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**Notebook:** D-modules
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# D-modules. Lecture 7.

1) Poisson structures from deformation.
2) Gabber's thm & consequences
3) Proof.

1) A general construction: let $\mathcal{A}_\varepsilon$ be an associative $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-algebra, free as a $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-module. Let $A := \mathcal{A}_\varepsilon/(\varepsilon)$ be comm've. $A$ comes w. a natural bracket satisf. Leibniz id.: note that $A \xrightarrow{\sim} \varepsilon\mathcal{A}_\varepsilon$.
Pick $\bar{a}, \bar{b} \in A$ and lift them to $a, b \in \mathcal{A}_\varepsilon$. Then $[a,b] \in \varepsilon\mathcal{A}_\varepsilon$ & depends only on $\bar{a}, \bar{b}$ not on choice of $a$ & $b$. Let $\{\bar{a},\bar{b}\}$ denote the image of $[a,b]$ in $A$ under the isomorphism $\varepsilon\mathcal{A}_\varepsilon \xrightarrow{\sim} A$.

See Remark on pages 3&4. → Exercise: $\{\cdot,\cdot\}$ is a well-defined bracket on $A$.

For example, let $\mathcal{A}$ be a filtered algebra s.t. $\mathrm{gr}(\mathcal{A})$ is comm've. Consider the Rees alg'a $R_\hbar(\mathcal{A})$ and its quotient $\mathcal{A}_\varepsilon := R_\hbar(\mathcal{A})/(\hbar^2)$ (w. $\varepsilon :=$ image $\hbar$ in $\mathbb{C}[\hbar]/(\hbar^2)$). Pick $a \in \mathcal{A}_{\leq i}$, $b \in \mathcal{A}_{\leq j}$. Then $[a\hbar^i, b\hbar^j] = \hbar\cdot(\hbar^{i+j-1}[a,b]) \in \hbar R_\hbar(\mathcal{A})$. The algebra $A := \mathrm{gr}(\mathcal{A})$ becomes Poisson w.r.t. $\{\cdot,\cdot\}$ given by $\{a + \mathcal{A}_{\leq i-1}, b + \mathcal{A}_{\leq j-1}\} := [a,b] + \mathcal{A}_{\leq i+j-2}$.

Now suppose that $\mathcal{A} = \mathcal{D}(X)$ for a smooth affine variety $X$. Then $A = \mathbb{C}[T^*X]$. Note that $T^*X$ is a symplectic variety, hence $A$ is equipped w. a Poisson bracket.

Lemma 1: This Poisson bracket coincides w. one induced by $\mathcal{A}$.

Proof: The bracket on $\mathbb{C}[T^*X]$ coming from the symplectic form is uniquely characterized by:

$$\{f_1, f_2\} = 0$$
$$\{v_1, f_1\} = \partial_{v_1} f_1$$
$$\{v_1, v_2\} = [v_1, v_2]$$

$\forall f_1, f_2 \in \mathbb{C}[X],\ v_1, v_2 \in Vect(X)$

Uniqueness follows b/c $\mathbb{C}[X]$, $Vect(X)$ generate $\mathbb{C}[T^*X]$. These identities for the bracket on $gr\,\mathcal{D}(X)$ coming from the deformation constr'n follow from:

$$[f_1, f_2] = 0$$
$$[v_1, f_1] = \partial_{v_1} f_1$$

$[v_1, v_2] =$ the bracket $[v_1, v_2] \in Vect(X)$, which are identities in $\mathcal{D}(X)$

2) Recall that a subvariety $Y \subseteq T^*X$ is

called **coisotropic** if one of the following equiv't conditions holds:

- $T_y Y \subset T_y(T^*X)$ is a coisotropic subspace $\forall y \in Y^{reg}$

- $\{I_Y, I_Y\} \subseteq I_Y$, where $I_Y := \{f \in \mathbb{C}[T^*X] \mid f|_Y = 0\}$

This was proved in Danny's seminar lecture.

Thm (Gabber): Let $\mathcal{A}_\varepsilon, A$ have the same meaning as before. Assume $A$ is fin. gen'd. Let $\mathcal{M}_\varepsilon$ be a finitely generated $\mathcal{A}_\varepsilon$-module free over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ and let $M := \mathcal{M}_\varepsilon / \varepsilon \mathcal{M}_\varepsilon$. Finally let $I := \sqrt{\mathrm{Ann}_A M} \subseteq A$. Then $\{I, I\} \subseteq I$.

