D-MODULES, HOMEWORK 1

Problem 1. Prove that the algebra $D(\mathbb{C}^n)$ is simple and the center is $\mathbb{C}$. Prove that the module $\mathbb{C}[x_1, \dots, x_n]$ is irreducible.

Problem 2. Let $\mathbb{F}$ be an algebraically closed field of characteristic p. For us, the algebra $D(\mathbb{F}^n)$ is defined by generators and relations.

a) Determine the center of this algebra and describe the kernel of the action of $D(\mathbb{F}^n)$ on $\mathbb{F}[x_1, \dots, x_n]$.

b) Further, prove that $D(\mathbb{F}^n)$ is a free module over the center. For every maximal ideal $\mathfrak{m}$ in the center prove that $D(\mathbb{F}^n)/D(\mathbb{F}^n)\mathfrak{m}$ is the matrix algebra of rank p^n.

Problem 3. Consider the algebra $A = \mathbb{Z}[x_1, \dots, x_n]$ and its algebra $D^G(A)$ of Grothendieck's differential operators. Prove that $D^G(A)$ is a free (left) A-module with basis given by the elements $\partial^{(\beta)}$, where

$$\partial^{(\beta)} = \prod_{i=1}^{n} \frac{\partial_i^{\beta_i}}{\beta_i!}.$$

What happens when we replace $\mathbb{Z}$ with $\mathbb{F}$?

Problem 4. Generalize Problem 1 to the algebra $D(X)$, where X is a smooth affine variety over $\mathbb{C}$.

Problem 5. Generalize Problem 2 to the algebra $D(X)$, where X is a smooth affine variety over $\mathbb{F}$.