0) Reminder on duality.
1) Duality on $\mathcal{O}$-coherent modules.
2) Duality vs Kashiwara's lemma.
3) Classification of simple holonomic $\mathcal{D}$-modules.

0) Let X be a smooth variety. We have defined a triangulated functor $\mathbb{D}: D^b(\mathrm{Coh}\,\mathcal{D}_X) \xrightarrow{\sim} D^b(\mathrm{Coh}\,\mathcal{D}_X^{opp})$ via $\mathcal{M} \mapsto K_X^{-1}\otimes_{\mathcal{O}_X} R\,\underline{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)[\dim X]$ w. following properties:

i) $\mathbb{D}$ maps $Hol(\mathcal{D}_X)$ to $Hol(\mathcal{D}_X)^{opp}$ (i.e. all cohomology sheaves except 0th vanish & 0th homology is holonomic.)

ii) $\mathbb{D}^2 \simeq id$.

Rem: The proof of ii) implied that
$$\mathbb{D}(?) \simeq R\,\underline{Hom}_{\mathcal{D}_X^{opp}}(K_X\otimes_{\mathcal{O}_X} ?, \mathcal{D}_X).$$

In this lecture we will establish two more important properties of $\mathbb{D}$:

1) If V is $\mathcal{O}$-coherent, then $\mathbb{D}(V)\simeq V^\vee$.

2) If $Y \overset{i}{\hookrightarrow} X$ is a closed smooth subvariety, then
$$\mathbb{D}_X \circ i_* \simeq i_* \circ \mathbb{D}_Y$$

In the 3rd part we will apply the duality (together with a fact about push-forward to be proved later) to classify irreducible holonomic $\mathcal{D}$-modules.

1) To compute $\mathbb{D}(V) = \underline{Ext}^{\dim X}_{\mathcal{D}_X}(K_X \otimes_{\mathcal{O}_X} V, \mathcal{D}_X)$ we will first consider the case of $V = \mathcal{O}_X$ and then deduce the general case. And for $V = \mathcal{O}_X$, we'll produce an explicit locally free resolution of $\mathcal{O}_X$.

1.1) De Rham complex of a $\mathcal{D}$-module. Let's pick $\mathcal{M} \in QCoh(\mathcal{D}_X)$. It carries a natural map $\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega^1_X$, $\langle \nabla(m), \xi \rangle := \xi m$, where on the l.h.s. we pair the Ω^1_X-factor w. $Vect_X$. Note that $\nabla(fm) = f\nabla m + m \wedge df$. This ∇ is known as a connection form.

Def'n: The de Rham complex $dR(\mathcal{M})$ has terms $\mathcal{M} \otimes_{\mathcal{O}_X} \Omega^i_X$ & differential $d: \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^i_X \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^{i+1}_X$ given by

$$d(m \otimes \alpha) = \nabla m \wedge \alpha + m \otimes d\alpha$$

Exer: $d^2 = 0$.

Of course, $dR(\mathcal{O}_X)$ is the usual algebraic de Rham complex. Now we note that any (local) endomorphism of $\mathcal{M}$ gives rise to a (local) endomorphism of $dR(\mathcal{M})$. We apply this to $\mathcal{M} := \mathcal{D}_X$ and see that $dR(\mathcal{D}_X)$ is a complex of right $\mathcal{D}_X$-modules (b/c $\mathcal{D}_X^{opp} = \underline{End}_{\mathcal{D}_X}(\mathcal{D}_X)$).

Below we usually shift the complex putting $\mathcal{D}_X \otimes_{\mathcal{O}_X} K_X$ in the homological degree 0.

1.2) Resolution of K_X.

Proposition 1. We have $H_i(dR(\mathcal{D}_X)) = \begin{cases} K_X, & i=0 \\ 0, & \text{else} \end{cases}$

Proof: Case $i=0$:

Note that, by the construction, the image of $d: \mathcal{D}_X \otimes \Omega^{\dim X - 1}_X \longrightarrow \mathcal{D}_X \otimes K_X$ lies in $(Vect_X \mathcal{D}_X) \otimes K_X$. So $H_0(dR(\mathcal{D}_X)) \twoheadrightarrow K_X$

To check this is iso, it suffices to assume X has an etale coordinate chart $x^1, \dots, x^n$. There $\nabla m = \sum_{i=1}^n \partial^i m \wedge dx^i$ and the check is left as an exercise.

