

ROUQUIER COMPLEXES

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1. ROUQUIER COMPLEXES

1.1. **Setup.** In this talk \mathbf{k} is a field of characteristic 0. Let (W, S) be a Coxeter group with $|S| = n < \infty$. Also we let $V = \sum_{s \in S} \mathbf{k}e_s$ be the reflection representation of W , and $R = \mathbf{k}[V]$. R is a graded algebra with V^* in degree 2. We abbreviate $M \otimes_R N = MN$ for a right R -module M and a left R -module N . We let $\{\alpha_s\}_{s \in S}$ be the dual basis of $\{e_s\}_{s \in S}$, which can be considered as elements in $V^* \subset R$. Thus $R \simeq \mathbf{k}[\alpha_1, \dots, \alpha_n]$ is a polynomial algebra of n variables.

The natural action of W on V induces that on R , i.e. $(w \cdot f)(v) := f(w^{-1}(v))$, and thus on $R - \text{mod}_{\text{gr}}$ (the category of graded R -module) and $D^b(R - \text{mod}_{\text{gr}})$ (its bounded derived category) in a way that for $f \in R$ and $m \in {}^w M$ we have $f \cdot m = (w^{-1} \cdot f)(m)$. Note that for $M \in R - \text{mod}_{\text{gr}}$, ${}^w M \simeq R_w M$, where R_w is a R -bimodule such that $R_w \simeq R$ as a left R -module and

$$m \cdot a = mw(a) \text{ for } m \in R_w, a \in R,$$

i.e. the right action is twisted by w . In other words,

$$W \rightarrow \text{End}(D^b(R - \text{mod}_{\text{gr}})) : w \mapsto R_w \otimes_R -$$

is a well-defined action of W on $D^b(R - \text{mod}_{\text{gr}})$. Also note that for any $w, w' \in W$, we have a canonical isomorphism $R_w R_{w'} \simeq R_{ww'} : 1 \otimes 1 \mapsto 1$ of R -bimodules.

Let $K^b(R - \text{mod}_{\text{gr}})$ be the bounded homotopy category of $R - \text{mod}_{\text{gr}}$. This is the category of bounded chain complexes in $R - \text{mod}_{\text{gr}}$ modulo homotopy equivalence: for $f, g : A^\bullet \rightarrow B^\bullet$, we say that f is homotopic to g , denoted $f \sim g$, if there exists $h : A^\bullet \rightarrow B^{\bullet-1}$ such that $f - g = dh + hd$. In $K^b(R - \text{mod}_{\text{gr}})$, $A^\bullet \simeq B^\bullet$ if there exist $f : A^\bullet \rightarrow B^\bullet$ and $g : B^\bullet \rightarrow A^\bullet$ such that $gf \sim id_A$ and $fg \sim id_B$. Note that this category is well-defined if one replaces $R - \text{mod}_{\text{gr}}$ by any additive category, not necessarily abelian. In particular, if we let SBim_{gr} be the category of graded Soergel bimodules and $R - \text{bim}_{\text{gr}}$ be the category of graded R -bimodules, then $K^b(\text{SBim}_{\text{gr}})$ and $K^b(R - \text{bim}_{\text{gr}})$ are well-defined and we have $K^b(\text{SBim}_{\text{gr}}) \subset K^b(R - \text{bim}_{\text{gr}})$.

1.2. **Definition of Rouquier complexes.** Our goal is to lift the action of W on $D^b(R - \text{mod}_{\text{gr}})$ mentioned above to $K^b(R - \text{mod}_{\text{gr}})$, using Soergel bimodules. How can we lift $R_w \otimes -$? We may consider:

$$W \rightarrow \text{End}(K^b(R - \text{mod}_{\text{gr}})) : w \mapsto R_w \otimes_R -$$

which gives a trivial lift, but this is not interesting... Here we introduce another ‘interesting’ lifting introduced by Rouquier. However, instead we should work with the braid group B_W of (W, S) instead of W itself.

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Any object $M \in K^b(R - \text{bim}_{\text{gr}})$ defines an endofunctor $K^b(R - \text{mod}_{\text{gr}}) \rightarrow K^b(R - \text{mod}_{\text{gr}}) : N \mapsto M \otimes_R N =: MN$. For a word \underline{w} with alphabet $S \cup S^{-1}$, we will define an invertible object $F_{\underline{w}} \in K^b(\text{SBim}_{\text{gr}}) \subset K^b(R - \text{bim}_{\text{gr}})$ which descends to $R_{\underline{w}} \in D^b(R - \text{bim}_{\text{gr}})$ (up to a shift). Here, $w \in W$ is \underline{w} considered as a product of simple reflections in W .

