Quantum Groups Seminar, IAS, Winter 2021

R-matrix for $\mathcal{U}_q(\mathfrak{g})$

References:

"Lectures on Quantum Groups" Jens Carsten Jantzen

"Lectures on Tensor Categories and Modular Functors" Bojko Bakalov and Alexander Kirillov Jr.

§0 Introduction

$\mathcal{U}$ is a Hopf algebra

Let $V, W \in \mathcal{U}\text{-mod}^{\text{type } 1}$

Consider $\quad V \otimes W \quad$ and $\quad W \otimes V$

For $u \in \mathcal{U}$, write $\Delta(u) = \sum_i u_i \otimes u_i'$

$$u \cdot v \otimes w = \sum u_i v \otimes u_i' w \qquad u \cdot w \otimes v = \sum u_i w \otimes u_i' v$$

<u>Def:</u> $\quad \Delta^{op} := P \circ \Delta \qquad$ (we write $P : A \otimes B \to B \otimes A$ for all A, B)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a \otimes b \mapsto b \otimes a$

$$\begin{array}{ccc} W \otimes V & \xrightarrow{\quad P \quad} & V \otimes W \\ \circlearrowleft \Delta & & \circlearrowleft \Delta^{op} \\ \mathcal{U} & & \mathcal{U} \end{array}$$

$$P(u \cdot w \otimes v) = \sum_i u_i' v \otimes u_i w = u \underset{op}{\bullet} v \otimes w$$

Since $\Delta \neq \Delta^{op}$, P will not be a $\mathcal{U}$-module intertwiner.

However, $\quad V \otimes W \cong W \otimes V \qquad$ (formal characters are determined by $\mathfrak{g}$ and for $\mathfrak{g}$ $\Delta = \Delta^{op}$)

Lets suppose there are isomorphisms

$$R_{V,W} : V\otimes W \longrightarrow W\otimes V$$

which are functorial in V and W for all V,W

motivation only

$$\begin{tikzcd}
V\otimes W \arrow[r, "R_{V,W}"] \arrow[d, "f\otimes g"'] & W\otimes V \arrow[d, "g\otimes f"] \\
V'\otimes W' \arrow[r, "R_{V',W'}"] & W'\otimes V'
\end{tikzcd}$$

Consider $R_{{}_u\mathcal{U},{}_u\mathcal{U}} : \mathcal{U}\otimes\mathcal{U} \longrightarrow \mathcal{U}\otimes\mathcal{U}$

$$1 \longmapsto R$$

Then for $M \in \mathcal{U}\text{-Mod}$ and $m \in M$ we get a $\mathcal{U}$-module map

$$\rho_m : {}_u\mathcal{U} \longrightarrow M$$
$$u \longmapsto u\cdot m$$

For V,W we compute

$$\begin{aligned}
R_{V,W}(v\otimes w) &= R_{V,W}\circ(\rho_v\otimes\rho_w)(1) \\
&= \rho_w\otimes\rho_v \circ R_{{}_u\mathcal{U},{}_u\mathcal{U}}(1) \\
&= \rho_w\otimes\rho_v\,(R) \\
&= R\cdot w\otimes v \\
&= RP(v\otimes w)
\end{aligned}$$

- Since $R_{(-),(-)}$ is a natural isomorphism, $R \in \mathcal{U}\otimes\mathcal{U}$ is a unit.
- Since $R_{(-),(-)}$ is a morphism of $\mathcal{U}$-modules, $R\,\Delta^{op}(u) = \Delta(u) R$.

$$
\begin{aligned}
R\,\Delta^{op}(u) &= R_{\mathcal{U},\mathcal{U}}(\Delta(u)) \\
&= R_{\mathcal{U},\mathcal{U}}(u\cdot 1) \\
&= u \cdot R_{\mathcal{U},\mathcal{U}}(1) \\
&= \Delta(u) R
\end{aligned}
$$

Drinfeld defined element $R \in \widetilde{\mathcal{U}\otimes\mathcal{U}}$

"completion" can think $\prod_{\lambda\in X_+} \mathrm{End}_{\mathcal{U}}(L(\lambda))$

$$R = \underbrace{q^{\sum h_\alpha \otimes h_\alpha'}}_{\substack{h_\alpha,\, h_\alpha' \text{ dual} \\ \text{basis of } \mathfrak{h}\subset\mathfrak{g} \\ \text{w/r/t } (-,-)}} \big(1 + \underbrace{\dots}_{\substack{\text{sum of} \\ \text{elts in} \\ \mathcal{U}^-\otimes\mathcal{U}^+}}\big)$$

$$q^h\big|_{V_\mu} = q^{\langle h,\mu\rangle}\cdot \mathrm{id}_{V_\mu} \quad \text{i.e. } q^{h_\alpha} = K_\alpha$$

Jantzen's book uses this all just as motivation.

§1 R matrix for $U_q(\mathfrak{sl}_2)$

$\Bbbk$ is field of characteristic zero and $q \in \Bbbk$ is transcendental

$$U = U_q(\mathfrak{sl}_2) = \Bbbk \langle E, F, K \mid \text{relns} \rangle$$

$\Delta(E) = E \otimes 1 + K \otimes E \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F \quad \Delta(K) = K \otimes K$

$V, W \in U\text{-mod}^{\text{type } 1}$

$$\Theta_{V,W} := 1 + \sum_{n \geq 1} (-1)^n q^{-\binom{n}{2}} \frac{(q - q^{-1})^n}{[n]!} F^n \otimes E^n \Big|_{V \otimes W}$$

Drinfeld's R matrix w/out the $q^{\sum h \otimes h'}$ part

is well defined since E, F act nilpotently in all finite dimensional reps.