Case $i > 0$. Again, it's enough to assume that X has an etale coordinate chart $x^1, \dots, x^n$, so that $d(m \otimes \alpha) = \sum_{i=1}^n \partial^i m \otimes (dx^i \wedge \alpha) + m \otimes d\alpha$. The terms in $dR(\mathcal{D}_X)$ inherit a filtration from $\mathcal{D}_X$. The 1st summand increases the filtration degree by 1 and the 2nd preserves it. So we can pass to the associated graded complex $\mathrm{gr}\, dR(\mathcal{D}_X)$ w. differential $\mathrm{gr}\, d(\underline{m} \otimes \alpha) = \sum_{i=1}^n \partial^i \underline{m} \otimes (dx^i \wedge \alpha)$. Here $\underline{m} \in \mathbb{C}[T^*X]$. What we get is the Koszul complex for the elements $\partial^1, \dots, \partial^n \in \mathbb{C}[T^*X]$ $(= \mathbb{C}[X][\partial^1, \dots, \partial^n])$. It's a classical fact that, since $\partial^1, \dots, \partial^n$ form a regular sequence, the Koszul complex has no higher homology. It's then an exercise to check that once $(\mathrm{gr}\, dR(\mathcal{D}_X), \mathrm{gr}\, d)$ doesn't have higher homology, neither does $(dR(\mathcal{D}_X), d)$. $\square$

1.3) Computation of $\mathbb{D}(\mathcal{O}_X)$.

Proposition 2: We have $\mathbb{D}(\mathcal{O}_X) \simeq \mathcal{O}_X$.

Proof: $\mathbb{D}(\mathcal{O}_X) \simeq R\,\underline{Hom}_{\mathcal{D}_X^{opp}}(K_X, \mathcal{D}_X)[\dim X] = [K_X \simeq_{q.is} dR(\mathcal{D}_X)] = \underline{Hom}_{\mathcal{D}_X^{opp}}(dR(\mathcal{D}_X), \mathcal{D}_X)[\dim X]$, the Hom is taken termwise so we get the complex of left modules w. terms $\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^i \mathrm{Vect}_X$ (in homological degree i). The homology is in deg 0 only, and is the cokernel of a homomorphism $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathrm{Vect}_X \xrightarrow{\varphi} \mathcal{D}_X$. We need to find the homomorphism. It is obtained from ∇: $\mathcal{D}_X \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega^1_X$ by applying $\underline{Hom}_{\mathcal{D}_X^{opp}}(?, \mathcal{D}_X)$. So take $\xi \in \mathrm{Vect}_X$. Let $\tilde{\xi}$ denote the corresponding element of $\underline{Hom}_{\mathcal{D}_X^{opp}}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega^1_X, \mathcal{D}_X)$;

$\tilde{\xi}(a\otimes\alpha) = \langle\alpha,\xi\rangle a$ (we multiply on the left b/c this is a right $\mathcal{D}$-module homomorphism). Then $(\tilde{\xi}\circ\nabla)(b) = (\varphi(\xi))b \ \forall b\in\mathcal{D}_X$.
$\Rightarrow \varphi(\xi) = (\tilde{\xi}\circ\nabla)(1)$. By the definition of ∇, we have $\varphi(\xi)=\xi$. We conclude that $\varphi(b\otimes\xi) = b\xi$ (recall that φ is a left $\mathcal{D}$-module homom'm). And coker $\varphi \simeq \mathcal{O}_X$. $\square$

1.4) Computation of $\mathbb{D}(V)$ for $\mathcal{O}$-coherent V.

We need to show $\mathbb{D}(V) = V^\vee (:= \underline{Hom}(V,\mathcal{O}_X))$. This will follow from Proposition 2 combined with:

Proposition 3: We have $\mathbb{D}(V\otimes ?) \simeq V^\vee \otimes \mathbb{D}(?)$

Proof: Consider the $\mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_X$-module $\mathcal{D}_X^{K^{-1}} := \mathcal{D}_X\otimes_{\mathcal{O}_X} K_X^{-1}$. This is analogous to $\mathcal{D}_X^{opp}\otimes_{\mathbb{C}}\mathcal{D}_X^{opp}$-module $\mathcal{D}_X^K = K_X\otimes_{\mathcal{O}_X}\mathcal{D}_X$ from Lec 12.