First we recall some morphisms defined in Boris' talk:

$$\begin{aligned} m_s : B_s \rightarrow R(1) : f \otimes g &\mapsto fg, & m_s^a : R(-1) \rightarrow B_s : f &\mapsto f\alpha_s \otimes 1 + f \otimes \alpha_s \\ j_s : B_s B_s(1) \rightarrow B_s : f \otimes g \otimes h &\mapsto \partial_s(g)f \otimes h & j_s^a : B_s(1) \rightarrow R : f \otimes g &\mapsto f \otimes 1 \otimes g \end{aligned}$$

(i) is the grading shift as a graded R -bimodule.) For $s \in S$, we define

$$\begin{aligned} F_s &:= 0 \rightarrow \boxed{B_s} \xrightarrow{m_s} R(1) \rightarrow 0 \in K^b(\text{SBim}_{\text{gr}}) \\ F_{s^{-1}} &:= 0 \rightarrow R(-1) \xrightarrow{m_s^a} \boxed{B_s} \rightarrow 0 \in K^b(\text{SBim}_{\text{gr}}) \end{aligned}$$

The box indicates cohomology degree 0. From the following exact sequences of graded R -bimodules

$$\begin{aligned} 0 \rightarrow R_s(-1) \xrightarrow{f \mapsto f \otimes \alpha_s - f \alpha_s \otimes 1} B_s \xrightarrow{m_s} R(1) \rightarrow 0 \\ 0 \rightarrow R(-1) \xrightarrow{m_s^a} B_s \xrightarrow{f \otimes g \mapsto f s(g)} R_s(1) \rightarrow 0 \end{aligned}$$

it easily follows that

$$R_s \simeq F_s(1) \simeq F_{s^{-1}}(-1) \quad \text{in } D^b(R - \text{bim}_{\text{gr}}).$$

However, they are not equal in $K^b(R - \text{bim}_{\text{gr}})$.

Exercise 1.1. Prove that $R_s \not\simeq F_s(1)$ and $R_s \not\simeq F_{s^{-1}}(-1)$.

In general, for a word $\underline{w} = abcd \dots$ with alphabet $S \cup S^{-1}$, we define

$$F_{\underline{w}} := F_a F_b F_c F_d \dots \in K^b(\text{SBim}_{\text{gr}}),$$

called the Rouquier complex corresponding to \underline{w} . In particular, we have $F_{\emptyset} = R$, $F_{\underline{s}} = F_s$, and $F_{\underline{s^{-1}}} = F_{s^{-1}}$.

Remark. Indeed, the split Grothendieck group of SBim_{gr} is isomorphic to the Hecke algebra of W . Under this correspondence, the image of the class of F_s is the standard basis T_s .

1.3. Some properties of Rouquier complexes. We claim that indeed this correspondence $\underline{w} \mapsto F_{\underline{w}}$ defines a weak action of B_W on $K^b(R - \text{mod}_{\text{gr}})$. For that, we need to check that $F_{\underline{w_1}} \simeq F_{\underline{w_2}}$ if $\underline{w_1}$ and $\underline{w_2}$ represent the same element in B_W . (Note that this does not hold if we only assume that $\underline{w_1}$ and $\underline{w_2}$ represent the same element in W , e.g. $F_s \not\simeq F_{s^{-1}}$.) To that end, it suffices to show the following.

- (1) $F_s F_{s^{-1}} \simeq F_{s^{-1}} F_s \simeq R$ for $s \in S$.
- (2) $F_s F_t F_s \dots \simeq F_t F_s F_t \dots$ where each side is the product of m terms, for $s, t \in S$ and $m \in \mathbb{N}$ such that $(st)^m = 1 \in W$.

First we show that $F_s F_{s-1} \simeq R$ in $K^b(R - \text{bim}_{\text{gr}})$. The tensor product of F_s and F_{s-1} is described as follows. (Here, we understand this diagram as a single chain complex, each of whose terms is the direct sum of objects on each column.)

$$F_s F_{s-1} = \begin{array}{ccccc} & & \boxed{B_s B_s} & & \\ & \nearrow^{1 \otimes m^a} & & \searrow^{m_s \otimes 1} & \\ B_s(-1) & & & & B_s(1) \\ & \searrow^m & & \nearrow^{-m^a} & \\ & & \boxed{R} & & \end{array}$$

We claim that it is homotopy equivalent to R . Indeed, if we define (we omit the subscript s)

- (1) $\iota_1 = (1 \otimes m^a, m) : B_s(-1) \rightarrow B_s B_s \otimes R$
- (2) $\iota_2 = (j^a, 0) : B_s(1) \rightarrow B_s B_s \otimes R$
- (3) $\iota_3 = (j^a m^a, Id) : R \rightarrow B_s B_s \otimes R$
- (4) $\rho_1 = j \oplus 0 : B_s B_s \otimes R \rightarrow B_s(-1)$
- (5) $\rho_2 = (m \otimes 1) \oplus (-m^a) : B_s B_s \otimes R \rightarrow B_s(1)$
- (6) $\rho_3 = -mj \oplus Id : B_s B_s \otimes R \rightarrow R$

then they give a decomposition of $B_s B_s \oplus R$ into $B_s(-1) \oplus B_s(1) \oplus R$, i.e. $\rho_i \iota_i = Id$ and $\rho_j \iota_i = 0$ if $i \neq j$.

$$\begin{array}{ccccc} & \xrightarrow{(1 \otimes m^a, m)} & & \xrightarrow{(m \otimes 1) \oplus (-m^a)} & \\ B_s(-1) & \xleftarrow{j \oplus 0} & B_s B_s \oplus R & \xleftarrow{(j^a, 0)} & B_s(1) \\ & & \uparrow (j^a m^a, id) & & \\ & & R & & \\ & & \downarrow -mj \oplus id & & \end{array}$$

