0) Equivariant coherent sheaves.
1) Equivariant $\mathcal{D}$-modules.

0) Throughout the lecture, X is an algebraic variety & H is an algebraic group that acts on X algebraically. Recall that when we say that H is an algebraic group, we mean that H is a group & an algebraic variety and these two structures are compatible in the sense that the multiplication map $H\times H \longrightarrow H$ & the inverse map $H\to H$ are morphisms. Examples are given by $G = GL_n(\mathbb{C}), SL_n(\mathbb{C}), O_n(\mathbb{C})$ etc. Similarly, an action $G \curvearrowright X$ is called algebraic if the action map $G\times X\to X$ is a morphism.

1.1) Definition of an equivariant sheaf.

Let $\mathcal{F}$ be a q.coh't sheaf (of $\mathcal{O}_X$-modules) on X. We say that $\mathcal{F}$ is <u>G-equivariant</u> if G acts on $\mathcal{F}$ so that the structure map $\mathcal{O}_X\otimes\mathcal{F}\longrightarrow\mathcal{F}$ is G-equivariant and the action is algebraic in a suitable sense. This is formalized via commutativity of certain diagrams, which we will not need explicitly. Here are two examples/special cases that explain what's going on.

Example 1: Let X be affine, and $A:=\mathbb{C}[X]$. Then G acts on A by algebra automorphisms and this action is rational: every $a\in A$ is contained in a finite dimensional subrep'n, say $V \subset A$, and the representation of G in V is algebraic. To give a quasi-coherent sheaf on X is the same thing as to give an A-module M. The equivariance condition translates to the condition that M is equipped with a rational rep'n of G s.t $A \otimes_{\mathbb{C}} M \to M$ is G-equivariant.

This definition can be extended to the case when X is not necessarily affine but every point has an open G-stable affine neighborhood - by gluing.

Example 2: Let $\mathcal{F}$ be a vector bundle. Then we can talk about its total space, say Y. To give a G-equivariant structure on $\mathcal{F}$ is the same as to equip Y with an algebraic G-action subject to the following properties:

- the projection map $\pi: Y \to X$ is G-equivariant.
- the action of G on Y is fiberwise linear: $\forall g \in G, x \in X$, the map $\pi^{-1}(x) \to \pi^{-1}(gx)$ induced by g is linear.

Concrete examples: 1) $\mathcal{O}_X$ has a natural G-equivariant structure. More generally, for a rat'l rep'n V of G, have a natural G-equiv. structure on $\mathcal{O}_X \otimes V$, so that $Y = X \times V$ w. diag. action. So a fixed sheaf can have several non-isomorphic equivariant structures.

2) $Vect_X$ is G-equivariant. So are more gen'l tensor bundles

3) The line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V)$ is $GL(V)$-equivariant

The category of G-equivariant quasi-coherent sheaves will be denoted by $QCoh^G(X)$. We have usual functors on $QCoh^G$; e.g. push-forwards & pullbacks for G-equivariant morphisms.

0.2) Equivariant sheaves on principal G-bundles.

Let's start by recalling a definition of a principal G-bundle. Let X_0 be a variety. By a principal G-bundle on X_0 we mean a pair (X,π) of a variety X w. action of G & G-invariant morphism $\pi: X \to X_0$ s.t.

(a) π is affine & surjective

(b) There is a <u>surjective</u> <u>etale</u> morphism $\rho: \tilde{X}_0 \twoheadrightarrow X_0$ s.t. we have a G-equivariant isomorphism $G\times \tilde{X}_0 \longrightarrow \tilde{X}_0 \times_{X_0} X$ that intertwines the projections $G\times \tilde{X}_0, \tilde{X}_0\times_{X_0} X \twoheadrightarrow \tilde{X}_0$.

