D-MODULES, HOMEWORK 3

Problem 1. Let $f : Y \to X$ be an etale morphism. Show that the sheaf theoretic push-forward $f_\bullet$ coincides with the D-module push-forward. This generalizes what we had for the case of open embedding.

Problem 2. Let M be a coherent $D(X)$-module, where X is affine. Prove that the following two numbers are equal (this was a premium exercise in the lecture):
a) $\operatorname{codim}_{T^*X} \operatorname{SS}(M)$,
b) the minimal number k such that $\operatorname{Ext}^k_{D(X)}(M, D(X)) \neq 0$.

Problem 3. Let $f : Y \to \mathrm{pt}$, where Y is affine. Compute the derived pushforward of $\mathcal{O}_Y$. What about general Y? A general O-coherent D-module?

Problem 4. Let M be a finitely generated $D(\mathbb{A}^n)$-module. Give a direct proof that the dimensions of the singular support of M w.r.t. the Bernstein filtration and the filtration by the order of differential operator are the same. Can you describe a relation between the corresponding singular supports?

Problem 5. Let i be the inclusion of $\mathbb{A}^{n-1}$ into $\mathbb{A}^n$. Prove that i^* preserves holonomicity. Try to prove the same for the pullback under the general morphism $f : Y \to X$.

Problem 6. Let A, B be $m \times m$-matrices. Consider the rank m O-coherent D-modules $\mathbb{C}[x^{\pm 1}]x^A, \mathbb{C}[x^{\pm 1}]x^B$. Suppose

$$\exp(2\pi\sqrt{-1}A), \exp(2\pi\sqrt{-1}B)$$

are conjugate (and hence the Deligne theorem tells us that these D-modules are isomorphic). Prove an isomorphism in an elementary way.

Problem 7. This problem explains why the intermediate extension functor is also called the minimal extension. Let $j : U \hookrightarrow X$ be open, let $\mathcal{F}_U \in \operatorname{Hol}(D_U)$. Show that there is a unique object $\mathcal{F} \in \operatorname{Hol}(D_U)$ with the following two properties:

- The restriction of $\mathcal{F}$ to U is $\mathcal{F}_U$.
- $\mathcal{F}$ has neither sub nor quotients supported on $X \setminus U$.

Furthermore, show that $\mathcal{F} = j_{!*}(\mathcal{F}_U)$.