructed in Zeyu's talk we will show that there is a natural isomorphism $\operatorname{Fun}\operatorname{Op}_{G^\vee}(D) \cong \bigcap_{i=1}^{l} \ker \overline{V_i}[1]$. This will yield an isomorphism between $\mathfrak{z}(\hat{\mathfrak{g}})$ and $\operatorname{Fun}\operatorname{Op}_{G^\vee}(D)$. Moreover, all our constructions will be $\operatorname{Aut}\mathcal{O}$-equivariant.

In my talk I will explain the first and the second steps of this plan, this is relatively quick. I will also explain steps 3-4 for in the case $\mathfrak{g} = \mathfrak{sl}_2$.

1.1 Steps 1 and 2

We want to prove that ω_{κ_c} is injective. First, we discuss a finite-dimensional analogue of this statement.

In Daishi's notes [K] there is a homomorphism of Lie algebras

$$\rho \colon \mathfrak{g} \to \operatorname{vect}(B_+)^H = \operatorname{vect}(N_+) \oplus (\mathbb{C}[N_+] \otimes \mathfrak{h}).$$

The right-hand side is contained in $\mathbb{C}[T^*N_+ \times \mathfrak{h}^*]$, so we can extend ρ to

$$\phi^* \colon \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*N_+ \times \mathfrak{h}^*].$$

Here $\phi \colon T^*N_+ \times \mathfrak{h}^* \to \mathfrak{g}^*$ is a morphism of varieties.

We can also extend ρ naturally to

$$\rho \colon U(\mathfrak{g}) \to D(N_+) \otimes U(\mathfrak{h}).$$

The map ω_{κ_c} is an affine analog of ρ. Namely, $U(\mathfrak{g})$ corresponds to $V_{\kappa_c}(\mathfrak{g})$, $U(\mathfrak{h})$ corresponds to $V_0(\mathfrak{h})$ and $D(N_+)$, differential operators, corresponds to $M_{\mathfrak{g}}$, that could be realized as chiral differential operators [CDO1].

The injectivity of ρ is proved in Remark 2.4 of [K] as follows: we take the associated graded map of ρ and get $\operatorname{gr}\rho = \phi^*$. After that we show that ϕ is dominant, equivalently, that ϕ^* is injective.

We will use a similar strategy below to prove that ω_{κ_c} is injective. In fact, we will prove a stronger statement:

Proposition 1.1. *The homomorphism* $\omega_\kappa\colon V_\kappa(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0^{\kappa-\kappa_c}$ *is injective for any* κ.

Proof. We will introduce filtrations on $V_\kappa(\mathfrak{g})$ and $M_\mathfrak{g}\otimes\pi_0^{\kappa-\kappa_c}$ such that ω_κ preserves filtrations and $\operatorname{gr}\omega_\kappa$ is injective. The PBW filtration on $U(\hat{\mathfrak{g}})$ induces a filtration on $V_\kappa(\mathfrak{g})$: $|0\rangle$ has degree zero and x_n with $n < 0$ has degree 1. The filtration on $W_{0,\kappa}$ is defined similarly with $|0\rangle$ in degree 0, and operators $a^*_{\alpha,n}$ with $n \leq 0$ in degree 0 and $a_{\alpha,n}$, b_n with $n < 0$ in degree 1.

Exercise. *1. T is filtration preserving on each of the vertex algebras.*

2. ω_κ is filtration preserving. Hint: look at the formulas in Section 4 of [W] or in Section 1.6 of [CDO2].

3. $\operatorname{gr}\omega_\kappa$ is a homomorphism of graded commutative algebras with differentials.

We know that $\operatorname{gr} V_\kappa(\mathfrak{g}) = \mathbb{C}[J\mathfrak{g}]$ and it can be checked similarly that $\operatorname{gr} W_{0,\kappa} = \mathbb{C}[J(T^*N_+ \times \mathfrak{h}^*)]$. These are the jet schemes of the varieties $\mathfrak{g} \cong \mathfrak{g}^*$, $T^*N_+ \times \mathfrak{h}^*$ considered above.

Exercise. *Prove that* $\operatorname{gr}\omega_\kappa = (J\phi)^*$, *where* ϕ *is defined above. Hint: for any affine variety* X *the algebra* $\mathbb{C}[JX]$ *is graded, the grading is unique such that* $\deg T = -1$, $\deg \mathbb{C}[X] = 0$. *Here* T *is the derivation of* $\mathbb{C}[JX]$. *For a morphism* $\varphi\colon X \to Y$ *the map*

$$(J\varphi)^*\colon \mathbb{C}[JY] \to \mathbb{C}[JX]$$

is a unique homomorphism such that

1. $(J\varphi)^*$ *restricts to* $\varphi^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$.
2. $(J\varphi)^*$ *interwtwines the derivations.*

Check that $\operatorname{gr}\omega_\kappa$ *satisfies the properties (1) and (2).*

It remains to prove that $(J\phi)^*$ is injective. Exercise 2.4 in [K] says that ϕ^* is injective, so that ϕ is dominant. Using Exercise 1.2.13 in Vanya's notes [KL] we get that $J\phi$ is dominant, hence $(J\phi)^*$ is injective. □

We move to the second step of the plan:

Lemma 1.2. $\omega_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}}))$ *is contained in* $\pi_0 \subset W_{0,\kappa_c}$.