In other words, the following diagram describes that $F_s F_{s-1} \simeq R$ in $K^b(R - \text{bim}_{\text{gr}})$. (One can easily show that the vertical maps are mutual inverses using ι_i and ρ_i for $1 \leq i \leq 3$.)

$$\begin{array}{ccccc} B_s(-1) & \xrightarrow{(1 \otimes m^a, m)} & \boxed{B_s B_s \oplus R} & \xrightarrow{(m \otimes 1) \oplus (-m^a)} & B_s(1) \\ \downarrow & & \downarrow -mj \oplus id & & \downarrow \\ 0 & \longrightarrow & \boxed{R} & \longrightarrow & 0 \\ \downarrow & & \downarrow (j^a m^a, id) & & \downarrow \\ B_s(-1) & \xrightarrow{(1 \otimes m^a, m)} & \boxed{B_s B_s \oplus R} & \xrightarrow{(m \otimes 1) \oplus (-m^a)} & B_s(1) \end{array}$$

$F_{s-1} F_s \simeq R$ can be proved similarly.

However, $F_s F_s$ is not homotopy equivalent to R , even up to a shift. Indeed, we have

$$F_s F_s = \begin{array}{ccccc} & & & B_s(1) & \\ & & 1 \otimes m & \nearrow & \\ & & & & m \\ \boxed{B_s B_s} & & & & R(2) \\ & & m \otimes 1 & \searrow & \\ & & & B_s(1) & \\ & & & & -m \end{array}$$

and it is homotopy equivalent to the following chain complex.

$$\boxed{B_s(-1)} \xrightarrow{1 \otimes 1 \mapsto \alpha_s \otimes 1 - 1 \otimes \alpha_s} B_s(1) \xrightarrow{m} R(2)$$

As $B_s B_s \simeq B_s(1) \oplus B_s(-1)$, the claim follows if we can “cancel out” $B_s(1)$ in cohomological degree 0 and 1, respectively. To be more precise, we use the following diagram. (The vertical maps are mutual inverses in $K^b(R - \text{bim}_{\text{gr}})$.)

$$\begin{array}{ccccc} \boxed{B_s B_s} & \xrightarrow{(1 \otimes m, m \otimes 1)} & B_s(1) \oplus B_s(1) & \xrightarrow{m \oplus (-m)} & R(2) \\ \downarrow -j^a & & \downarrow (f, g) \mapsto f - g & & \downarrow Id \\ \boxed{B_s(-1)} & \xrightarrow{1 \otimes 1 \mapsto \alpha_s \otimes 1 - 1 \otimes \alpha_s} & B_s(1) & \xrightarrow{m} & R(2) \\ \downarrow 1 \otimes 1 \mapsto \alpha_s \otimes 1 \otimes 1 - 1 \otimes \alpha_s \otimes 1 & & \downarrow id \oplus 0 & & \downarrow Id \\ \boxed{B_s B_s} & \xrightarrow{(1 \otimes m, m \otimes 1)} & B_s(1) \oplus B_s(1) & \xrightarrow{m \oplus (-m)} & R(2) \end{array}$$

Also, there exists a quasi-isomorphism

$$\begin{array}{ccc} \boxed{B_s(-1)} & \xrightarrow{1 \otimes 1 \mapsto \alpha_s \otimes 1 - 1 \otimes \alpha_s} & B_s(1) \xrightarrow{m} R(2) \\ m^a \uparrow & & \\ R(-2) & & \end{array}$$

Thus, $F_s F_s \simeq R(-2)$ in $D^b(R - \text{bim}_{\text{gr}})$. However, it is not a homotopy equivalence. Indeed, otherwise we would have $F_s \simeq F_{s-1}(-2)$, but it can be easily seen to be false.

Exercise 1.2. Prove that $F_s \simeq F_{s-1}(-2)$. (Hint: try to construct homotopy equivalences between F_s and $F_{s-1}(-2)$. What are the degree zero endomorphisms of B_s ?)

Exercise 1.3. Show that $F_s F_s \cdots F_s$ (m terms) is homotopy equivalent to

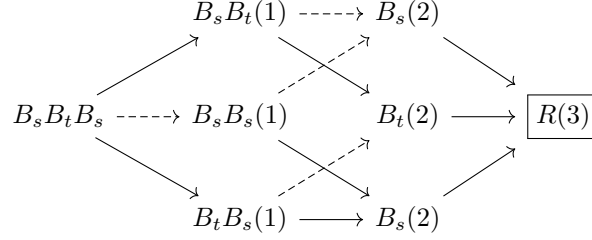
$$\boxed{B_s(1-m)} \xrightarrow{d_0} B_s(3-m) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-2}} B_s(m-1) \xrightarrow{m} R(m)$$

