D-MODULES, HOMEWORK 4

Problem 1. Let Fl_n denote the variety of complete flags in $\mathbb{C}^n$ and let K be one of the symmetric subgroups of GL_n: $\mathrm{GL}_k \times \mathrm{GL}_{n-k}$, SO_n or Sp_n (the latter for even n).
a) Use Linear algebra to show that K acts on Fl_n with finitely many orbits.
b) In each of the cases, identify the open K orbits and compute the stabilizer.
c) What about the closed K-orbits?
d) Complete the classification of irreducible K-equivariant D-modules on Fl_3.

Problem 2. Let n be a positive integer and a be a residue mod n coprime to n. Further, let $\mathcal{N}$ denote the nilpotent cone for $\mathfrak{sl}_n$. Show that

(1) there is a unique irreducible SL_n-equivariant D-module $\mathcal{M}$ on $\mathcal{N}$ such that $\mathrm{diag}(z, \ldots, z)$, a typical element in the center of SL_n, acts on $\mathcal{M}$, by z^a,
(2) this D-module is associated to the principal nilpotent orbit,
(3) It coincides with both $*$- and !-pushforward of an irreducible $\mathcal{O}$-coherent D-module from the principal orbit.
(4) The category of equivariant D-modules supported on $\mathcal{N}$ with the specified action of the center is semisimple.

Problem 3. Consider the action of the maximal unipotent subgroup N on G/B.
a) Let χ be a non-degenerate character of $\mathfrak{n}$. Show that there is a unique irreducible (N, χ)-equivariant D-module on G/B, that it is associated to the open N-orbit and that the category $\mathrm{Coh}^{N,\chi}(D_{G/B})$ is semisimple.
b) Classify the irreducible (N, χ)-equivariant D-modules in the case when χ is an arbitrary character.

Problem 4. Let X be a smooth variety and $\mathcal{L}$ is a line bundle on X. Show that the categories $\mathrm{Coh}(D_X)$ and $\mathrm{Coh}(D_X^{\mathcal{L}})$ are equivalent and on the level of quasicoherent $\mathcal{O}_X$-modules the equivalence is given by tensoring with $\mathcal{L}$.

Problem 5. Explain how to view D_X^{opp} as a sheaf of TDO on X and identify it with $D_X^{K_X}$.

Problem 6. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be line bundles on a smooth variety X_0. Let X denote the corresponding principal $T = (\mathbb{C}^\times)^n$-bundle and $\chi = \sum_{i=1}^n z_i \mathbf{1}_i \in \mathfrak{t}^*$. Prove that the cohomology class corresponding to $\pi_\bullet(R_X^\chi)^T$ is $\sum_{i=1}^n z_i c_1(\mathcal{L}_i)$.