Proof. We will use the results from Ivan's notes [CDO1, CDO2] that provide an alternative construction of $M_{\mathfrak{g}}$ and ω_{κ_c} using chiral differential operators.

Recall that $\mathfrak{z}(\hat{\mathfrak{g}})$ is the $\mathfrak{g}[[t]]$-invariants in $V_{\kappa_c}(\mathfrak{g})$. It is enough to prove that

$$\omega_{\kappa_c}(V_{\kappa_c}(\mathfrak{g})^{b_+[[t]]}) \subset \pi_0.$$

Note that $V_{\kappa_c}(\mathfrak{g})^{b_+[[t]]} = V_{\kappa_c}(\mathfrak{g})^{JB_+}$. The chiral differential operator of realization of W_{0,κ_c} provides a natural action of JB_+ on W_{0,κ_c}, explained in Section 1.2-1.3 of [CDO2]. With this action the map ω_{κ_c} is JB_+-equivariant, it is an exercise just before section 1.5 of [CDO2].

It follows that

$$\omega_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}})) \subset W_{0,\kappa_c}^{JB_+}.$$

Specializing the results of section 1.5 of [CDO2] to $P_+ = B_+$, $\mathfrak{m} = \mathfrak{h}$, we get

$$\omega_{\kappa_c}(\mathfrak{z}(\hat{\mathfrak{g}})) \subset V_0(\mathfrak{h}) = \pi_0.$$

□

2 Screening operators for $\mathfrak{sl}_2$

For $\lambda \in \mathbb{C}$ let M_λ, M_λ^* denote, respectively, the Verma and the dual Verma module over $\mathfrak{sl}_2$ with highest weight λ. We have a short exact sequence

$$0 \to M_{-2} \to M_0 \to L_0 = \mathbb{C} \to 0,$$

where L_0 is the trivial representation. Applying the duality functor we get a short exact sequence

$$0 \to \mathbb{C} \to M_0^* \to M_{-2}^* \to 0.$$

We want an affine analogue of this short exact sequence. We will define a homomorphism of $\hat{\mathfrak{sl}}_2$-modules

$$S_k \colon W_{0,k} \to W_{-2,k}$$

for non-critical level k and prove the following

Proposition 2.1. *When $k+2$ is not a nonnegative rational number, we have a short exact sequence*

$$0 \to V_k(\mathfrak{sl}_2) \to W_{0,k} \xrightarrow{S_k} W_{-2,k} \to 0.$$

2.1 Modules over vertex algebras

We will need the notion of a module over a vertex algebra V. This is a vector space M with a map $Y_M\colon V \to \mathrm{End}_M[[z^{\pm 1}]]$ such that

1. $Y_M(|0\rangle, z) = \mathrm{Id}_M$
2. For any $u, v \in V$, $m \in M$ the expressions

$$Y_M(u,z)Y_M(v,t)m, \qquad Y_M(v,t)Y_M(u,z)m, \qquad Y_M(Y(u,z-t)v,t)m$$

are expansions of the same element of $M[[z,t]][z^{-1},t^{-1},(z-t)^{-1}]$, similarly to the associativity condition for vertex algebras, [D]

We have the following example. Let $\mathfrak{h} = \mathbb{C}h$ be one-dimensional commutative Lie algebra, so that $\hat{\mathfrak{h}}_\kappa$ is a Heisenberg Lie algebra for nonzero κ and an abelian Lie algebra for $\kappa = 0$. Let $V_\kappa(\mathfrak{h})$ be the corresponding vertex algebra. Consider $M = M_\lambda = \mathrm{Ind}_{\mathfrak{h}[[t]]}^{\hat{\mathfrak{h}}_\kappa} \mathbb{C}_\lambda$, a Verma module over $\hat{\mathfrak{h}}_\kappa$. For $a_1, \ldots, a_k < 0$ we define

$$Y_M(h_{a_1}h_{a_2}\cdots h_{a_k}|0\rangle) = \frac{1}{(-a_1-1)!\cdots(-a_k-1)!}\partial_z^{-a_1-1}h(z)\cdots\partial_z^{-a_k-1}h(z):,$$

similarly to $Y(h_{a_1}\cdots h_{a_k}|0\rangle)$. It can be checked that conditions 1 and 2 are satisfied.

We can upgrade this example. Let $W_{0,k} = M_{\mathfrak{sl}_2} \otimes V_{k+2}(\mathfrak{h})$. Setting $\lambda = -2$ and tensoring by $M_{\mathfrak{sl}_2}$ we get a module $W_{-2,k} = M_{\mathfrak{sl}_2} \otimes \pi_{-2}^{k+2}$ over $W_{0,k}$.