where differentials are given by

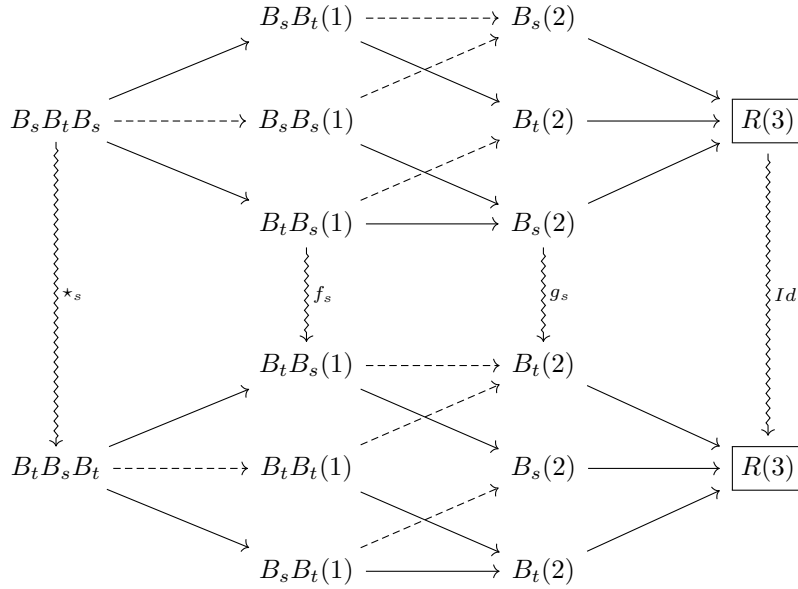
$$d_{m-2}, d_{m-4}, \dots = 1 \otimes 1 \mapsto \alpha_s \otimes 1 - 1 \otimes \alpha_s, \quad d_{m-3}, d_{m-5}, \dots = 1 \otimes 1 \mapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s.$$

Now we assume $s, t \in S$ are given. If $m_{st} = 2$, then indeed it is not hard to show that $F_s F_t \simeq F_t F_s \in K^b(R - \text{bim}_{\text{gr}})$, since $B_s B_t \simeq B_t B_s \simeq B_{st}$. What if $m_{st} = 3$? We want to show that $F_s F_t F_s \simeq F_t F_s F_t \in K^b(R - \text{bim}_{\text{gr}})$. Equivalently, we will show that $F_s[1]F_t[1]F_s[1] \simeq$

$F_t[1]F_s[1]F_t[1]$, where $[i]$ is the cohomology shift.¹ $F_s[1]F_t[1]F_s[1]$ is expressed as the following cube-shaped diagram.



The solid arrows are multiplication maps and the dashed ones are their negatives. From [EK10], we have a homotopy equivalence which is denoted by squiggling arrows in the following diagram.



Here, \star_s , f_s , g_s are defined as follows.

- \star_s is the projection $B_s B_t B_s \rightarrow B_{sts}$ composed with the injection $B_{sts} \hookrightarrow B_t B_s B_t$. It satisfies

$$\begin{aligned}
 1 \otimes 1 \otimes 1 \otimes 1 &\mapsto 1 \otimes 1 \otimes 1 \otimes 1, \\
 1 \otimes (2\alpha_s + \alpha_t) \otimes 1 \otimes 1 &\mapsto (\alpha_s + 2\alpha_t) \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes (\alpha_s + 2\alpha_t).
 \end{aligned}$$

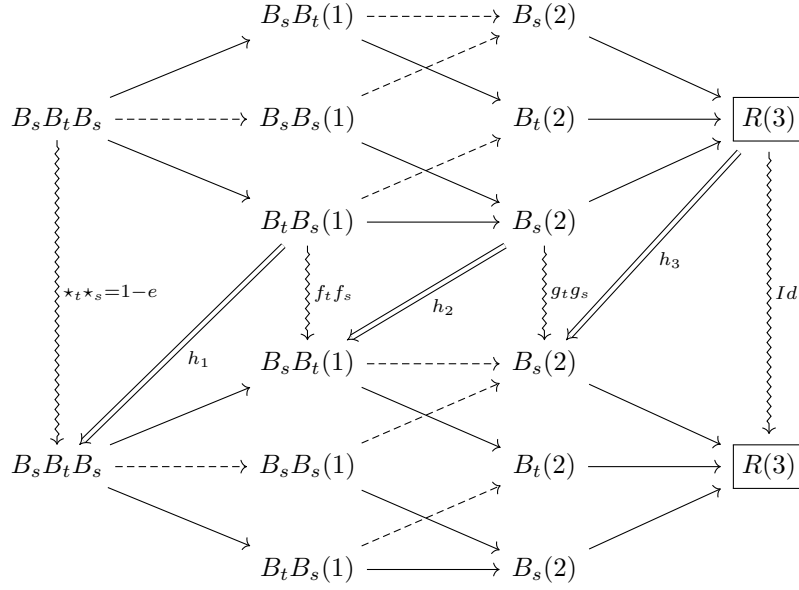
Note that $1 \otimes 1 \otimes 1 \otimes 1, 1 \otimes (2\alpha_s + \alpha_t) \otimes 1 \otimes 1$ generate $B_s B_t B_s$ as an R -bimodule. The reason we choose $1 \otimes (2\alpha_s + \alpha_t) \otimes 1 \otimes 1$ as a second generator is that $1 \otimes (2\alpha_s + \alpha_t) \otimes 1 \otimes 1 = 1 \otimes 1 \otimes (2\alpha_s + \alpha_t) \otimes 1$, i.e. $2\alpha_s + \alpha_t$ is fixed by t . Also, we have that $\star_t \star_s = (1 - e_s) = Id + m_t^a j_s^a j_s m_t$, which is the projection idempotent $B_s B_t B_s \rightarrow B_{sts}$ defined in Boris' talk. (There, we defined $e = -m_t^a j_s^a j_s m_t$ and the corresponding idempotent was written $1 - e$.)