Now $R\underline{Hom}_{\mathcal{D}_X}(V\otimes ?, \mathcal{D}_X^{K^{-1}}) \simeq R\underline{Hom}_{\mathcal{D}_X}(?, V^\vee\otimes\mathcal{D}_X^{K^{-1}})$; here we've multiplied on the side of the original left action, where we take Hom. In Lecture 12, we've seen that $(\mathcal{D}_X^K)^\sigma \simeq \mathcal{D}_X^K$, where σ corresponds to swapping the two right actions. This was done by analyzing the relations for the generating subsheaf $K_X\subset\mathcal{D}_X$ and showing that they don't change when we swap the actions. We will use the same strategy to show $(V^\vee\otimes\mathcal{D}_X^{K^{-1}})^\sigma = V^\vee\otimes\mathcal{D}_X^{K^{-1}}$. Take $V\otimes K_X^{-1}$ as a generating subsheaf. Let ξ^L, ξ^R denote the images of $\xi\in Vect_X$ under the left & twisted right actions of $\mathcal{D}_X$. Then for $\nu\in V^\vee$ & $\eta\in K_X^{-1}$ have $\xi^L\cdot(\nu\otimes\eta) = (\xi\nu)\otimes\eta + \nu\otimes\xi\eta$ (w. $\xi\eta = \xi\otimes\eta\in\mathcal{D}_X\otimes K_X^{-1}$).
Now $\xi^R\cdot(\nu\otimes\eta) = \nu\otimes\xi^R\eta$. To compute $\xi^R\eta$ we note that for any $\omega\in K_X$ we must have $(\eta\omega)\xi = -\eta L_\xi\omega - (\xi^R\eta)\omega \Leftrightarrow \xi(\eta\omega) - L_\xi(\eta\omega)$
$= -\eta L_\xi\omega - (\xi^R\eta)\omega \Leftrightarrow [L_\xi(\eta\omega) = (L_\xi\eta)\omega + \eta(L_\xi\omega)] \Leftrightarrow \xi\eta\omega - (L_\xi\eta)\omega =$

$= (-\xi^R \eta)\omega \Leftrightarrow \xi^R\eta = -\xi\eta + L_\xi \eta$. So the relation becomes: $(\xi^L + \xi^R)(\vartheta\otimes\eta) = (\xi\vartheta)\otimes\eta + \vartheta\otimes L_\xi\eta$ - and it is symmetric.

So $R\underline{Hom}_{\mathcal{D}_X}(?, V^\vee\otimes \mathcal{D}_X^{k-1}) \simeq R\underline{Hom}_{\mathcal{D}_X}(?, (V^\vee\otimes\mathcal{D}_X^{k-1})^{\sigma}) \simeq$ [now tensoring w. $V^\vee$ doesn't interfere w. taking $\underline{Hom}$] $V^\vee\otimes R\underline{Hom}_{\mathcal{D}_X}(?, \mathcal{D}_X^{k-1})$. We conclude that $\mathbb{D}(V\otimes ?) \simeq V^\vee\otimes\mathbb{D}(?)$ $\square$

2) Duality vs Kashiwara's lemma.

Theorem 1: Let $Y\subset X$ be a closed irreducible smooth subvariety of X, and $i: Y\hookrightarrow X$ be the inclusion. Then $i_*\circ\mathbb{D}_Y \simeq \mathbb{D}_X\circ i_*$.

Sketch of proof: we'll produce an iso of functors in the case when X is affine.

To show an isomorphism of functors for affine X we'll show that both sides are $RHom_{D(Y)}$ to the same object in $D^b(D(Y)\text{-}D(X)\text{-bimod})$. Let's start with $i_*\circ\mathbb{D}_Y$.

$$i_*\circ\mathbb{D}_Y = \underbrace{(K_X^{-1}\otimes_{\mathbb{C}[X]} D_{Y\to X}\otimes_{\mathbb{C}[Y]} K_Y)\otimes_{D(Y)}}_{i_*} \underbrace{K_Y^{-1}\otimes_{\mathbb{C}[Y]} RHom_{D(Y)}(?, D(Y))[\dim Y]}_{\mathbb{D}_Y}$$