Example 1: Let X be affine, G be reductive. The algebra $\mathbb{C}[X]^G$ of G-invariants is known to be finitely generated. Set $X_0 := \operatorname{Spec} \mathbb{C}[X]^G$, so that we have a natural morphism $\pi: X \to X_0$. It is clearly affine and can be shown to be surjective. Moreover, and this is a harder result, if G acts on X freely (= w/o stabilizers), then (b) holds as well.

Example 2: Let G be an algebraic subgroup of another algebraic group, $\hat{G}$. Consider the action of G on $\hat{G}$ by right translations. Then there's a unique algebraic variety structure on $X_0 := \hat{G}/G$ such that the natural map $\pi: \hat{G} \twoheadrightarrow \hat{G}/G$ is a morphism. The pair $(\hat{G},\pi)$ is a principal G-bundle on X_0.

It turns out that the categories $QCoh^G(X)$ & $QCoh(X_0)$ are equivalent. Namely, note that G naturally acts on $\pi^*\mathcal{F}_0$ for $\mathcal{F}_0 \in QCoh(X_0)$. So we can view π^* as a functor $QCoh(X_0) \longrightarrow QCoh^G(X)$. Conversely, G acts on $\pi_*(\mathcal{F})$ for

$\mathcal{F} \in QCoh^G(X)$ by automorphisms of an $\mathcal{O}_{X_0}$-module.

Proposition 1: The functors $\pi^*(\cdot) : QCoh(X_0) \rightleftarrows QCoh^G(X) : \pi_*(\cdot)^G$ are mutually quasi-inverse equivalences.

Example: Let's see use a closely related equivalence to describe the category $Coh^G(G/H)$ for an algebraic subgroup $H \subset G$. Namely, if we let π to denote the natural projection $G \to G/H$, then π^* gives an equivalence $Coh^G(G/H) \xrightarrow{\sim} Coh^{G\times H}(G)$. Now let $\rho: G \twoheadrightarrow pt$, a G-equivariant map. It gives rise to an equivalence $\rho^*: Coh^H(pt) \xrightarrow{\sim} Coh^{G\times H}(G)$. Note that $Coh^H(pt)$ is nothing else but the category of finite dimensional rational reps of H, $Rep(H)$. So we get a composed equivalence

$$Coh^G(G/H) \xrightarrow{\sim} Coh^{G\times H}(G) \xrightarrow{\sim} Rep(H)$$

Exercise: Show that this equivalence sends $V \in Coh^G(G/H)$ (which is forced to be a vector bundle) to its fiber at $\pi(1)$ with its natural representation of H.

0.3) Lie algebra actions.

The G-action on X gives rise to a Lie algebra homomorphism $\mathfrak{g} \to Vect(X)$, the Lie algebra of derivations of $\mathcal{O}_X$, to be denoted by $\xi \mapsto \xi_X$. If $\mathcal{F}$ is a G-equivariant q. coh't sheaf on X we have a Lie algebra homomorphism $\xi \mapsto \xi_{\mathcal{F}} : \mathfrak{g} \to End_{\mathbb{C}}(\mathcal{F})$ also coming from the G-action on $\mathcal{F}$. These two homomorphisms are compatible via $\xi_{\mathcal{F}}(am) = \xi_X(a)m + a\,\xi_{\mathcal{F}}(m)$

Example: $\cdot$ Let X be affine. Then $\mathbb{C}[X]$ & $\Gamma(\mathcal{F})$ are rational representations of G, hence have natural representations of

$\mathfrak{g}$, which are the homomorphisms above.

· For $\mathcal{F} = \mathrm{Vect}_X$ & more general tensor bundles we have $\xi_{\mathcal{F}} = L_\xi$, the Lie derivative.

1) Equivariant $\mathcal{D}$-modules

Now we assume X is smooth. We note that $\mathcal{D}_X$ is G-equivariant as the quotient of $T_{\mathcal{O}_X}(\mathrm{Vect}_X)$. We have $\xi_{\mathcal{D}_X} = [\xi_X, \cdot]$.

1.1) Definitions.

