1) Equivariant $\mathcal{D}$-modules vs finite # of orbits.
2) Applications to Representation theory.

1.1) Equivariant $\mathcal{D}$-modules on G/H. Let G be an algebraic group, $H\subset G$ its closed subgroup. Let $H^\circ\subset H$ denote the connected (=irreducible) component of 1, this is an algebraic normal subgroup & H/H° is a finite subgroup

Theorem 1: We have a category equivalence $Coh^G(G/H)\xrightarrow{\sim} Rep(H/H^\circ)$.

Proof: Recall that $\pi: G\twoheadrightarrow G/H$ is a principal H-bundle. Hence $\pi^*: Coh(\mathcal{D}_{G/H})\xleftrightarrow{\sim} Coh^H(\mathcal{D}_G): \pi_\bullet(?)^H$. In fact, the functors $\pi^*, \pi(?)^H$ lift to functors between $Coh^G(\mathcal{D}_{G/H}), Coh^{G\times H}(\mathcal{D}_G)$ (more precisely, they lift to functors between the categories of weakly G-equivariant modules - and then restrict to subcategories of strongly equivariant modules). This an exercise. The lifts are again mutually quasi-inverse equivalences. Similarly, $Coh^{G\times H}(\mathcal{D}_G)\xleftrightarrow{\sim} Coh^H(\mathcal{D}_{pt})$. What remains to show is that $Coh^H(\mathcal{D}_{pt})\xleftrightarrow{\sim} Rep(H/H^\circ)$. For this we notice that for $M\in Coh^H(\mathcal{D}_{pt})$, the equivariance condition just say $\xi_M=0$ $\forall \xi\in\mathfrak{h}$. So H° must act trivially This establishes an equivalence we need. $\square$

Remark: The equivalence in the theorem should be compared to $Coh^G(G/H)\xleftrightarrow{\sim} Rep(H)$. Also note that, for a G-variety X, one can talk about its G-equivariant fundamental group,

$\pi_1^G(X)$ that controls G-equivariant covers. We have $\pi_1^G(G/H)$ $\simeq H/H^\circ$ (exercise). For reasons explained in the previous lecture, we should expect the irreducible $\mathcal{O}$-coherent G-equivariant $\mathcal{D}_{G/H}$-modules to be classified by $\mathrm{Irrep}(\pi_1^G(G/H))$. And this is indeed the case: from $V \in \mathrm{Rep}(H/H^\circ)$, we can explicitly construct the corresponding $\mathcal{D}$-module. This is done as follows: take $\mathcal{O}_{G/H^\circ}$. This is a $G\times H/H^\circ$-equivariant $\mathcal{D}$-module on G/H°. Let $\underline{\pi}: G/H^\circ \twoheadrightarrow G/H$ be the natural map. It's a cover w. Galois group H/H°. Then $\underline{\pi}_*(\mathcal{O}_{G/H^\circ})$ is a $G\times H/H^\circ$-equivariant $\mathcal{D}$-module on G/H. Our equivalence sends V to $(\underline{\pi}_*(\mathcal{O}_{G/H^\circ})\otimes V)^{H/H^\circ}$. In particular, this $\mathcal{D}$-module is $\mathcal{O}$-coherent. So, for $X=G/H$, every G-equivariant coherent $\mathcal{D}$-module is $\mathcal{O}$-coherent and hence holonomic.

1.2) Equivariant $\mathcal{D}$-modules on G-varieties w. fin. many orbits.

This is the case where we can completely describe irreducible G-equivariant $\mathcal{D}$-modules. Let $X=\bigsqcup_{i=1}^k \mathbb{O}_i$ be the decomposition into G-orbits & $\mathbb{O}_i \simeq_G G/H_i$.

Theorem 2: 1) Every coherent G-equivariant $\mathcal{D}_X$-module is holonomic.

2) Irreducible G-equivariant $\mathcal{D}_X$-modules are classified by $\coprod_{i=1}^k \mathrm{Irrep}(H_i/H_i^\circ)$: to $V\in \mathrm{Irrep}(H_i/H_i^\circ)$ we assign $IC(\mathbb{O}_i, \mathcal{M}_V)$, where $\mathcal{M}_V$ is the irreducible $\mathcal{O}$-coherent equivariant $\mathcal{D}$-module on $\mathbb{O}_i$ constructed in the remark.