$= K_X^{-1}\otimes_{\mathbb{C}[X]} D_{Y\to X}\otimes_{D(Y)} RHom_{D(Y)}(?, D(Y))[\dim Y] =$ [$D_{Y\to X} = \mathbb{C}[Y]\otimes_{\mathbb{C}[X]} D(X)$ is a flat $D(Y)$-module, this follows from its explicit form in etale chart, so $\bullet\otimes_{D(Y)} D_{Y\to X} = \bullet\otimes^L_{D(Y)} D_{Y\to X}$] $= K_X^{-1}\otimes_{\mathbb{C}[X]} RHom_{D(Y)}(?, D(Y)\otimes^L_{D(Y)} D_{Y\to X})[\dim Y] = K_X^{-1}\otimes_{\mathbb{C}[X]} RHom_{D(Y)}(?, D_{Y\to X}[\dim Y])$.

Now we do the same for $\mathbb{D}_X\circ i_*$:

$\mathbb{D}_X\circ i_* = K_X^{-1}\otimes_{\mathbb{C}[X]} RHom_{D(X)}(i_*(?), D(X))[\dim X] = K_X^{-1}\otimes_{\mathbb{C}[X]} RHom_{D(X)}(D_{X\leftarrow Y}\otimes^L_{D(Y)} ?, D(X)[\dim X]) = K_X^{-1}\otimes_{\mathbb{C}[X]} RHom_{D(Y)}(?, RHom_{D(X)}(D_{X\leftarrow Y}, D(X)[\dim X]))$.

So we need to establish an isomorphism

$\mathcal{D}_{Y\to X} \xleftarrow{\sim} R\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{D}_{X\leftarrow Y}, \mathcal{D}(X))[\operatorname{codim}_X Y]$

of objects in $D^b(\mathcal{D}(Y)\text{-}\mathcal{D}(X)\text{-bimod})$. It will be more convenient to prove an equivalent iso in $D^b(\mathcal{D}(X)\text{-}\mathcal{D}(Y)\text{-bimod})$:

$$\mathcal{D}_{X\leftarrow Y} \xleftarrow{\sim} R\mathrm{Hom}_{\mathcal{D}(X)^{opp}}(\mathcal{D}_{Y\to X}, \mathcal{D}(X))[\operatorname{codim}_X Y] \qquad (1)$$

In the proof of (1), we first assume that X is the total space of a vector bundle, say $\mathcal{E}$ ($= T_Y X$, the normal bundle). Then $\mathbb{C}[X] = S_{\mathbb{C}[Y]}(\mathcal{E}^\vee)$ & we can write the Kostul resolution of $\mathbb{C}[Y]$:

$$\to \Lambda^i \mathcal{E}^\vee \otimes_{\mathbb{C}[Y]} \mathbb{C}[X] \to \ldots \to \mathcal{E}^\vee \otimes_{\mathbb{C}[Y]} \mathbb{C}[X] \to \mathbb{C}[X]$$

Then $R\mathrm{Hom}_{\mathcal{D}(X)^{opp}}(\mathcal{D}_{Y\to X}, \mathcal{D}(X)) = R\mathrm{Hom}_{\mathcal{D}(X)^{opp}}(\mathbb{C}[Y]\otimes^L_{\mathbb{C}[X]}\mathcal{D}(X), \mathcal{D}(X))$

$= R\mathrm{Hom}_{\mathbb{C}[X]}(\mathbb{C}[Y], \mathcal{D}(X)) = \left(\mathcal{D}(X) \to \mathcal{E}\otimes_{\mathbb{C}[Y]}\mathcal{D}(X) \to \ldots \to \ldots \to \Lambda^m \mathcal{E}\otimes_{\mathbb{C}[Y]}\mathcal{D}(X)\right)$,

where $m = \operatorname{codim}_X Y$. Our final expression is the Koszul complex for the $\mathbb{C}[Y]$-module $\Lambda^m\mathcal{E}\otimes_{\mathbb{C}[Y]}(\mathbb{C}[Y]\otimes_{\mathbb{C}[X]}\mathcal{D}(X))$, hence is a resolution of that module. Note that $\Lambda^m\mathcal{E} = \Lambda^m(T_X|_Y/T_Y) = K_X^{-1}|_Y\otimes K_Y$. So, we have an isomorphism of $\mathbb{C}[Y]$-modules $R\mathrm{Hom}_{\mathcal{D}(X)^{opp}}(\mathcal{D}_{Y\to X}, \mathcal{D}(X)) \xrightarrow{\sim} \mathcal{D}_{X\to Y}$. That it is $\mathcal{D}(X)$ & $\mathcal{D}(Y)$-linear can be checked in local coordinate charts and is left as a (premium) exercise.