Definition 1: By a weakly equivariant $\mathcal{D}_X$-module we mean a G-equivariant q. coh't sheaf $\mathcal{M}$ that is also a $\mathcal{D}_X$-module s.t. $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}$ is G-equivariant.

Examples: $\mathcal{D}_X$ and $\mathcal{O}_X$.

In particular, every $\xi \in \mathfrak{g}$ gives an operator $\xi_{\mathcal{M}}$ on $\mathcal{M}$. On the other hand, have $\xi_X \in \Gamma(\mathrm{Vect}_X) \subset \Gamma(\mathcal{D}_X)$.

Definition 2: We say $\mathcal{M}$ is (strongly) G-equivariant $\mathcal{D}_X$-module if it's weakly equivariant & $\xi_{\mathcal{M}} m = \xi_X m$ $\forall$ local section m of $\mathcal{M}$.

Examples: · $\mathcal{O}_X$ is strongly equivariant.

· $\mathcal{D}_X$ is not equivariant b/c $\xi_{\mathcal{D}_X} = [\xi_X, \cdot] \neq \xi_X \cdot$

· The quotient $R_X := \mathcal{D}_X / \mathcal{D}_X \mathrm{Span}(\xi_X \mid \xi \in \mathfrak{g})$ is strongly equiv't.

Rem: · if G is connected, then there's at most one equivariant structure on each $\mathcal{M}$ b/c $\xi_{\mathcal{M}}$'s are uniquely specified

· We can also define G-equivariant right $\mathcal{D}_X$-modules, the condition becomes $\xi_{\mathcal{M}} m = -\xi_X m$.

Important exercise 1: Tensoring w. $K_X \in \mathrm{Coh}^G(X)$ defines an equivalence between the categories of left & right equivariant $\mathcal{D}_X$-mod's.

We use the notation $Coh^G(\mathcal{D}_X)$ for the category of (strongly) equivariant $\mathcal{D}_X$-modules.

1.2) Pullbacks & push-forwards.

Proposition 2: Let $f: Y \to X$ be a G-equivariant morphism. Then f^* gives a functor $QCoh^G(\mathcal{D}_X) \to QCoh^G(\mathcal{D}_Y)$.

Proof: Recall that $f^*\mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{f^\cdot(\mathcal{D}_X)} f^\cdot(\mathcal{M})$, where $\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{f^\cdot(\mathcal{O}_X)} f^\cdot(\mathcal{D}_X)$. If $\mathcal{M}$ is G-equivariant, then $f^*\mathcal{M}$ acquires a natural G-equivariant structure. We need to show that ξ_Y acts as $\xi_{f^*\mathcal{M}}$ on $f^*\mathcal{M}$: $\xi_Y(a\otimes m) = \xi_{f^*\mathcal{M}}(a\otimes m)$ for local sections a of $\mathcal{D}_{Y\to X}$ & m of $\mathcal{M}$. We have $\xi_{f^*\mathcal{M}}(a\otimes m) = \xi_{\mathcal{D}_{Y\to X}} a\otimes m + a\otimes \xi_{\mathcal{M}} m$. Analogously to $\xi_{\mathcal{D}_X} = [\xi_X, \cdot]$, we have:

$$\xi_{\mathcal{D}_{Y\to X}} a = \xi_Y a - a\xi_X$$

Also $\xi_{\mathcal{M}} m = \xi_X m$. Now we use the identity $a\xi_X \otimes m = a\otimes \xi_X m$ to conclude that $\xi_Y(a\otimes m) = \xi_{f^*\mathcal{M}}(a\otimes m)$ $\square$

Important exercise 2: Suppose f is affine or is an open embedding. Then f_* sends $QCoh^G(\mathcal{D}_Y)$ to $QCoh^G(\mathcal{D}_X)$.

Lemma 1: Let $i: Y \hookrightarrow X$ be a closed embedding of a smooth G-stable subvariety. Then i_* is an equivalence $Coh^G(\mathcal{D}_Y) \xrightarrow{\sim} Coh^G_Y(\mathcal{D}_X)$.