Proof: 1): We can order the orbits $\mathbb{O}_i$ in such a way that $\overline{\mathbb{O}_i}\supseteq \mathbb{O}_j \Rightarrow i\leq j$. In particular, $\mathbb{O}_1$ is the open orbit. Then

$X_\ell = \bigsqcup_{i=1}^{\ell} \mathbb{O}_i$ is an open G-stable subvariety. We prove $Coh^G(\mathcal{D}_{X_i}) = Hol^G(\mathcal{D}_{X_i})$ by induction on i. The case of $i=1$ follows from the remark above. Now suppose we know $Coh^G(\mathcal{D}_{X_i}) = Hol^G(\mathcal{D}_{X_i})$ and want to prove $Coh^G(\mathcal{D}_{X_i}) = Hol^G(\mathcal{D}_{X_i})$. Let $j: X_i \hookrightarrow X_{i+1}$ denote the inclusion map. Let $\mathcal{F} \in Coh^G(\mathcal{D}_{X_i})$. Then $j^*\mathcal{F} \in Coh^G(\mathcal{D}_{X_i}) = Hol^G(\mathcal{D}_{X_i}) \Rightarrow j_* j^* \mathcal{F} \in Hol^G(\mathcal{D}_{X_i})$ by preservation of holonomicity. The kernel of $\mathcal{F} \longrightarrow j_* j^* \mathcal{F}$ is supported on $X_{i+1} \setminus X_i = \mathbb{O}_{i+1}$. It's G-equivariant. By the equivariant version of Kashiwara's lemma, it's a pushforward of an equivariant coherent $\mathcal{D}$-module from $\mathbb{O}_{i+1}$. The latter must be holonomic hence the kernel is holonomic, and therefore $\mathcal{F}$ is holonomic. This finishes the proof of 1).

2): We know that the irreducible holonomic G-equivariant $\mathcal{D}$-modules on X are precisely $IC(Z, V)$, where Z is a locally closed G-irreducible smooth subvariety in X and V is an irreducible $\mathcal{O}$-coherent equivariant $\mathcal{D}$-module on Z. The subvariety Z has a unique open G-orbit, say $\mathbb{O}_i$. The $\mathcal{D}$-module $V|_{\mathbb{O}_i}$ is still irreducible so comes from $V \in Irrep(H_i/H_i^\circ)$. It's easy to see $(\mathbb{O}_i, V)$ is uniquely recovered from the irreducible $\mathcal{D}$-module. This gives the required bijection. $\square$

Rem: This theorem generalizes to the case when X is a closed (not necessary smooth) subvariety in a smooth variety $\tilde{X}$ w. a G-action, and still the number of G-orbits in X is finite.

2) Applications to Representation theory.

We are going to consider specific examples of $H \curvearrowright X$ (w. fin. many orbits) that are of relevance for Representation theory. There will be two families of examples.

1) G is a connected semisimple algebraic group, $X = G/B$, its flag variety & $H \subset G$ is a connected subgroup acting on X w. finitely many orbits (such subgroups are called "spherical"). Two examples of H we consider are: $H = N$, the unipotent radical of B, and $H = (G^\sigma)^\circ$, where $\sigma: G \to G$ is an involution.

2) $X = \mathcal{N}$ is the nilpotent cone (= the subvariety of all nilpotent matrices) in $\mathfrak{g} = \mathfrak{sl}_n$ & $G = PGL_n$ or SL_n acting by conjugations.

2.1) Localization theorems.

It turns out that the $\mathcal{D}$-modules on G/B are closely related to representations of $\mathfrak{g} = Lie(G)$. This is the content of the localization theorem(s) due to Beilinson & Bernstein.

Recall that a representation of $\mathfrak{g}$ is the same thing as a representation of $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y] \mid x,y \in \mathfrak{g})$, the universal enveloping algebra. The Lie algebra homomorphism $\mathfrak{g} \to Vect(G/B)$ coming from $G \curvearrowright G/B$ extends to a homomorphism of associative algebras $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}(G/B) := \Gamma(\mathcal{D}_{G/B})$. It turns out one can describe $\mathcal{D}(G/B)$ using this homomorphism.

For this we need a description of the center of $\mathcal{U}(\mathfrak{g})$ due to Harish-Chandra. We can decompose $B = T \ltimes N$, where T is a maximal torus and N is the maximal unipotent subgroup. Let $\mathfrak{t}, \mathfrak{n}$ be the Lie algebras. Let $W = N_G(T)/T$ be the Weyl group. Finally, consider the system of positive roots $\Delta^+ \subset \mathfrak{t}^*$ – the weights of T in $\mathfrak{n}$. Set $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$.