Now we sketch how to deal with the case of general X. Let X_0 be the total space of $T_Y X$. Let I, I_0 denote the ideals of Y in $\mathbb{C}[X], \mathbb{C}[X_0]$. Then we have an algebraic version of the tubular neighborhood theorem: the completions $\varprojlim \mathbb{C}[X]/I^k$, $\varprojlim \mathbb{C}[X_0]/I_0^k$ are isomorphic. Let $\hat{A}$ denote this algebra & $\hat{X} := \operatorname{Spec}(\hat{A})$. We have natural morphisms $\hat{X} \to X, X_0$, they are etale. We can still consider the algebra $\mathcal{D}(\hat{X})$, bimodules $\mathcal{D}_{Y\to\hat{X}}$, $\mathcal{D}_{\hat{X}\to Y}$ etc.

Then we deduce (1) for $\hat{X}$ from (1) for X_0, and then (1) for X from (1) for $\hat{X}$. $\square$

3) Classification of simple holonomic D-modules.

Here is the main result.

Theorem 2: 1) Let Z be an irreducible <u>locally</u> closed smooth subvariety in X, V an irreducible $\mathcal{O}$-coherent $\mathcal{D}_Z$-module. Let $U\subset X$ be open w. $U\cap\overline{Z}=Z$ & $i: Z\hookrightarrow U$ be the closed embedding, and $j: U\hookrightarrow X$ be the open embedding. Then there is a unique irreducible holonomic $\mathcal{D}_X$-module $\mathcal{F}$ s.t. $j^*\mathcal{F}\simeq i_*V$. Denote this $\mathcal{F}$ be $IC(Z,V)$ ("IC" from "intersection cohomology")

2) $\forall$ simple holonomic $\mathcal{F}$ $\exists$ Z, V as above w. $\mathcal{F}\simeq IC(Z,V)$.

3) Let Z_i, V_i, $i=1,2$, be as above. Then TFAE

a) $IC(Z_1,V_1)\simeq IC(Z_2,V_2)$

b) $Z:=Z_1\cap Z_2$ is open & dense in both Z_1, Z_2 & $V_1|_Z\simeq V_2|_Z$

4) $\mathbb{D}(IC(Z,V))\simeq IC(Z,V^\vee)$.

3.1) j_* of simple holonomic module.

Large part of Thm 2 has to do with extending simple holonomic modules from U to X. We start with a fact to be proved in the next lecture.

Fact: j_* maps $Hol(\mathcal{D}_U)$ to $Hol(\mathcal{D}_X)$.

Proposition 4: a) The adjunction counit $j^*j_*\to id_{Coh(\mathcal{D}_U)}$ is an iso.

b) Let $\mathcal{F}_U$ be a simple holonomic $\mathcal{D}_U$-module. Then there is a unique irreducible submodule $\mathcal{F}\subset j_*\mathcal{F}_U$. We have $j^*\mathcal{F}\simeq\mathcal{F}_U$ & $j^*(j_*\mathcal{F}_U/\mathcal{F})=0$.

Proof: a) is a general result about quasi-coherent sheaves. To prove b) note that $j_* \mathcal{F}_U$ is holonomic (by Fact) hence has finite length. So $\exists$ an irreducible submodule. To prove the uniqueness, note that $Hom_{\mathcal{D}_X}(\mathcal{F}, j_* \mathcal{F}_U) \simeq Hom_{\mathcal{D}_U}(j^* \mathcal{F}, \mathcal{F}_U)$. Assume that we have two distinct (possibly isomorphic) irreducible submodules $\mathcal{F}, \mathcal{F}' \subset j_* \mathcal{F}_U$. Then $\mathcal{F} \oplus \mathcal{F}' \hookrightarrow j_* \mathcal{F}_U \Rightarrow [j^* \text{ is exact}]\ j^* \mathcal{F} \oplus j^* \mathcal{F}' \hookrightarrow j^* j_* \mathcal{F}_U = [(a)] = \mathcal{F}_U$. Since $\mathcal{F}_U$ is irreducible, we conclude that, say, $j^* \mathcal{F}' = 0$. But then $Hom_{\mathcal{D}_X}(\mathcal{F}', j_* \mathcal{F}_U) = 0$, a contradiction.