Cor: Let $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$. Then $SS(\mathcal{M}) \subset T^*X$ is coisotropic.

Proof: Can assume $X$ is affine. Then we apply the theorem to $\mathcal{A}_\varepsilon := R_\hbar(\mathcal{D}(X))/(\hbar^2)$ & $\mathcal{M}_\varepsilon := R_\hbar(\mathcal{M})/\hbar^2 R_\hbar(\mathcal{M})$. $\square$

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Remark (on Poisson structures for Section 1)

Our deformation construction produces the bracket on $A$ that may fail to be Poisson. In order to get a Poisson bracket ($\Leftrightarrow$ Jacobi identity to holds) we need to start with an

algebra / $\mathbb{C}[\varepsilon]/(\varepsilon^3)$. Let $\tilde{\mathcal{A}}_\varepsilon$ be a $\mathbb{C}[\varepsilon]/(\varepsilon^3)$-algebra (associative & unital) that is a free $\mathbb{C}[\varepsilon]/(\varepsilon^3)$-module. Then set $\mathcal{A}_\varepsilon := \tilde{\mathcal{A}}_\varepsilon/(\varepsilon^2)$ and assume that $A := \mathcal{A}_\varepsilon/(\varepsilon) = \tilde{\mathcal{A}}_\varepsilon/(\varepsilon)$ is comm've. Then the bracket on $A$ coming from $\mathcal{A}_\varepsilon$ is Poisson: we have $[[a,b],c]+[b,[c,a]]+[c,[a,b]] =$
$\underline{\varepsilon^2(\{\{\bar{a},\bar{b}\},\bar{c}\}+\{\bar{b},\{\bar{c},\bar{a}\}\}+\{\bar{c},\{\bar{a},\bar{b}\}\})}$

3) Proof of Gabber's Thm.

First we will introduce notation that we'll use. The ideal $I \subset A$ is radical hence is the intersection of its minimal primes. If we know that $\{\mathfrak{p},\mathfrak{p}\} \subset \mathfrak{p}$ $\forall$ min. primes $\mathfrak{p}$, then $\{I, I\} \subset I$. So fix a min. prime $\mathfrak{p}$. Let $\ell =$ $= \dim A/\mathfrak{p}$. Choose $x_1, \ldots x_\ell \in \mathcal{A}_\varepsilon$ s.t. $\bar{x}_i + \mathfrak{p}$, $i = 1, \ldots, \ell$ are alg. indep't ($\bar{x}_i := x_i + \varepsilon\mathcal{A}_\varepsilon$). We can assume $\bar{x}_1$ lies in every min prime of $I$ but $\mathfrak{p}$ b/c we can choose any nonzero element for $\bar{x}_1 + \mathfrak{p}$. Let $R := \mathbb{C}[\bar{x}_1, \dots, \bar{x}_\ell] \hookrightarrow \bar{A}_i$

We can choose $\bar{f} \in R$ w. the following properties:

ii) $\bar{f}$ lies in all min. primes of $I$ but $\mathfrak{p}$.

i) $A_{\bar{f}}/\mathfrak{p}_{\bar{f}}$ is a free finite rk $R_{\bar{f}}$-module

For some $s \in \mathbb{Z}_{>0} \Rightarrow \mathfrak{p}_{\bar{f}}^s M_{\bar{f}} = 0$.

iii) Each $\mathfrak{p}_{\bar{f}}^i M_{\bar{f}} / \mathfrak{p}_{\bar{f}}^{i+1} M_{\bar{f}}$ is a free fin. rk $A_{\bar{f}}/\mathfrak{p}_{\bar{f}}$-module. (hence a free finite rk $R_{\bar{f}}$-module).

Pick $m_1, \dots m_r \in M_\varepsilon$ s.t. their images $\bar{m}_i$ in $M_{\bar{f}}$ form an $R_{\bar{f}}$-basis of $M_{\bar{f}}$ compatible w. the filtration $\mathfrak{p}_{\bar{f}}^i M_{\bar{f}} \subset M_{\bar{f}}$. This gives a partial order on the indexes $1, \dots r$: $i \prec j$ if $\bar{m}_i \in \mathfrak{p}_{\bar{f}}^k M_{\bar{f}}$ but $\bar{m}_j \notin \mathfrak{p}_{\bar{f}}^k M_{\bar{f}}$. Then $\mathfrak{p}_{\bar{f}} \bar{m}_j \subset \sum_{i \prec j} R_{\bar{f}} \bar{m}_i$.