Proof: It's enough to show this for right $\mathcal{D}$-modules. Using the proof of Proposition 2, we see that $Coh^G(\mathcal{D}_Y^{opp}) \to Coh^G_Y(\mathcal{D}_X^{opp})$. Recall that, in the non-equivariant setting, a quasi-inverse of i_* is given by $i^!(\mathcal{N}) = \underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{N})$, i.e. $\mathcal{N}$ is sent to its subsheaf of all sections annihilated by the ideal sheaf I_Y. Note that ξ_Y coincides with ξ_X at the points of Y. So ξ_Y acts on $i^!(N)$ in the same way as ξ_X does. Therefore if N is equivariant, then so is $i^!(N)$. We see that i_* is indeed an equivalence $Coh^G(\mathcal{D}_Y) \xrightarrow{\sim} Coh_Y^G(\mathcal{D}_X)$ $\square$

Corollary 1: The irreducible G-equivariant holonomic $\mathcal{D}$-modules are of the form $IC(Z, V)$, where Z is a locally closed smooth G-irreducible (= G permutes the components transitively) subvariety in X & V is an irreducible G-equivariant $\mathcal{O}$-coherent $\mathcal{D}$-module on Z.

Proof: exercise.

1.3) Equivariant $\mathcal{D}$-modules on principal bundles.

Let X be a principal G-bundle over X_0. We want to compare $Coh^G(\mathcal{D}_X)$ & $Coh(\mathcal{D}_{X_0})$. The result looks in the same way as for $\mathcal{O}_X$-modules.

Theorem 1: The functor π^* is an equivalence $QCoh(\mathcal{D}_{X_0}) \xrightarrow{\sim} QCoh^G(\mathcal{D}_X)$. A quasi-inverse equivalence is given by $\pi_\bullet(?)^G$, where $\pi_\bullet$ is the usual sheaf-theoretic push-forward.

Modulo Proposition 1, there are two things we need to check in order to prove Thm 1:

(a) For $M \in QCoh^G(\mathcal{D}_X)$, $\pi_\bullet(M)^G$ has a nat'l $\mathcal{D}_{X_0}$-module str're

(b) The functor $\pi^*: QCoh(\mathcal{D}_{X_0}) \longrightarrow QCoh^G(\mathcal{D}_X)$ is left adj't to $\pi_\bullet(?)^G$.

Proposition 1 then shows that π^* & $\pi_\bullet(?)^G$ are mutually quasi-inverse. In fact, techniques used to prove (a) & (b) are also used to prove Proposition 1.

The proofs of (a), (b) use an important construction known as quantum Hamiltonian reduction

1.4) Quantum Hamiltonian reduction

Our setting here is as follows. Let $\mathcal{A}$ be an associative unital algebra and G be an algebraic group acting on $\mathcal{A}$ rationally & by algebra automorphisms.

Definition: This action is called Hamiltonian if $\exists$ a G-equivariant linear map $\Phi: \mathfrak{g} \to \mathcal{A}$ (quantum comoment map) s.t. $[\Phi(\xi), \cdot] = \xi_{\mathcal{A}}\ \forall \xi \in \mathfrak{g}$.

Exercise: Φ is a Lie algebra homomorphism.

Example: Let X be an affine variety w. G-action. Then the G-action on $\mathcal{D}(X)$ is Hamiltonian w. $\Phi(\xi) = \xi_X$.

Similarly to the $\mathcal{D}$-module case we can talk about strongly G-equivariant $\mathcal{A}$-modules, the corresponding category will be denoted by $\mathcal{A}\text{-Mod}^G$. An example of an equivariant module is provided by $R := \mathcal{A}/\mathcal{A}\Phi(\mathfrak{g})$.

Important exercise 3: (i) R^G has well-def'd associative product given by $(a + \mathcal{A}\Phi(\mathfrak{g}))(b + \mathcal{A}\Phi(\mathfrak{g})) := ab + \mathcal{A}\Phi(\mathfrak{g})$.