¹This shift is introduced so that we can directly use the result of [EK10]. If one wants to compute $F_s F_t F_s$, signs of some maps should be changed accordingly.

$$\bullet f_s = \begin{bmatrix} 0 & -m_t^a \otimes j_s & Id \\ -(1-x)m_s \otimes j_t^a & j_t^a m_t^a m_s j_s & -xj_t^a \otimes m_s \\ Id & -j_s \otimes m_t^a & 0 \end{bmatrix}$$

$$\bullet g_s = \begin{bmatrix} 0 & (1-x)Id & 0 \\ Id & 0 & Id \\ 0 & xId & 0 \end{bmatrix}$$

Here, $x \in \mathbf{k}$ is arbitrary. The homotopy inverse is similarly defined, by switching s and t . We need to show that their composition is homotopy equivalent to the identity; the corresponding chain homotopy is given by the arrows with double lines in the following diagram.



Here, we have

$$f_t f_s = \begin{bmatrix} 0 & -m_s^a \otimes j_t & Id \\ -(1-x)m_t \otimes j_s^a & j_s^a m_t^a m_t j_t & -xj_s^a \otimes m_t \\ Id & -j_t \otimes m_s^a & 0 \end{bmatrix} \begin{bmatrix} 0 & -m_t^a \otimes j_s & Id \\ -(1-x)m_s \otimes j_t^a & j_t^a m_t^a m_s j_s & -xj_t^a \otimes m_s \\ Id & -j_s \otimes m_t^a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Id & -j_s \otimes m_t^a & 0 \\ -xj_s^a \otimes m_t & 2(1-x)L_{\alpha_t} j_s^a j_s + 2xj_s^a j_s R_{\alpha_t} & -(1-x)m_t \otimes j_s^a \\ 0 & -m_t^a \otimes j_s & Id \end{bmatrix},$$

$$g_t g_s = \begin{bmatrix} 0 & (1-x)Id & 0 \\ Id & 0 & Id \\ 0 & xId & 0 \end{bmatrix} \begin{bmatrix} 0 & (1-x)Id & 0 \\ Id & 0 & Id \\ 0 & xId & 0 \end{bmatrix} = \begin{bmatrix} (1-x)Id & 0 & (1-x)Id \\ 0 & Id & 0 \\ xId & 0 & xId \end{bmatrix}.$$

Here, $L_{\alpha_t}, R_{\alpha_t}$ are left and right multiplication by α_t , respectively. Chain homotopy maps are defined as follows. ($y \in \mathbf{k}$ is arbitrary.)

$$h_1 = [0 \quad m_t^a j_s^a j_s \quad 0], \quad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ -yj_s & 0 & (1-y)j_s \\ 0 & 0 & 0 \end{bmatrix}, \quad h_3 = 0$$

This proves that $F_s F_t F_s \simeq F_t F_s F_t$ for $m_{st} = 3$. See [Rou04, Section 3] for the proof in general cases.

Remark. Indeed, $F_s F_t F_s$ is homotopy equivalent to

$$\begin{array}{ccccc}
 & & B_s B_t(1) & \longrightarrow & B_s(2) \\
 & \nearrow & & \searrow & \nearrow \\
 B_{sts} & & & & \\
 & \searrow & & \nearrow & \searrow \\
 & & B_t B_s(1) & \longrightarrow & B_t(2) \\
 & & & & \nearrow \\
 & & & & R(3)
 \end{array}$$

with certain chain maps, which are symmetric with respect to s and t . Since $B_{sts} \simeq B_{tst}$, the relation $F_s F_t F_s \simeq F_t F_s F_t$ follows directly. However, it is not easy to show that the chain above is indeed homotopy equivalent to $F_s F_t F_s$.

In sum, we have the following theorem.

Theorem 1.4. [Rou04, Proposition 3.4] *The map $s \mapsto F_s$ extends to a morphism from B_W to the group of isomorphism classes of invertible objects of $K^b(R\text{-bim}_{\text{gr}})$. Furthermore, its image lies in $K^b(\text{SBim}_{\text{gr}})$, the category of graded Soergel bimodules.*

1.4. **A strict monoidal category \mathcal{B}_W .** In fact, we can proceed further; for $\tilde{w} \in B_W$, let $t_1 \cdots t_r, u_1 \cdots u_s$ be two words of \tilde{w} with alphabet $S \cup S^{-1}$. Then $F_{t_1} \cdots F_{t_m} \simeq F_{u_1} \cdots F_{u_n}$, and furthermore

$$\begin{aligned}
 \text{Hom}_{K^b(R\text{-bim}_{\text{gr}})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n}) &\simeq \text{End}_{K^b(R\text{-bim}_{\text{gr}})}(R) \simeq \mathbf{k} \\
 \text{Hom}_{D^b(R\text{-bim}_{\text{gr}})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n}) &\simeq \text{End}_{D^b(R\text{-bim}_{\text{gr}})}(R) \simeq \mathbf{k}
 \end{aligned}$$