Now we describe basic properties of modules over vertex algebras similar to the associativity and its corollaries for vertex algebras.

If M is a module over V and U is a vertex subalgebra of V, then M is a module over U. In particular, if V is a conformal vertex algebra with central charge c, we get an action of Virasoro vertex algebra Vir_c on M. If $\omega \in V$ is a conformal vector we define endomorphisms L_n^M of M via

$$Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^M z^{-n-2}.$$

We denote L_{-1}^M by T.

Recall the skew-symmetry property for vertex algebras:

$$Y(A,z)B = e^{zT}Y(B,-z)A.$$

Motivated by this we define a map $Y_{V,M}\colon M \to \mathrm{Hom}(V,M)[[z^{\pm 1}]]$ by

$$Y_{V,M}(B,z)A = e^{zT} Y_M(A,-z)B. \tag{1}$$

The following lemma is proved similarly to the associativity property of vertex algebras, [D]:

Lemma 2.2. *For any $A, C \in V$, $B \in M$ there exists an element $f \in M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ such that the formal power series*

$$Y_M(A,z)Y_{V,M}(B,w)C, \qquad Y_{V,M}(B,w)Y(A,z)C,$$

$$Y_{V,M}(Y_{V,M}(B,w-z)A,z)C, \qquad Y_{V,M}(Y_M(A,z-w)B,w)C.$$

are expansions of f in

$$M((z))((w)), \quad M((w))((z)), \quad M((z))((z-w)), \quad M((w))((z-w))$$

respectively.

Abusing the notation, for $A \in V$ we write

$$Y(A,z) = \sum A_{(n)} z^{-n-1}, \qquad Y_M(A,z) = A_{(n)} z^{-n-1}.$$

Similarly, for $B \in M$ we write

$$Y_{V,M}(B,w) = \sum B_{(n)} w^{-n-1}.$$

Similarly to the formula for the commutators of fields in vertex algebras [D] we have

$$[B_{(m)}, A_{(k)}] = \sum_{n \geq 0} \binom{m}{n} (B_{(n)}A)_{(m+k-n)}$$

and the same formula with A, B switched:

$$[A_{(m)}, B_{(k)}] = \sum_{n \geq 0} \binom{m}{n} (A_{(n)}B)_{(m+k-n)} \tag{2}$$

It can also be checked that

$$Y_{V,M}(TB,z) = \partial_z Y_{V,M}(B,z). \tag{3}$$

Remark 2.3. Let M be a vector space, V be a vertex algebra. One can show that to give a structure of a module over V on M is the same as to extend a vertex algebra structure from V to $V \oplus M$ such that

1. M is an ideal (this means for any $v \in V$, $m \in M$ and integer i we have $v_{(i)}m \in M$ and $m_{(i)}v \in M$.)
2. For any $m, n \in M$ and integer i we have $m_{(i)}n = 0$.

This is similar to the situation with modules over a commutative algebra: an A-module structure on a vector space M is the same as an algebra structure on $A \oplus M$ such that A is its subalgebra, $AM \subset M$, $M^2 = \{0\}$.

2.2 Definition of S_k and intertwining property

Definition 2.4. The *screening operator* S_k is the residue of

$$Y_{W_{0,k},W_{-2,k}}(a_{-1}|-2\rangle).$$

We will write an explicit formula for

$$S_k(z) = Y_{W_{0,k},W_{-2,k}}(a_{-1}|-2\rangle)\colon W_{0,k} \to W_{-2,k}$$

and prove that $S_k = \operatorname{Res} S_k(z)$ intertwines the action of $\hat{\mathfrak{sl}}_2$.

Lemma 2.5. *We have*

$$S_k(z) = a(z) \otimes \Big(T_{-2} \exp\big(\frac{1}{k+2}\sum_{n<0}\frac{b_n}{n}z^{-n}\big) \exp\big(\frac{1}{k+2}\sum_{n>0}\frac{b_n}{n}z^{-n}\big)\Big), \quad (4)$$

where $T_{-2}\colon \pi_0^{k+2} \to \pi_{-2}^{k+2}$ sends $|0\rangle$ to $|-2\rangle$ and commutes with the action of $b_n, n \neq 0$.

Proof. Since $W_{0,k} = M_{\mathfrak{sl}_2} \otimes \pi_0^{k+2}$ and $|-2\rangle_{W_{0,k}} = |0\rangle_{M_{\mathfrak{sl}_2}} \otimes |-2\rangle_{\pi_0^{k+2}}$, we have

$$Y_{W_{0,k},W_{-2,k}}(a_{-1}|-2\rangle) = Y_{M_{\mathfrak{sl}_2}}(a_{-1}|0\rangle, z) \otimes Y_{\pi_0^{k+2},\pi_{-2}^{k+2}}(|-2\rangle, z) = a(z) \otimes Y_{\pi_0^{k+2},\pi_{-2}^{k+2}}(|-2\rangle, z). \quad (5)$$

Let

$$V_{-2}(z) = Y_{\pi_0^{k+2},\pi_{-2}^{k+2}}(|-2\rangle, z).$$

It remains to compute $V_{-2}(z)$ in order to prove the lemma. We will do this in two steps. First, we will express $V_{-2}(z)$ via $V_{-2}(z)|0\rangle$. Then we will compute $V_{-2}(z)|0\rangle$.