The isomorphism $j^* \mathcal{F} \simeq \mathcal{F}_U$ has been established in the proof of uniqueness of $\mathcal{F}$. Then $j^*(j_* \mathcal{F}_U / \mathcal{F}) = 0$ by (a). $\square$

3.2) Functors $j_!$ & $j_{!*}$.

Definition: $j_! := \mathbb{D}_X \circ j_* \circ \mathbb{D}_U : Hol(\mathcal{D}_U) \longrightarrow Hol(\mathcal{D}_X)$ (makes sense b/c of the fact).

Properties of $j_!$: **I)** $j^* j_! \simeq id_{Coh(\mathcal{D}_U)}$ - b/c of (a) of Proposition 4 & observation that $j^* \circ \mathbb{D}_X \simeq \mathbb{D}_U \circ j^*$.

II) $j_!$ is right exact - b/c j_* is left exact & $\mathbb{D}_X, \mathbb{D}_U$ are exact & contravariant.

III) We have a functor morphism $j_! \xrightarrow{\alpha} j_*$ w. $j^*(\alpha) = id$: follows from II) & $Hom_{\mathcal{D}_X}(?, j_*(\cdot)) = Hom_{\mathcal{D}_U}(j^*(?), \cdot)$.

Definition: Define $j_{!*} : Hol(\mathcal{D}_U) \longrightarrow Hol(\mathcal{D}_X)$ as $\operatorname{im} \alpha$.

This functor is neither left nor right exact but has the following important property. See example on last page

Proposition 5: Let $\mathcal{F}_U \in Hol(\mathcal{D}_U)$ be simple. Then $j_{!*}(\mathcal{F}_U)$ is simple.

Moreover, if $\mathcal{F}' \in Hol(\mathcal{D}_X)$ is simple & $j^*\mathcal{F}' \simeq \mathcal{F}_U$, then $\mathcal{F}' \simeq j_{!*}(\mathcal{F}_U)$.
Proof: Let $\mathcal{F}$ be the unique simple submodule of $j_*\mathcal{F}_U$, so that $j^*(j_*\mathcal{F}_U/\mathcal{F})=0$. We claim that $j_!\mathcal{F}_U \twoheadrightarrow \mathcal{F}$ & $j^*(\ker)=0$. Since both $\mathbb{D}_X, \mathbb{D}_U$ are contravariant abelian equivalences, we see that $j_!\mathcal{F}_U$ has a unique simple quotient, say $\mathcal{F}^\circ$. Moreover, $\mathbb{D}_U \circ j^* = j^* \circ \mathbb{D}_X$
$\Rightarrow j^*(\ker[j_!\mathcal{F}_U \twoheadrightarrow \mathcal{F}^\circ])=0$. It follows that $\alpha_{\mathcal{F}_U}: \mathcal{F}^\circ \to \mathcal{F}$.
And $j^*(\alpha_{\mathcal{F}_U}) = id_{\mathcal{F}_U} \Rightarrow \alpha_{\mathcal{F}_U}: \mathcal{F}^\circ \xrightarrow{\sim} \mathcal{F}$. We conclude that $j_{!*}(\mathcal{F}_U)\ (\simeq \mathcal{F})$ is simple.

Now $j^*\mathcal{F}' = \mathcal{F}_U \Rightarrow Hom_{\mathcal{D}_X}(\mathcal{F}', j_*\mathcal{F}_U) \neq 0$. If $\mathcal{F}'$ is simple it must be isomorphic to $\mathcal{F}$ □

Rem: The functor $j_{!*}$ is called the intermediate (or minimal) extension.

3.3) Proof of Theorem 2.

1): Note that i_*V is simple thx to Kashiwara's lemma. Now 1) follows from Proposition 5.