Thus the canonical map

$$\text{Hom}_{K^b(R\text{-bim}_{\text{gr}})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n}) \rightarrow \text{Hom}_{D^b(R\text{-bim}_{\text{gr}})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n})$$

is an isomorphism. Recall that we already have a canonical isomorphism

$$R_{t_1} \cdots R_{t_m} \rightarrow R_{u_1} \cdots R_{u_n} : 1 \otimes \cdots \otimes 1 \mapsto 1 \otimes \cdots \otimes 1.$$

For each $\tilde{w} \in B_W$, we consider the system $\{F_{t_1} \cdots F_{t_m}\}$ of functors with isomorphisms $F_{t_1} \cdots F_{t_m} \rightarrow F_{u_1} \cdots F_{u_n}$ which descends to such a canonical isomorphism as above.

We define G_w as the limit of this direct system, which is unique up to a unique isomorphism. Then for $\tilde{w}, \tilde{w}' \in B_W$, there are unique isomorphisms $G_{\tilde{w}} G_{\tilde{w}'} \rightarrow G_{\tilde{w}\tilde{w}'}$ and $G_{\emptyset} \rightarrow R$. We consider the full subcategory \mathcal{B}_W of $K^b(R\text{-bim}_{\text{gr}})$ with objects $\{G_w\}_{w \in B_W}$. Also define $G_v^* := G_{v^{-1}}$. Then, we have the following theorem.

Theorem 1.5. [Rou04, Theorem 3.7] *The category \mathcal{B}_W is a strict rigid monoidal category, and $\tilde{w} \mapsto G_{\tilde{w}}$ gives a strong action of B_W on $K^b(R\text{-mod}_{\text{gr}})$.*

As a result, the ‘‘decategorification’’ of \mathcal{B}_W is a quotient of B_W . Rouquier conjectured that it is indeed the same as B_W , i.e. the action is faithful. Some partial answers to this conjecture are known: [KS02] for type A , [BT11] for type ADE , and [Jen17] for finite type in general.

2. KHOVANOV-ROZANSKY HOMOLOGY AND KNOT INVARIANTS

In this section, we relate Rouquier complexes and Khovanov-Rozansky homology, which is a certain invariant of links. We will see that one can calculate HOMFLY-PT polynomials from such complexes.

2.1. Hochschild homology. We define $M_R := M/[R, M] = R \otimes_{R^{en}} M$, i.e. taking coinvariants. Here, $R^{en} := R \otimes_{\mathbf{k}} R$ is the enveloping algebra of R . Then the functor $M \mapsto M_R$ is right exact and we define

$$\mathrm{HH}_i(R, M) := \mathrm{Tor}_i^{R^{en}}(R, M), \quad \mathrm{HH}(R, M) := \bigoplus_{i \geq 0} \mathrm{HH}_i(R, M).$$

called Hochschild homology of M . Similarly, one can define Hochschild cohomology as a derived functor of taking invariants. Indeed, for a polynomial algebra R we have

$$\mathrm{HH}_i(R, M) \simeq \mathrm{HH}^{n-i}(R, M),$$

which comes from the self-duality of the Koszul resolution of R . But this isomorphism does not hold for general R .

How can we calculate such homology? Recall that $R = \mathbf{k}[\alpha_1, \dots, \alpha_n]$. Henceforth we identify $R^{en} = \mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_n]$ and $R = R^{en}/(x_1 - y_1, \dots, x_n - y_n)$. We have a Koszul complex

$$0 \rightarrow \Lambda^n (R^{en})^n(-2n) \rightarrow \dots \rightarrow \Lambda^2 (R^{en})^n(-4) \rightarrow (R^{en})^n(-2) \xrightarrow{\clubsuit} \boxed{R^{en}} \rightarrow 0$$

where

$$\clubsuit = (f_1, \dots, f_n) \mapsto (x_1 - y_1)f_1 + \dots + (x_n - y_n)f_n.$$

It gives a projective resolution of R as an R^{en} -module. Therefore for any R^{en} -module M , $\mathrm{HH}(M)$ is the homology of

$$0 \rightarrow \Lambda^n (R^{en})^n \otimes_{R^{en}} M(-2n) \rightarrow \dots \rightarrow \Lambda^2 (R^{en})^n \otimes_{R^{en}} M(-4) \rightarrow M^n(-2) \rightarrow M \rightarrow 0.$$

2.2. Hochschild homology of Rouquier complexes. Now let \underline{w} be a word with alphabet $S \cup S^{-1}$ and consider $F_{\underline{w}}$. If $l(\underline{w}) = r$, then $F_{\underline{w}}$ consists of $r + 1$ terms, i.e.

$$F_{\underline{w}} = \dots \rightarrow 0 \rightarrow F_{\underline{w}}^{-r-1} \rightarrow \dots \rightarrow \boxed{F_{\underline{w}}^0} \rightarrow 0 \rightarrow \dots$$