Harish-Chandra proved that the center $Z \subset U(\mathfrak{g})$ is identified with the algebra of invariants $\mathbb{C}[\mathfrak{t}^*]^{(W,\cdot)}$ for the shifted action of W on $\mathfrak{t}^*$: $w\cdot\lambda = w(\lambda+\rho)-\rho$. For $z\in Z$, let f_z denote the corresponding element. Note that Z acts by scalars on every irreducible module. For each $\lambda\in\mathfrak{h}^*$ $\exists!$ $\mathfrak{g}$-irrep $L(\lambda)$ w. highest wt. λ. By the constr'n of f_z, $z\in Z$ acts on $L(\lambda)$ by $f_z(\lambda)$.

Set $\mathfrak{m}_0 := \{z\in Z \mid f_z(0)=0\}$ $(= Z\cap U(\mathfrak{g})\mathfrak{g})$, $U_0 := U(\mathfrak{g})/(\mathfrak{m}_0)$. This is the quotient of $U(\mathfrak{g})$ whose repr'n th'y is the most interesting. Here's the first of localization theorems:

Theorem 2: The homomorphism $U(\mathfrak{g}) \longrightarrow \mathcal{D}(G/B)$ descends to an isomorphism $U_0 \xrightarrow{\sim} \mathcal{D}(G/B)$.

We write U_0-mod for the category of finitely generated U_0-modules. We have the global section functor

$$\Gamma: Coh(\mathcal{D}_{G/B}) \longrightarrow U_0\text{-mod}.$$

As with every global section functor, we have $\Gamma := Hom_{\mathcal{D}_{G/B}}(\mathcal{D}_{G/B}, \cdot)$. So Γ has left adjoint: $Loc: U_0\text{-mod} \to Coh(\mathcal{D}_{G/B})$ given by $\mathcal{D}_{G/B}\otimes_{U_0} ?$.

Here's the "main" localization theorem:

Theorem 3: a) Every object in $Coh(\mathcal{D}_{G/B})$ has no higher cohomology & is generated by its global sections.

6) $\Gamma: \mathrm{Coh}(\mathcal{D}_{G/B}) \leftrightarrows \mathcal{U}_0\text{-mod}: \mathrm{Loc}$ are mutually quasi-inverse equivalences.

Rem: For an algebraic subgroup $H \subset G$, Γ & Loc give mutually quasi-inverse equivalences between $\mathrm{Coh}^H(\mathcal{D}_{G/B})$ & $\mathcal{U}_0\text{-mod}^H$. It's the latter category we'd like to understand and we do this by understanding the former.

2.2) Category $\mathcal{O}$. Consider the case when $H=N$, the maximal unipotent subgroup of G. We have the Bruhat decomposition: $G = \coprod_{w\in W} N\dot{w}B$ (where $\dot{w}$ is a lift of w to $N_G(T)$). This gives rise to $G/B = \coprod_{w\in W} N\dot{w}B/B$. Note that $N\dot{w}B/B$ is identified with $N/(N\cap \dot{w}B\dot{w}^{-1})$ as an N-variety (in fact, this is an affine space). Every closed subgroup of a unipotent algebraic group is unipotent, in particular, connected. Using Theorem 1, we see that the irreducibles in $\mathrm{Coh}^N(\mathcal{D}_{G/B})$ are classified by the elements of W. More precisely, let's write X_w for $N\dot{w}B/B$ & $\mathcal{O}_w$ for the $\mathcal{D}$-module $\mathcal{O}_{X_w}$ on X_w. Then the irreducibles in $\mathrm{Coh}^N(\mathcal{D}_{G/B})$ are precisely $IC(X_w, \mathcal{O}_w)$.

The functor Γ identifies $\mathrm{Coh}^N(\mathcal{D}_{G/B})$ with $\mathcal{U}_0\text{-mod}^N$. The irreducibles in $\mathcal{U}_0\text{-mod}^N$ are precisely $L(\lambda)$ w. $\lambda \in W\cdot 0$. In fact, $\Gamma(IC(X_w,\mathcal{O}_w)) = L(w\cdot(-2\rho))$.

Rem: • Let j_w be the inclusion $X_w \subset G/B$. The objects $\Gamma(j_{w*}\mathcal{O}_w)$, $\Gamma(j_{w!}\mathcal{O}_w)$ are the dual Verma and Verma modules w. highest wt $w\cdot(-2\rho)$.

• The equivalence $\mathrm{Coh}^N(\mathcal{D}_{G/B}) \xrightarrow{\sim} \mathcal{U}_0\text{-mod}^N$ is the first step in the proof of Kazhdan-Lusztig conjectures on the characters of the modules $L(w\cdot(-2\rho))$ (by Beilinson-Bernstein & Brylinski-Kashiwara).

2.3) Harish-Chandra modules. The classification of irreducibles in $\mathcal{U}_0\text{-mod}^N$ can be done by elementary representation theoretic tools. Here's another example of an important representation theoretic category, where the classification of irreducibles requires to use equivariant $\mathcal{D}$-modules: categories of Harish-Chandra (HC) $\mathcal{U}_0$-modules. Let G be a connected reductive alg'c group.