Apply (2) to $A = b_{-1}|0\rangle$, $B = |-2\rangle$ to get

$$[b_m, B_{(k)}] = \sum_{n \geq 0} \binom{m}{n} (b_n|-2\rangle)_{m+k-n} = -2B_{(m+k)},$$

since $b_n|-2\rangle$ is zero for $n > 0$ and $-2|-2\rangle$ for $n = 0$. It follows that

$$[b_m, V_{-2}(z)] = -2z^m V_{-2}(z). \tag{6}$$

Since vectors $b_{n_1} \cdots b_{n_l}|0\rangle$ span π_0^{k+2}, the action of $V_{-2}(z)$ is determined by $V_{-2}(z)|0\rangle \in \pi_{-2}^{k+2}[[z]]$. Namely,

$$V_{-2}(z) = V_{-2}(z)|0\rangle \exp\Big(\frac{1}{k+2} \sum_{n>0} \frac{b_n}{n} z^{-n}\Big), \tag{7}$$

where $\exp(\frac{1}{k+2} \sum_{n>0} \frac{b_n}{n} z^{-n}) \in \mathrm{End}(\pi_0^k)[[z^{-1}]]$ is a field and $V_{-2}(z)|0\rangle$ is a shorthand for the operator that sends $b_{a_1} \cdots b_{a_k}|0\rangle \in \pi_0^{k+2}$ to

$$b_{a_1} \cdots b_{a_k} V_{-2}(z)|0\rangle \in \pi_{-2}^{k+2}[[z]]$$

for any $a_1, \ldots, a_k < 0$. This operator is uniquely defined by the Taylor series $V_{-2}(z)|0\rangle \in \pi_{-2}^{k+2}[[z]]$ that we will compute below.

Now we use equation (3) for $B = |-2\rangle$ to get

$$\partial_z V_{-2}(z) = Y_{V,M}(T|-2\rangle, z).$$

We have the following property of vertex algebras (Corollary 2.3.3 in Frenkel's book or [D]): for any $n, m < 0$ and $A, B \in V$

$$Y(A_{(n)}B_{(m)}, z) = \frac{1}{(-n-1)!(-m-1)!} : \partial_z^{-n-1} Y(A, z) \partial_z^{-m-1} Y(B, z) : .$$

Using Lemma 2.2 for $A = b_{-1}|0\rangle$, $B = |-2\rangle$ and expanding

$$Y_M(A, z) Y_{V,M}(B, w) C = Y_{V,M}(Y_M(A, z-w)B) C$$

in powers of $z - w$ similarly to [D] we get

$$Y_{V,M}(b_{-1}|-2\rangle, z) =: b(z) V_{-2}(z) : .$$

Using Proposition 6.2.2 in Frenkel's book or the third section of [W] we see that the action of $T = L_{-1} = Y(\mathbf{S}_k, z)_{-1}$ on π_0 is given by

$$T = \frac{1}{4(k+2)} \sum_{n \in \mathbb{Z}} b_n b_{-n-1}.$$

Hence

$$-b_{-1}|-2\rangle = (k+2)T|-2\rangle. \tag{8}$$

It follows that

$$(k+2)\partial_z V_{-2}(z) = - : b(z) V_{-2}(z) : . \tag{9}$$

Using (1) for $A = |0\rangle$ we see that for any vertex algebra V, module M over V and $B \in M$ we have

$$Y_{V,M}(B)|0\rangle \in B + zM[[z]] \tag{10}$$

Applying both sides of (9) to $|0\rangle$ we get

$$(k+2)\partial_z (V_{-2}(z)|0\rangle) = -b_+(z) V_{-2}(z)|0\rangle.$$

This is a differential equation for the power series

$$V_{-2}(z)|0\rangle = Y_{\pi_0^{k+2}, \pi_{-2}^{k+2}}(|-2\rangle, z)|0\rangle$$

with constant term $|-2\rangle$, the solution is

$$V_{-2}(z)|0\rangle = \exp(\frac{1}{k+2} \sum_{n<0} \frac{b_n}{n} z^{-n})|-2\rangle.$$

Comparing this with (7) we get

$$V_{-2}(z) = T_{-2} \exp\big(\frac{1}{k+2} \sum_{n<0} \frac{b_n}{n} z^{-n}\big) \exp\big(\frac{1}{k+2} \sum_{n>0} \frac{b_n}{n} z^{-n}\big). \tag{11}$$