(ii) R is an $\mathcal{A}$-R^G-bimodule.

(iii) $R \otimes_{R^G} \cdot$ is a functor $R^G\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}^G$

(iv) $?^G$ is a functor $\mathcal{A}\text{-Mod}^G \longrightarrow R^G\text{-Mod}$

(v) We have $?^G = \mathrm{Hom}_{\mathcal{A}\text{-Mod}^G}(R, ?)$. In particular, $?^G$

is right adjoint to $R\otimes_{R^G}\bullet$.

The algebra R is called the quantum Hamiltonian reduction of $\mathcal{A}$. We also say that the functor $?^G$ is a Hamiltonian reduction functor.

1.5) Proof of Theorem 1.

The following two claims imply (a), (b) thanks to Important Exercise 3:

(a'). For X affine, have nat'l isomorphism $\mathcal{D}(X_0)\xrightarrow{\sim}(R_X)^G$.

(b') For X affine, have a nat'l isomorphism of functors: $\pi^*(?)\simeq R_X\otimes_{R_X^G}?$

Naturality here implies, in particular, that the functors are compatible w. gluing.

Proof of (a'): The proof is in two parts: we first construct an algebra homomorphism $\mathcal{D}(X_0)\longrightarrow R_X^G$ & then check it's an isomorphism. First we need a lemma.

Lemma 2: We have an isomorphism (w. $\varphi(\xi)=\xi_X$)
$$\alpha^X:[\mathrm{Vect}(X)/\mathbb{C}[X]\varphi(\mathfrak{g})]^G\xrightarrow{\sim}\mathrm{Vect}(X_0)$$

Proof of Lemma 2: Consider the inclusion $\mathbb{C}[X_0]=\mathbb{C}[X]^G\hookrightarrow\mathbb{C}[X]$. Then $\mathbb{C}[X]\varphi(\mathfrak{g})$ annihilates $\mathbb{C}[X_0]$. So $v\in[\mathrm{Vect}(X)/\mathbb{C}[X]\varphi(\mathfrak{g})]^G$ gives a well-defined map $\alpha^X(v):\mathbb{C}[X_0]\to\mathbb{C}[X_0]$ (not just to $\mathbb{C}[X]$ b/c v is G-invariant). It's easy to see that $\alpha_X(v)$ is derivation. This is the map we need.

Now we prove α^X an isomorphism. Let $\tilde{X}_0\twoheadrightarrow X_0$ be a surj've etale morphism as in (6) of 0.2. Then

$$\mathrm{Vect}(\tilde{X}_0) \simeq \mathbb{C}[\tilde{X}_0] \otimes_{\mathbb{C}[X_0]} \mathrm{Vect}(X_0), \quad \mathrm{Vect}(\tilde{X}) \simeq \mathbb{C}[\tilde{X}_0] \otimes_{\mathbb{C}[X_0]} \mathrm{Vect}(X)$$

$$[\mathrm{Vect}(\tilde{X})/\mathbb{C}[\tilde{X}]\varphi(\mathfrak{g})]^G \simeq \mathbb{C}[\tilde{X}_0] \otimes_{\mathbb{C}[X_0]} [\mathrm{Vect}(X)/\mathbb{C}[X]\varphi(\mathfrak{g})]^G$$

Moreover, $d^{\tilde{X}}$ is obtained from d^X by base change from $\mathbb{C}[X_0]$ to $\mathbb{C}[\tilde{X}_0]$ as well. The $\mathbb{C}[X_0]$-module $\mathbb{C}[\tilde{X}_0]$ is faithfully flat ("faithfully" means that $\mathbb{C}[\tilde{X}_0]\otimes_{\mathbb{C}[X_0]}\cdot$ doesn't kill nonzero modules). So to check that d^X is an isomorphism, it's enough to show $d^{\tilde{X}}$ is. But $\tilde{X} = G \times \tilde{X}_0$, $\mathbb{C}[\tilde{X}]\varphi(\mathfrak{g})$ consists precisely of vector fields tangent to fibers of $G \times \tilde{X}_0 \to \tilde{X}_0$ and the claim that $d^{\tilde{X}}$ is an isomorphism follows $\square$