Definition: A symmetric subgroup of G is a subgroup of the form $(G^\sigma)^\circ$, where $\sigma: G\to G$ is an involution of G.

Example: • Let $G=GL_n(\mathbb{C})$. The inner involutions σ are conjugate to $A\mapsto \Sigma A\Sigma^{-1}$ w $\Sigma=\operatorname{diag}(1,..1,-1,..,-1)$. The corresponding symmetric subgroups are $GL_k\times GL_{n-k}$ (block-diagonal matrices). The outer involutions are $A\mapsto (A^*)^{-1}$ w.r.t. orthogonal or (for even n) symplectic forms. The corresponding subgroups are SO_n & Sp_n (the latter for even n).

• Let $G=K\times K$, where K is a connected reductive group. Then $(k_1,k_2)\mapsto(k_2,k_1)$ is an involution & the corresponding symmetric subgroup is K embedded into G diagonally.

Definition: Let $K\subset G$ be a symmetric subgroup. By a HC $(\mathcal{U}_0,K)$-module we mean an object in $\mathcal{U}_0\text{-mod}^K$.

Such modules appear in the study of infinite dimensional representations of reductive real Lie groups.

Here's a key fact for geometric understanding of irreducible HC modules.

Fact: K acts on G/B with finitely many orbits.

Example: Let $G=PGL_2$. It has a unique involution: conjugation with $\mathrm{diag}(1,-1)$. The subgroup K is T, the one dimensional torus of diagonal matrices. We have $G/B=\mathbb{P}^1$ & 3 orbits for T: $\{0\},\{\infty\}$ & the complement.

Premium exercise (on Linear algebra!): check the finiteness claim for the action of K on G/B for $G=GL_n$ & K listed above.

So the localization theorem allows to classify the irreducible HC $(\mathcal{U}_0, K)$-modules in geometric terms.

Example: Let's see what happens in the PGL_2 example. We have 2 T-fixed points, 0 and ∞ - with connected stabilizers. They contribe two irreducible HC modules (positive & negative anti-dominant Verma modules). The complement is a free orbit. It contributes one irreducible HC module.

Anti-premium exercise: what is this module?

In general, however, there will be disconnected stabilizers, for example for the open SO_n-orbit on $Fl_n=SL_n/B$, the stabilizer is $\{\pm 1\}^{n-1}$ (exercise).

2.4) Equivariant $\mathcal{D}$-modules on the nilpotent cone.

Let $\mathcal{N}$ be the subvariety of nilpotent matrices in $\mathfrak{sl}_n$. The group GL_n acts on $\mathfrak{sl}_n$ by conjugation stabilizing $\mathcal{N}$. The orbits in $\mathcal{N}$ are labelled by the partitions of n - via JNF.

The stabilizers in GL_n are all connected: for $e \in \mathcal{N}$ its stabilizer is all non-degenerate matrices in the centralizer: $\{A \in \mathfrak{gl}_n \mid [A,e]=0\}$. So the irreducible GL_n-equivariant $\mathcal{D}$-modules on $\mathcal{N}$ are naturally labelled by the partitions of n. This is closely related to the Springer theory that we tried to study at the seminar.

Since the center of GL_n acts trivially on $\mathcal{N}$ (& on $\mathfrak{gl}_n$) the GL_n-equivariant $\mathcal{D}$-modules are the same as PGL_n-equivariant. If we replace PGL_n with SL_n, we get more equivariant $\mathcal{D}$-modules. To an irreducible SL_n-equivariant $\mathcal{D}$-module we can assign a residue mod n that encodes the scalar by which the center of SL_n ($\simeq \mathbb{Z}/n\mathbb{Z}$) acts on that $\mathcal{D}$-module.

The most interesting case is when the residue is coprime to n. Denote the residue by a.

Premium exercise: There is a unique such $\mathcal{D}$-module, it's associated to the principal nilpotent orbit (i.e. the one with a single block). Denote it by $\mathcal{M}_{a,n}$.

The $\mathcal{D}$-modules in the exercise (one for each a mod n coprime to n) are very interesting – and there are still things to be understood about objects closely related to them. For example, the irreducible representation $L(\mu)$ of SL_n with highest weight μ appears in $\mathcal{M}_{a,n} \Leftrightarrow \mu_1+\dots+\mu_n \equiv a \mod n$ ($\Rightarrow$ is a tautology) and the multiplicity is $\frac{1}{n}\dim L(\mu)$ (which is an integer – an exercise).