Using (5) and (7) we get

$$S_k(z) = a(z) \otimes \Big(T_{-2} \exp\big(\frac{1}{k+2} \sum_{n<0} \frac{b_n}{n} z^{-n}\big) \exp\big(\frac{1}{k+2} \sum_{n>0} \frac{b_n}{n} z^{-n}\big)\Big),$$

as claimed. $\square$

Proposition 2.6. *The map S_k is a homomorphism of $\hat{\mathfrak{sl}}_2$-modules.*

Proof. The plan of the proof is as follows:

1. We will compute the action of $e_n, f_n, h_n, n \geq 0$ on $a_{-1}|-2\rangle$.

2. Using (2) for $A = x_{-1}|0\rangle$, $B = a_{-1}|-2\rangle$, where $x = e, f, h$, we will show that
$$[A_{(n)}, B_{(0)}] = 0.$$
Since $A_{(n)} = x_n$, $B_{(0)} = S_k$, this will prove the proposition.

We move to the first step of the plan. Recall that e_n is sent to a_n. Using $\hat{\mathfrak{sl}}_2$ relations we get

$$[e_n, a_{-1}] = 0, \qquad [h_n, a_{-1}] = 2a_{n-1}, \qquad [f_n, a_{-1}] = -h_{n-1} + k\delta_{n,1}.$$

Recall the formulas for other generators (6.2.3 in Frenkel's book, follows from formulas in section 2 of [W]):

$$h(z) \mapsto -2 : a^*(z)a(z) : +b(z), \tag{12}$$
$$f(z) \mapsto : a^*(z)^2 a(z) : +k\partial_z a^*(z) + a^*(z)b(z). \tag{13}$$

Using these formulas and the grading on the Wakimoto module by degree of t we get

$$e_n|-2\rangle = a_n|-2\rangle = 0, \qquad n \geq 0; \qquad h_n|-2\rangle = f_n|-2\rangle = 0, \qquad n > 0,$$
$$h_0|-2\rangle = -2|-2\rangle.$$

It follows that

$$e_n a_{-1}|-2\rangle = h_n a_{-1}|-2\rangle = 0, \qquad n \geq 0.$$

We also have

$$f_n a_{-1}|-2\rangle = 0, \qquad n \geq 1; \qquad f_1 a_{-1}|-2\rangle = (-h_0 + k)|-2\rangle = (k+2)|-2\rangle. \tag{14}$$

To compute the action of f_0 we have to look more carefully at (13). First we use the $\hat{\mathfrak{sl}}_2$ relation to get

$$f_0 a_{-1}|-2\rangle = a_{-1} f_0|-2\rangle - h_{-1}|-2\rangle.$$

Using (12), (13) and the fact that $a_m|-2\rangle = a^*_{m+1}|-2\rangle = b_{m+1}|-2\rangle = 0$ for $m \geq 0$ we get

$$f_0|-2\rangle = a^*_0 b_0|-2\rangle = -2a^*_0|-2\rangle,$$
$$h_{-1}|-2\rangle = (-2a_{-1}a^*_0 + b_{-1})|-2\rangle.$$

It follows that

$$f_0 a_{-1}|-2\rangle = -b_{-1}|-2\rangle.$$

Using (8) we get

$$f_0 a_{-1}|-2\rangle = (k+2)T|-2\rangle. \tag{15}$$

Now we will check that S_k is $\hat{\mathfrak{sl}}_2$-linear. Set $B = a_{-1}|-2\rangle$ for the computations below. By definition, $S_k = B_{(0)}$.

Using equation (2) for $A = a_{-1}|0\rangle$ we have

$$[A_{(m)}, B_{(k)}] = \sum_{n\geq 0}\binom{m}{n}(A_{(n)}B)_{(m+k-n)} = \sum_{n\geq 0}\binom{m}{n}(a_n a_{-1}|-2\rangle)_{(m+k-n)} = 0.$$

Using (2) for $A = h_{-1}|0\rangle$ we have

$$[A_{(m)}, B_{(k)}] = \sum_{n\geq 0}\binom{m}{n}(A_{(n)}B)_{(m+k-n)} = \sum_{n\geq 0}\binom{m}{n}(h_n a_{-1}|-2\rangle)_{(m+k-n)} = 0.$$

Using (2) for $A = f_{-1}|0\rangle$, $B = a_{-1}|-2\rangle$ we have

$$\begin{aligned}[A_{(m)}, B_{(l)}] &= \sum_{n\geq 0}\binom{m}{n}(A_{(n)}B)_{(m+l-n)} = \sum_{n\geq 0}\binom{m}{n}(f_n a_{-1}|-2\rangle)_{(m+l-n)} \\ &= m(f_1\cdot a_{-1}|-2\rangle)_{(m+l-1)} + (f_0\cdot a_{-1}|-2\rangle)_{(m+l)} = (14),\ (15) \\ &= (k+2)m|-2\rangle_{(m+l-1)} + (k+2)(T|-2\rangle)_{(m+l)}\end{aligned}$$

Now we write

$$\begin{aligned}(T|-2\rangle)_{(m+l)} =& [z^{-1-m-l}]Y(T|-2\rangle, z) \\ =& [z^{-1-m-l}]Y(|-2\rangle, z)' = (-m-l)(|-2\rangle)_{(m+l-1)}.\end{aligned}$$

It follows that

$$[A_{(m)}, B_{(l)}] = -(k+2)l(|-2\rangle)_{(m+l-1)}.$$

In particular, for $l = 0$ we get zero.