• Construction of $\mathcal{D}(X_0) \to R_X^G$: The algebra $\mathcal{D}(X_0)$ is generated by $\mathbb{C}[X_0]$, $\mathrm{Vect}(X_0)$. On $\mathbb{C}[X_0]$ our homomorphism is $\mathbb{C}[X_0] \hookrightarrow \mathcal{D}(X)^G \longrightarrow R_X^G$. On $\mathrm{Vect}(X_0)$ it is:

$$\mathrm{Vect}(X_0) \xrightarrow{(d^X)^{-1}} [\mathrm{Vect}(X)/\mathbb{C}[X]\varphi(\mathfrak{g})]^G \longrightarrow R_X^G.$$

To show that these two maps indeed lead to an algebra hom'm, one needs to check relations, which is left as an exercise

• Proving $\mathcal{D}(X_0) \xrightarrow{\sim} (R_X)^G$ is similar to the proof of that part in Lemma 2 - and is again left as an exercise. $\square$

Proof of (6'): We have two $\mathcal{D}(X)$-$\mathcal{D}(X_0)$-bimodules: $\mathcal{D}_{X\to X_0} = \mathbb{C}[X]\otimes_{\mathbb{C}[X_0]}\mathcal{D}(X_0)$, R_X. The $\mathcal{D}(X)$-module $\mathcal{D}_{X\to X_0}$ is generated by 1 & $\varphi(\mathfrak{g})1 = 0$ b/c the image of ξ_X in $\mathbb{C}[X]\otimes_{\mathbb{C}[X_0]}\mathrm{Vect}(X_0)$ is 0. So we get a $\mathcal{D}(X)$-linear map $R_X \longrightarrow \mathcal{D}_{X\to X_0}$. We need to check ~~that~~:

(i) This map is an isomorphism,

(ii) and that it's $\mathcal{D}(X_0)$-linear.

To prove (i) we can replace (X, X_0) w. $(\tilde{X}, \tilde{X}_0)$. We have $\mathcal{D}_{\tilde{X}\to\tilde{X}_0} = \mathbb{C}[\tilde{X}_0]\otimes_{\mathbb{C}[X_0]}\mathcal{D}_{X\to X_0}$ & same for $R_{\tilde{X}}$ vs R_X. Next, $\mathcal{D}(\tilde{X}) = \mathcal{D}(G)\otimes\mathcal{D}(\tilde{X}_0)$, $\mathcal{D}_{\tilde{X}\to\tilde{X}_0} = \mathbb{C}[G]\otimes\mathcal{D}(X_0)$, $R_{\tilde{X}} = [\mathcal{D}(G)/\mathcal{D}(G)\varphi(\mathfrak{g})]\otimes\mathcal{D}(\tilde{X}_0) = \mathbb{C}[G]\otimes\mathcal{D}(\tilde{X}_0)$ & $R_{\tilde{X}}\longrightarrow\mathcal{D}_{\tilde{X}\to\tilde{X}_0}$ is the identity.

To prove (ii) we notice that $\mathcal{D}_{X\to X_0}$ is flat over $\mathbb{C}[X]$ b/c $\mathcal{D}(X_0)$ is flat over $\mathbb{C}[X_0]$. So R_X is also flat over $\mathbb{C}[X]$. So $\mathcal{D}_{X\to X_0}\hookrightarrow\mathcal{D}_{\tilde{X}\to\tilde{X}_0}$, $R_X\hookrightarrow R_{\tilde{X}}$ and because of this it's enough to check (ii) for $(\tilde{X}, \tilde{X}_0)$, where it's clear. $\square$