Pick a lift $f$ of $\bar{f}$ to $\mathcal{A}_\varepsilon$. Set $\vec{m} = (m_1, \dots, m_r)^T$.

<u>Technical lemma:</u> Let $a, b \in \mathcal{A}_\varepsilon$ be s.t. $\bar{a}, \bar{b} \in \mathfrak{p}$. Then $\exists\, n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$, $X \in Mat_r(\mathcal{A}_\varepsilon)$ s.t.

- $f^{n_1}[f^{n_2} a, f^{n_3} b]\vec{m} = \varepsilon X \vec{m}$
- $\bar{X} \in Mat_r(R)$ & $\operatorname{tr} \bar{X} = 0$

Let's explain how this lemma implies the theorem. Let $n = n_1 + n_2 + n_3$. Using the Leibniz identity, we

see that $f^{n_1}[f^{n_2}a, f^{n_3}b] = \varepsilon c$ for $c \in \bar{f}^n\{\bar{a}, \bar{b}\} + \mathfrak{p} \subset A$.
Lemma means that $c$ acts on the free $R_{\bar{f}}$ module $M_{\bar{f}}$ by $\bar{X}$. Since $\mathfrak{p}_{\bar{f}}$ acts nilpotently, we see that
$0 = tr_{M_{\bar{f}}} \bar{f}^n\{\bar{a}, \bar{b}\} = \bar{f}^n tr_{M_{\bar{f}}}\{\bar{a}, \bar{b}\} = 0$ (in $R_{\bar{f}}$) $\Rightarrow$

$tr_{M_{\bar{f}}}\{\bar{a}, \bar{b}\} = 0$

Now pick arbitrary $\bar{g} \in A_{\bar{f}}$. We get $tr_{M_{\bar{f}}}\{\bar{g}\bar{a}, \bar{b}\} = 0$
$\Rightarrow$ $[\{\bar{g}\bar{a}, \bar{b}\} \in \bar{g}\{\bar{a}, \bar{b}\} + \mathfrak{p}_{\bar{f}}]$ $tr_{M_{\bar{f}}} \bar{g}\{\bar{a}, \bar{b}\} = 0$.

Let $gr M_{\bar{f}} := \bigoplus_i \mathfrak{p}_{\bar{f}}^i M_{\bar{f}} / \mathfrak{p}_{\bar{f}}^{i+1} M_{\bar{f}} \Rightarrow tr_{M_{\bar{f}}} \bar{g}\{\bar{a}, \bar{b}\} =$

$= tr_{gr M_{\bar{f}}} \bar{g}\{\bar{a}, \bar{b}\}$. But $gr M_{\bar{f}}$ is a free rk $r/q$

module over $A_{\bar{f}}/\mathfrak{p}_{\bar{f}}$, where $q$ is the rank of $A_{\bar{f}}/\mathfrak{p}_{\bar{f}}$ over $R_{\bar{f}}$. So

$$tr_{gr M_{\bar{f}}} \bar{g}\{\bar{a}, \bar{b}\} = \left(tr_{A_{\bar{f}}/\mathfrak{p}_{\bar{f}}} \bar{g}\{\bar{a}, \bar{b}\}\right) \cdot \frac{r}{q}$$

The trace form $tr(\cdot ?) : A_{\bar{f}}/\mathfrak{p}_{\bar{f}} \times A_{\bar{f}}/\mathfrak{p}_{\bar{f}} \longrightarrow R_{\bar{f}}$
has no kernel $\Rightarrow \{\bar{a}\ \bar{b}\} \in \mathfrak{p}_{\bar{f}}$. Since $\mathfrak{p}_{\bar{f}} \cap A = \mathfrak{p}$,
we get $\{\bar{a}, \bar{b}\} \in \mathfrak{p}$, q.e.d. This finishes the proof of Thm modulo lemma.

Proof of Lemma: Let $R_\varepsilon$ be the preimage of $R$ in $\mathcal{A}_\varepsilon$. We claim $\exists\, U^{(0)}, U^{(1)} \in Mat_r(R_\varepsilon)$ & $n_2 \in \mathbb{Z}_{\geq 0}$ s.t.