We checked that $B_{(0)}$ commutes with the action of e_m, f_m, h_m for all m. Hence $B_{(0)} = \operatorname{Res} S_k(z)$ is an intertwining operator. □

Now we prove that we indeed have a short exact sequence of $\hat{\mathfrak{sl}}_2$-modules.

Proposition 2.7. *For $k \notin -2 + \mathbb{Q}_{\geq 0}$, the sequence*

$$0 \to V_k(\mathfrak{sl}_2) \to W_{0,k} \to W_{-2,k} \to 0$$

is exact.

Proof. Writing $V_{-2}(z) = \sum V_{-2,-n} z^n$ we get

$$S_k = \operatorname{Res} S_k(z) = \sum_n a_n V_{-2,-n}. \tag{16}$$

Recall (11):

$$V_{-2}(z) = T_{-2} \exp\left(\frac{1}{k+2} \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(\frac{1}{k+2} \sum_{n>0} \frac{b_n}{n} z^{-n}\right).$$

It follows that $V_{-2,-n}|0\rangle = 0$ when $n < 0$. We also have $a_n|0\rangle = 0$ for $n \geq 0$. Hence $S_k|0\rangle = 0$. Since S_k commutes with the action of $\hat{\mathfrak{sl}}_2$ and the image $\omega_\kappa(V_k(\mathfrak{sl}_2))$ is the $\hat{\mathfrak{sl}}_2$-submodule generated by $|0\rangle$, the image of $V_k(\mathfrak{sl}_2)$ lies in the kernel of S_k.

We know that $W^+_{0,k}$ is isomorphic to the Verma module $\mathbf{M}_{0,k}$, this is Theorem 2.3 in [S] and Proposition 6.3.3 in Frenkel's book. Similarly to section 2.1 of [S] we can compute the formal character of $W_{\lambda,k}$ as follows. Let α be the positive root for $\mathfrak{sl}_2$, δ be the grading operator by the degree of t. Let $q = e^{-\delta}$, $u = e^{\alpha}$. Then the operators a_n, a^*_{n+1}, b_n, $n < 0$ have weight uq^{-n}, uq^{-1-n}, q^{-n} respectively. It follows that

$$\operatorname{ch} W_{\lambda,k} = u^{\frac{\lambda}{2}} \prod_{n>0} (1-q^n)^{-1}(1-uq^n)^{-1}(1-u^{-1}q^{n-1})^{-1}.$$

Therefore $\operatorname{ch} W_{\lambda,k} = \operatorname{ch} \mathbf{M}_{\lambda,k}$, where $\mathbf{M}_{\lambda,k}$ is Verma module of level k with h_0-weight λ.

Using Proposition 3.1 in [KK], it can be shown that, for $k \notin -2 + \mathbb{Q}_{\geq 0}$, the Verma module $\mathbf{M}_{-2,k}$ is irreducible. Since $\operatorname{ch} W_{-2,k} = \operatorname{ch} \mathbf{M}_{-2,k}$, the $\hat{\mathfrak{sl}}_2$-module $W_{-2,k}$ is also irreducible. Using (16) and the fact that $V_{-2,-n}$ commutes with a, a^* we have

$$S_k(a^*_0|0\rangle) = \sum a_n a^*_0 V_{-2,-n}|0\rangle = \sum_{n\geq 0} a_n a^*_0|-2\rangle = a_0 a^*_0|-2\rangle = |-2\rangle,$$

so S_k is nonzero, hence surjective.

It follows that the character of $\ker S_k$ equals to $\operatorname{ch} \mathbf{M}_{0,k} - \operatorname{ch} \mathbf{M}_{-2,k}$.

Exercise. $\operatorname{ch}\mathbf{M}_{0,k} - \operatorname{ch}\mathbf{M}_{-2,k} = \operatorname{ch} V_k(\mathfrak{sl}_2)$, *hence* $V_k(\mathfrak{sl}_2)$ *coincides with the kernel of* S_k.

□

3 Friedan-Martinec-Shenker bosonization

3.1 The vertex algebra Π_0

A note on terminology. One may think that bosonization means boson-fermion correspondence. This is not the case: FMS bosonization is boson-boson correspondence.

We want to construct objects similar to $M_{\mathfrak{sl}_2}$ and $W_{k,\lambda}$ that depend on two parameters. This will be convenient for taking the limit $k \to -2$.