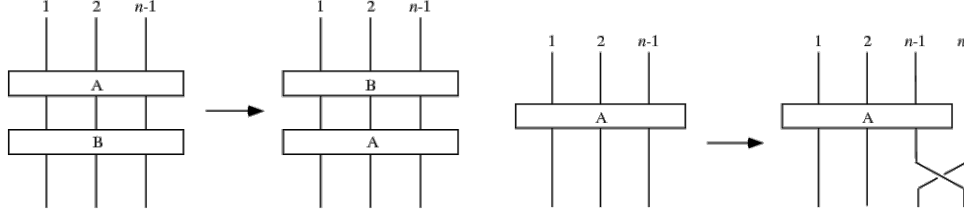
By applying HH on this sequence, we have

$$\mathrm{HH}(F_{\underline{w}}) = \dots \rightarrow 0 \rightarrow \mathrm{HH}(R, F_{\underline{w}}^{-r-1}) \rightarrow \dots \rightarrow \boxed{\mathrm{HH}(R, F_{\underline{w}}^0)} \rightarrow 0 \rightarrow \dots$$

where each $\mathrm{HH}(R, F_{\underline{w}}^i)$ is a bigraded \mathbf{k} -vector space, where the bigrading comes from (1) the graded R -bimodule structure of $F_{\underline{w}}^i$ and (2) the grading of Hochschild homology. Now we define $\mathrm{HHH}(F_{\underline{w}})$ as the cohomology of this sequence. This is then a triply-graded \mathbf{k} -vector space with (bimodule grading, Hochschild grading, cohomological grading). This turns out to be strongly related to the geometry of the knot (or link in general) represented by \underline{w} as the following theorem shows.

Theorem 2.1. [Kho07, Theorem 1] *Up to a grading shift, $\mathrm{HHH}(F_{\underline{w}})$ only depends on the closure of the braid represented by \underline{w} , say σ . This homology theory is isomorphic to the reduced link homology $\bar{H}(\sigma)$ defined by Khovanov and Rozansky (up to an affine transformation on the tri-grading.)*

One can check that \overline{H} and HHH are defined in an analogous manner. The proof of the second part in [Kho07] uses this similarity. The first part follows from this result and the fact that \overline{H} only depends on the closure of the braid σ . Indeed, we already know that Rouquier complexes only depend on the image of the word \underline{w} in B_W . However, if we take the closure, then we also need to consider the following ‘‘Markov moves’’, which result in the same link if one takes the closure of braids.



It is easy to prove the first part concerning the first Markov move; it suffices to show that $\text{HHH}(A^\bullet F_s) \simeq \text{HHH}(F_s A^\bullet)$ for any $A^\bullet \in K^b(\text{SBim}_{\text{gr}})$ and $s \in S$. But we have

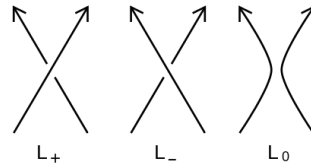
$$A^\bullet F_s = \dots \rightarrow A^{i-1} B_s \oplus A^{i-2}(1) \rightarrow A^i B_s \oplus A^{i-1}(1) \rightarrow A^{i+1} B_s \oplus A^i(1) \rightarrow \dots$$

$$F_s A^\bullet = \dots \rightarrow B_s A^{i-1} \oplus A^{i-2}(1) \rightarrow B_s A^i \oplus A^{i-1}(1) \rightarrow B_s A^{i+1} \oplus A^i(1) \rightarrow \dots$$

Now the result follows from the fact that $(MN)_R \simeq (NM)_R$ canonically (i.e. the coinvariants of MN and NM are isomorphic).

2.3. Connection with HOMFLY-PT polynomials. For each link σ , one can attach a unique Laurent polynomial $P(\sigma) = P(\sigma)(a, z)$ with certain properties, called the HOMFLY-PT polynomial [FYH⁺85]. It is characterized by the following properties.

- (1) $P(\text{unknot}) = 1$
- (2) $aP(L_+) - a^{-1}P(L_-) = zP(L_0)$ for links L_+, L_-, L_0 that are isomorphic to one another except one part, which is described as follows.



Jones polynomials and Alexander polynomials are the specializations of HOMFLY-PT polynomials: $P(t^{-1}, t^{1/2} - t^{-1/2}), P(1, t^{1/2} - t^{-1/2})$, respectively.

One property of \overline{H} is that its Euler characteristic gives the HOMFLY-PT polynomial of the corresponding link. Here, we deal with a simple example. (This is the dual version of [Kho07, pp.15–16].) Suppose $n = 1$, thus $R = \mathbf{k}[\alpha]$. We also identify $R^{en} = \mathbf{k}[x, y]$. Let $S = \{s\}$. We calculate the HOMFLY-PT polynomial corresponding to $s^m \in B_W$. We assume that m is odd, so that the closure of the braid is a knot, not just a link.