(1) $f^{n_2} a \vec{m} = (U^{(0)} + \varepsilon U^{(1)})\vec{m}$ where $U^{(0)} = (u_{ij}^{(0)})$ has $u_{ij}^{(0)} \neq 0 \Rightarrow i<j$. Indeed $\bar{a}\,\vec{m} = \bar{U}'\vec{m}$ for $\bar{U}' \in Mat_r(R_{\bar{p}})$ w. $u'_{ij} \neq 0 \Rightarrow i<j$ b/c $\bar{a} \in \mathfrak{b}$. Then we find $n_2'$ s.t. $\bar{f}^{n_2'}\bar{U}' \in Mat_r(R)$ and lift $\bar{f}^{n_2'}\bar{U}'$ to $U^{(0)\prime}$ w. $u_{ij}^{(0)\prime} \neq 0 \Rightarrow i<j$. Then $\tilde{f}^{n_2'} a\vec{m} - U^{(0)\prime}\vec{m} \in \varepsilon\mathcal{M}_\varepsilon \simeq M \Rightarrow \exists\, n_2''$ s.t. $f^{n_2''}(f^{n_2'} a\vec{m} - U^{(0)\prime}\vec{m}) \in \varepsilon\, Span_R(\bar{m}_1, \dots, \bar{m}_r)$ & we are done getting (1).

Similarly, get

(2) $f^{n_3} b\,\vec{m} = (V^{(0)} + \varepsilon V^{(1)})\vec{m}$.

We get $f^{n_2} a f^{n_3} b\,\vec{m} = f^{n_2} a\,(V^{(0)} + \varepsilon V^{(1)})\vec{m} =$
$= [[\mathcal{A}_\varepsilon, \mathcal{A}_\varepsilon] \subset \varepsilon A] = (V^{(0)} + \varepsilon V^{(1)}) f^{n_2} a\vec{m} +$
$\varepsilon\{\bar{f}^{n_2}\bar{a}, \bar{V}^{(0)}\}\vec{m} = [(V^{(0)} + \varepsilon V^{(1)})(U^{(0)} + \varepsilon U^{(1)}) \cdot$
$+ \varepsilon\{\bar{f}^{n_2}\bar{a}, \bar{V}^{(0)}\}]\,\vec{m}$.

$\Rightarrow$

$$[f^{n_2}a, f^{n_3}b]\vec{m} = [V^{(0)} + \varepsilon V^{(1)}, U^{(0)} + \varepsilon U^{(1)}]\vec{m} + \varepsilon(\{\bar{f}^{n_2}\bar{a}, \bar{V}^{(0)}\} - \{\bar{f}^{n_3}\bar{b}, \bar{U}^{(0)}\})\vec{m}$$

Note that $[f^{n_2}a, f^{n_3}b] \in \varepsilon A \subset \mathcal{A}_\varepsilon$. So $[V^{(0)}, U^{(0)}] \in Mat_r(\varepsilon A)$.

Set $W^{(0)} := [V^{(0)}, U^{(0)}] + \varepsilon(\{\bar{f}^{n_2}\bar{a}, \bar{V}^{(0)}\} - \{\bar{f}^{n_3}\bar{b}, \bar{U}^{(0)}\})$

This is a strictly upper triangular w.r.t. $\preceq$ matrix in $Mat_r(\varepsilon A)$. Therefore $\exists\ n_1 \in \mathbb{Z}_{\geq 0}$ & strictly upper triangular $\bar{Y} \in Mat_r(R)$ s.t.

$$\bar{f}^{n_1} W^{(0)} \vec{m} = \varepsilon \bar{Y} \vec{m}.$$

Finally, set $W^{(1)} = [V^{(1)}, U^{(0)}] + [V^{(0)}, U^{(1)}]$. So we get

$$f^{n_1}[f^{n_2}a, f^{n_3}b]\vec{m} = f^{n_1}(W^{(0)} + \varepsilon W^{(1)})\vec{m} =$$
$$= \varepsilon(Y + \bar{f}^{n_1}\bar{W}^{(1)})\vec{m}.$$

We set $\bar{X} = \bar{Y} + \bar{f}^{n_1}\bar{W}^{(1)}$. The equality $\operatorname{tr}\bar{X} = 0$ holds b/c $\bar{X}$ is the sum of a strictly upper triangular matrix, $\bar{Y}$, and the sum of two matrix commutators, $\bar{W}^{(1)}$. $\square$