Consider the Heisenberg Lie algebra with generators $p_n, q_m, \mathbf{1}$ such that $\mathbf{1}$ is central and relations

$$[p_n, p_m] = n\delta_{n,-m}\mathbf{1}, \quad [q_n, q_m] = -n\delta_{n,-m}\mathbf{1}, \quad [p_n, q_m] = 0.$$

We set

$$p(z) = \sum p_n z^{-n-1}, \quad q(z) = \sum q_n z^{-n-1}.$$

We define Fock representation $\Pi_{\lambda,\mu}$ of this algebra as usual: it is generated by $|\lambda, \mu\rangle$ such that

$$p_n|\lambda,\mu\rangle = \lambda\delta_{n,0}|\lambda,\mu\rangle, \quad q_n|\lambda,\mu\rangle = \mu\delta_{n,0}|\lambda,\mu\rangle, \quad n \geq 0, \quad \mathbf{1}|\lambda,\mu\rangle = |\lambda,\mu\rangle.$$

Recall (11):

$$V_{-2}(z) = T_{-2}\exp(\frac{1}{k+2}\sum_{n<0}\frac{b_n}{n}z^{-n})\exp(\frac{1}{k+2}\sum_{n>0}\frac{b_n}{n}z^{-n})\colon \pi_0^{k+2} \to \pi_{-2}^{k+2}$$

Similarly to V_{-2} we define $V_{\lambda,\mu}\colon \Pi_{\lambda',\mu'} \to \Pi_{\lambda+\lambda',\mu+\mu'}$ given by

$$V_{\lambda,\mu}(z) = T_{\lambda,\mu}z^{\lambda\lambda'-\mu\mu'}\exp\left(-\sum_{n<0}\frac{\lambda p_n + \mu q_n}{n}z^{-n})\exp(-\sum_{n>0}\frac{\lambda p_n + \mu q_n}{n}z^{-n}\right),$$

where $T_{\lambda,\mu}$ is defined similarly to T_{-2}: it sends highest weight vector to the highest weight vector and commutes with all p_n, q_n for nonzero n.

We claim that

$$\Pi_0 = \oplus_{n\in\mathbb{Z}}\Pi_{n,n}$$

has a vertex algebra structure such that $\Pi_{0,0}$ is its vertex subalgebra. First, we define $T\colon \Pi_0 \to \Pi_0$ using the fact that each $\Pi_{n,n}$ is a module over $\Pi_{0,0}$. Then we define

$$Y(p_{-1}|0\rangle, z) = p(z), \quad Y(q_{-1}|0\rangle, z) = q(z)$$
$$Y(|1,1\rangle, z) = V_{1,1}(z), \quad Y(|-1,-1\rangle, z) = V_{-1,-1}(z)$$

Using a formula similar to (6) we can check that $V_{\lambda,\mu}$ is mutually local with $p(z)$ and $q(z)$. Hence the third condition of the strong reconstruction theorem is satisfied. We also see that the four fields above generate Π_0. Equation (3) is exactly the second condition of the reconstruction theorem. The first condition can also be checked. Hence all four conditions of the strong reconstruction theorem are satisfied and Π_0 gets a structure of a vertex algebra.

Similarly, we can check that for any $\gamma \in \mathbb{C}$

$$\Pi_\gamma = \oplus_{n\in\mathbb{Z}}\Pi_{n+\gamma,n+\gamma}$$

is a module over Π_0.

Let $u(z)$ be the "integral" of $p(z)$, similarly with v and q:

$$u(z) = -\sum_{n\neq 0}\frac{p_n}{n}z^{-n} + p_0 \log z, \qquad v(z) = -\sum_{n\neq 0}\frac{q_n}{n}z^{-n} + q_0 \log z.$$

Abusing notation we will write $V_{\lambda,\mu} = e^{\lambda u+\mu v}$.

FMS bosonization is the following realization of the vertex algebra $M_{\mathfrak{sl}_2}$.

Theorem 3.1. *There is a unique embedding of vertex algebras $M \hookrightarrow \Pi_0$ such that the fields $a(z)$, $a^*(z)$ are mapped to*

$$\tilde{a}(z) = e^{u+v}, \qquad \tilde{a}^*(z) = (\partial_z e^{-u})e^{-v} = -:p(z)e^{-u-v}: .$$

Furthermore, the image of M in Π_0 is the kernel of $\operatorname{Res} e^u$.