From Exercise 1.3, $F_s \cdots F_s$ (m terms) is equivalent to the following chain complex.

$$\boxed{B_s(1-m)} \xrightarrow{d_0} B_s(3-m) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-2}} B_s(m-1) \xrightarrow{d_{m-1}} R(m)$$

We take HH on this sequence. First, the Koszul complex of R is very simple in this case.

$$0 \rightarrow \mathbf{k}[x, y](-2) \xrightarrow{1 \mapsto x-y} \boxed{\mathbf{k}[x, y]} \rightarrow 0$$

Tensoring with R over R^{en} , we have $0 \rightarrow R(-2) \xrightarrow{0} \boxed{R} \rightarrow 0$, thus

$$\mathrm{HH}_0(R, R) = R, \quad \mathrm{HH}_1(R, R) = R(-2), \quad \mathrm{HH}_i(R, R) = 0 \text{ otherwise.}$$

If we tensor with B_s over R^{en} , then we have $0 \rightarrow B_s(-2) \xrightarrow{1 \otimes 1 \mapsto \alpha \otimes 1 - 1 \otimes \alpha} \boxed{B_s} \rightarrow 0$, thus

$$\mathrm{HH}_0(R, B_s) = R(1), \quad \mathrm{HH}_1(R, B_s) = R(-3), \quad \mathrm{HH}_i(R, B_s) = 0 \text{ otherwise.}$$

Therefore, the zeroth and the first homology of $\mathrm{HHH}(F_s \cdots F_s)$ is

$$\boxed{R(2-m)} \xrightarrow{\mathrm{HH}_0(d_0)} R(4-m) \xrightarrow{\mathrm{HH}_0(d_1)} \cdots \xrightarrow{\mathrm{HH}_0(d_{m-2})} R(m) \xrightarrow{\mathrm{Id}} R(m)$$

$$\boxed{R(-2-m)} \xrightarrow{\mathrm{HH}_1(d_0)} R(-m) \xrightarrow{\mathrm{HH}_1(d_1)} \cdots \xrightarrow{\mathrm{HH}_1(d_{m-2})} R(m-4) \xrightarrow{2\alpha} R(m-2)$$

where

$$\begin{aligned} \mathrm{HH}_0(d_{m-2}), \mathrm{HH}_0(d_{m-4}), \cdots = 0, & \quad \mathrm{HH}_0(d_{m-3}), \mathrm{HH}_0(d_{m-5}), \cdots = 1 \mapsto 2\alpha, \\ \mathrm{HH}_1(d_{m-2}), \mathrm{HH}_1(d_{m-4}), \cdots = 0, & \quad \mathrm{HH}_1(d_{m-3}), \mathrm{HH}_1(d_{m-5}), \cdots = 1 \mapsto 2\alpha. \end{aligned}$$

Thus, we have nontrivial homology isomorphic to \mathbf{k} at $(m-4, 0, 1), (m-8, 0, 3), \cdots, (2-m, 0, m-2), (m, 1, 1), (m-4, 1, 3), \cdots, (2-m, 1, m)$. (Recall that the trigrading is given by (bimodule grading, Hochschild grading, cohomological grading).) Therefore, if we let

$$P_m(x, y, z) := \sum_{i,j,k} x^i y^j z^k \dim \mathrm{HHH}_{(i,j,k)}(F_s \cdots F_s),$$

then it follows that

$$P_m(x, y, z) = \frac{x^{-m} (zx^{2m} (x^4 y + 1) - x^2 z^m (yz^2 + 1))}{x^4 - z^2}.$$

Thus the Euler characteristic of HHH with respect to the second grading is

$$P_m(x, -1, z) = \frac{x^{-m} (x^2 (z^2 - 1) z^m - (x^4 - 1) z x^{2m})}{x^4 - z^2}.$$

We put $x \mapsto x^{-1/2} z^{-1/4}$ and $z \mapsto x^{-1} z^{1/2}$, and multiply $-x^{\frac{3m}{2}} z^{-\frac{m}{4}}$ to get

$$f(m) := (-x^{\frac{3m}{2}} z^{-\frac{m}{4}}) P_m(x^{-1/2} z^{-1/4}, -1, x^{-1} z^{1/2}) = \frac{x^{m-1} z^{\frac{1-m}{2}} (-1 + z^{m+1} + x^2 (z - z^m))}{z^2 - 1}.$$

For example, we have

$$\begin{aligned} f(1) &= 1, & f(3) &= -x^4 + x^2 z + \frac{x^2}{z}, & f(5) &= -x^6 z - \frac{x^6}{z} + x^4 z^2 + \frac{x^4}{z^2} + x^4 \\ f(7) &= -x^8 z^2 - \frac{x^8}{z^2} - x^8 + x^6 z^3 + \frac{x^6}{z^3} + x^6 z + \frac{x^6}{z} \end{aligned}$$

We claim that these are HOMFLY-PT polynomials. Indeed, if we let $g(m)$ be the usual HOMFLY-PT polynomial of the knot $(m, 1)$, then from [kno] we have

$$g(1) = 1, \quad g(3) = -a^4 + a^2z^2 + 2a^2, \quad g(5) = -a^6z^2 - 2a^6 + a^4z^4 + 4a^4z^2 + 3a^4, \\ g(7) = -a^8z^4 - 4a^8z^2 - 3a^8 + a^6z^2 + 6a^6z^4 + 10a^6z^2 + 4a^6$$

If we substitute $a \mapsto x$ and $z \mapsto z^{1/2} - z^{1/2}$, then they become the same as $f(m)$.

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