2): We had this argument as a motivation to consider push-forwards: Let $\mathcal{F} \in Hol(\mathcal{D}_X)$ be simple. Let Z' be an irreducible component of $Supp_X(\mathcal{F})$. Let $U' \subset X$ be open such that $U' \cap Supp_X(\mathcal{F}) = Z'^{reg}$. Then $Supp_{U'}(j'^*\mathcal{F}) = Z'^{reg}$. Let $\mathcal{G} \in Hol(\mathcal{D}_{Z'})$ be the image of $j'^*(\mathcal{F})$ under the Kashiwara equivalence $Hol(\mathcal{D}_{Z'}) \xrightarrow{\sim} Hol_{Z'}(\mathcal{D}_{U'})$. Then $Supp_{Z'^{reg}}(\mathcal{G}) = Z'^{reg}$, hence $Z'^{reg} \subset T^*Z'^{reg}$ is an irreducible component. We can find an open subvariety $Z \hookrightarrow Z'^{reg}$ s.t $SS(\mathcal{G}|_Z) = Z$. It follows that $V := \mathcal{G}|_Z$ is $\mathcal{O}$-coherent. By the construction, $j^*\mathcal{F} \simeq i_*V$. It remains to show that V is irreducible. Assume the contrary, let $V_0 \subsetneq V$ be an irreducible submodule. Then we have a nonzero monomorphism $i_* V_0 \hookrightarrow i_* V \rightsquigarrow$ nonzero homomorphism $j_{!*}(i_* V_0) \to j_{!*}(i_* V)$. It's nonzero b/c applying j^* we get $i_* V_0 \hookrightarrow i_* V$ back. For the same reason, it's not an isomorphism. But $\mathcal{F} = j_{!*}(i_* V)$ is simple. A contradiction.

3) Set $\mathcal{F}_k := IC(Z_k, V_k)$. Assume $\mathcal{F}_1 \simeq \mathcal{F}_2$. Note that $\bar{Z} = \mathrm{Supp}_X(\mathcal{F}_k) \Rightarrow \bar{Z}_1 = \bar{Z}_2 \Rightarrow Z := Z_1 \cap Z_2$ is open and dense in both Z_1, Z_2. Let U be open in X s.t $U \cap \bar{Z}_i = Z$. Then $\mathcal{F}_i|_U = i_*(V_k|_Z)$. From Kashiwara's lemma we deduce $V_1|_Z \simeq V_2|_Z$.

Now suppose we have Z & $V := V_k|_Z$ as in the statement. We claim $\mathcal{F}_k \simeq IC(Z, V)$. This is a direct consequence of the uniqueness part in 1).

4) We have $IC(Z,V) = j_{!*}(i_* V)$. Then $\mathbb{D}\, IC(V,Z) = [\mathbb{D}_X j_{!*} \simeq j_{!*} \mathbb{D}_U$ - exercise$] = j_{!*} \mathbb{D}_U i_* V = [$section 2: $\mathbb{D} i_* = i_* \mathbb{D}] = j_{!*} i_* \mathbb{D} V = [$section 1$] = j_{!*} i_* V^\vee = IC(Z, V^\vee)$.

Example for $j_!, j_{!*}$: Let $X = \mathbb{C}$, $U = \mathbb{C}^\times$, $\mathcal{F}_U = \mathcal{O}_U$ $(= \mathbb{C}[x^{\pm 1}])$, then $j_* \mathcal{F}_U = \mathbb{C}[x^{\pm 1}]$ fits into an exact sequence

$$0 \to \mathcal{O}_X \to j_* \mathcal{O}_U \to \delta_0 \to 0$$

Note that $\mathbb{D}_U \mathcal{O}_U = \mathcal{O}_U$, $\mathbb{D}_X \mathcal{O}_X = \mathcal{O}_X$, $\mathbb{D}_X \delta_0 = \delta_0$. So we get the following exact sequence for $j_! \mathcal{O}_U$:

$$0 \to \delta_0 \to j_! \mathcal{O}_U \to \mathcal{O}_X \to 0$$

End $\alpha_{\mathcal{O}_U}$ is the composition $j_! \mathcal{O}_U \twoheadrightarrow \mathcal{O}_X \hookrightarrow j_* \mathcal{O}_U$ hence $j_{!*} \mathcal{O}_U = \mathcal{O}_X$.