Proof. We have $[a(z), a^*(t)] = \delta(z-t)$. Since

$$[p(z), e^{u(t)+v(t)}] = \delta(z-t)e^{u+v},$$

it can be checked that

$$[\tilde{a}(z), \tilde{a}^*(t)] = \delta(z-t).$$

This map is an embedding because M is a simple vertex algebra. In this talk we will not need the description of the image of M. □

The Virasoro field for M is $T(z) =: \partial_z a^*(z)a(z) :$, this can be checked using formulas from Section 3 in [W]. This field maps to

$$\tfrac{1}{2}\big(: p(z)^2 : -\partial_z p(z) - : q(z)^2 : +\partial_z q(z)\big).$$

The main advantage of FMS bosonization for us is that we can define $a(z)^\gamma$ for any complex number γ as

$$\tilde{a}(z)^\gamma = e^{\gamma(u+v)} \colon \Pi_0 \to \Pi_\gamma.$$

3.2 Screening operators of the second kind

Let $k \neq -2$. Consider $\Pi_0 \otimes \pi_0^{k+2}$. This is a vertex algebra that contains $M \otimes \pi_0^{k+2} = W_{0,k}$ and hence $V_k(\mathfrak{sl}_2)$ as a vertex subalgebra. For $\lambda, \gamma \in \mathbb{C}$ the tensor product $\Pi_\gamma \otimes \pi_\lambda^{k+2}$ is a module over $V_k(\mathfrak{sl}_2)$, hence over $\hat{\mathfrak{sl}}_2$. We denote this module by $\widetilde{W}_{\gamma,\lambda,k}$. We also introduce the operator

$$V_{2k+2}(z) = T_{2(k+2)} \exp(-\sum_{n<0} \frac{b_n}{n} z^{-n}) \exp(-\sum_{n>0} \frac{b_n}{n} z^{-n}) \colon \pi_0^{k+2} \to \pi_{2k+2}^{k+2}.$$

Set

$$\tilde{S}_k(z) = \tilde{a}(z)^{-(k+2)} \otimes V_{2k+2}(z).$$

Reasoning similarly to the proof of Proposition 2.6 we have the following proposition.

Proposition 3.2. *The residue* $\tilde{S}_k = \operatorname{Res} \tilde{S}_k(z) \colon W_{0,0,k} \to W_{-(k+2),2(k+2),k}$ *intertwines the* $\hat{\mathfrak{sl}}_2$*-actions.*

The map $\tilde{S}_k$ or its restriction to $W_{0,k} \subset \tilde{W}_{0,0,k}$ is called the *screening operator of the second kind* for $\hat{\mathfrak{sl}}_2$.

The main result of this subsection is the following:

Proposition 3.3. *For generic* k *the* $\hat{\mathfrak{sl}}_2$*-submodule* $V_k(\mathfrak{sl}_2) \subset W_{0,k}$ *is equal to the kernel of*

$$\tilde{S}_k \colon W_{0,k} \to \tilde{W}_{-(k+2),2(k+2),k}.$$

Proof. We sketch the proof, see Proposition 7.2.5 in Frenkel's book for details.

It is enough to prove that $\operatorname{Ker} \tilde{S}_k = \operatorname{Ker} S_k$. We can extend

$$S_k \colon W_{0,k} \to W_{-2,k}$$

to
$$S'_k\colon \tilde{W}_{0,0,k} \to \tilde{W}_{1,-2,k}$$
by changing $a(z)$ to $\tilde{a}(z)$. We will prove a stronger statement: $\operatorname{Ker}\tilde{S} = \operatorname{Ker} S'_k$. Let $\phi(z) = -\sum_{n\neq 0}\frac{b_n}{n}z^{-n}$. Then similarly to the notation for $V_{\lambda,\mu}$ we will write
$$V_{-2}(z) = e^{-(k+2)^{-1}\phi}, \qquad V_{2(k+2)}(z) = e^{\phi}.$$
With this notation we have
$$S'_k(z) = e^{u+v-(k+2)^{-1}\phi}, \qquad \tilde{S}_k(z) = e^{-(k+2)u-(k+2)v+\phi}. \tag{17}$$

Now the fact that $\operatorname{Ker}\operatorname{Res} S'_k(z) = \operatorname{Ker}\operatorname{Res}\tilde{S}_k(z)$ follows from equation (15.4.10) in [FBZ], as explained in Proposition 7.2.5 of Frenkel's book. □

References

[D] Ilya Dumanski, Vertex Algebras II, https://gauss.math.yale.edu/ il282/Ilya2.pdf

[FBZ] E. V. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, vol. 88, Second ed., Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2004. MR 2005d:17035

[KK] V.G Kac, D.A Kazhdan, Structure of representations with highest weight of infinite-dimensional Lie algebras, Advances in Mathematics, Volume 34, Issue 1, 1979.

[KL] Ivan Karpov, Ivan Losev, Invariants of jets and the center for $\hat{\mathfrak{sl}}_2$, https://gauss.math.yale.edu/ il282/Vanya.pdf

[K] Daishi Kiyohara, Free Field Realization, https://gauss.math.yale.edu/ il282/Daishi.pdf

[CDO1] Ivan Losev, Chiral differential operators 1, https://gauss.math.yale.edu/ il282/CDO1.pdf

[CDO2] Ivan Losev, Chiral differential operators 2, https://gauss.math.yale.edu/ il282/CDO2.pdf

[S] Kenta Suzuki, Parabolic Wakimoto modules and applications, https://gauss.math.yale.edu/ il282/Kenta.pdf

[W] Zeyu Wang, Wakimoto modules, https://gauss.math.yale.edu/ il282/Zeyu.pdf