$\Theta_{V,W}$ is also unipotent, i.e. $1 +$ nilpotent, and therefore is invertible.

example

$V = L(\varpi) \in \mathcal{U}\text{-mod}^{\text{type } 1}$

$$\begin{tikzcd} Fv \arrow[r, bend right, "E"'] \arrow[loop left, "K=q^{-1}"] & v \arrow[l, bend right, "F"'] \arrow[loop right, "K=q"] \end{tikzcd}$$

$$\Theta_{V,V} = 1 + (-1)\cdot(q-q^{-1})\, F\otimes E|_{v\otimes v}$$

$$\begin{array}{cccc} v\otimes v & v\otimes Fv & Fv\otimes v & Fv\otimes Fv \\ \end{array}$$

$$\left(\begin{array}{c|cc|c} 1 & & & \\ \hline & 1 & & \\ & (q^{-1}-q) & 1 & \\ \hline & & & 1 \end{array}\right)$$

Jantzen's approach to $q^{\sum h_i \otimes h^i}$ is as follows.

note for sl_2 $(\alpha,\alpha)=2$ so use $q^{h_\alpha \otimes h_\alpha/2}$

Let Λ = weight lattice i.e. $\Lambda = \mathbb{Z}\varpi$

Assume q has a square root in k

Define $f : \Lambda \times \Lambda \longrightarrow k$

by $f(a,b) = q^{-ab/2}$

For $V, W \in \mathcal{U}\text{-mod}^{\text{type } 1}$ define

$$\tilde{f} : V \otimes W \longrightarrow V \otimes W$$

by $\tilde{f}\big|_{V_\lambda \otimes W_\mu} = f(\lambda,\mu) \cdot id_{V_\lambda \otimes W_\mu}$.

Claim:

$$\left.\begin{aligned} f(\lambda,\mu+\nu) &= f(\lambda,\mu) f(\lambda,\nu) \\ f(\lambda+\mu,\nu) &= f(\lambda,\nu) f(\mu,\nu) \end{aligned}\right\} \text{needed for hexagon identity}$$

$$\left.\begin{aligned} f(\lambda,\mu+\nu) &= q^{-(\nu,\lambda)} f(\lambda,\mu) \\ f(\lambda+\nu,\mu) &= q^{-(\nu,\mu)} f(\lambda,\mu) \end{aligned}\right\} \text{needed for } \Theta\Delta^{\sigma} = \Delta\Theta$$

Define $\quad R_{V,W} := \underbrace{\Theta_{W,V} \circ \tilde{F} \circ P}_{\text{our version of Drinfeld's } R}$

__Thm 1 ($\mathfrak{sl}_2$)__ $R_{V,W} : V \otimes W \longrightarrow W \otimes V$ is an isomorphism of $\mathcal{U}$-modules and is functorial in V and W.

__Proof__ Postponed

__example__ $V = L(\varpi)$

$$R_{V\otimes V} = \overset{\Theta}{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & & & 1 \end{pmatrix}} \overset{\tilde{f}}{\begin{pmatrix} q^{-1/2} & & & \\ & q^{1/2} & & \\ & & q^{1/2} & \\ & & & q^{-1/2} \end{pmatrix}} \overset{P}{\begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & q^{1/2} & q^{-1/2} - q^{3/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}$$

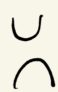
__exercise__ The maps $\mathrm{cap} : V \otimes V \longrightarrow \mathbb{1}$ $\qquad \mathrm{cup} : \mathbb{1} \longrightarrow V \otimes V$

$$v \otimes v \longmapsto 0$$
$$v \otimes Fv \longmapsto 1$$
$$Fv \otimes v \longmapsto -q$$
$$Fv \otimes Fv \longmapsto 0$$

$$1 \longmapsto -q^{-1} v \otimes Fv + Fv \otimes v$$

commute with $\mathcal{U}$. Verify the following:

$$R_{V,V} = q^{-1/2}\, \mathrm{id}_{V\otimes V} + q^{1/2}\, \mathrm{cup} \circ \mathrm{cap}$$

 $= q^{-1/2}\ ||\ + q^{1/2}$ 

Skein relation for Jones polynomial.

Write $\quad \mathrm{can} : (A\otimes B)\otimes C \xrightarrow{\sim} A\otimes(B\otimes C)$

$$(a\otimes b)\otimes c \longmapsto a\otimes(b\otimes c)$$

can is a $\mathcal{U}$ module iso. by coassociativity

and $\quad \mathrm{can} : A\otimes(B\otimes C) \xrightarrow{\sim} (A\otimes B)\otimes C$

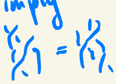
<u>Thm</u>2 ($\mathfrak{sl}_2$) The following diagrams are commutative

i)
$$\begin{tikzcd}
 & V\otimes(X\otimes W) \arrow[r, "\mathrm{can}"] & (V\otimes X)\otimes W \arrow[dr, "R_{V,X}\otimes \mathrm{id}_W"] & \\
V\otimes(W\otimes X) \arrow[ur, "\mathrm{id}_V\otimes R_{W,X}"] \arrow[dr, "\mathrm{can}"'] & & & (X\otimes V)\otimes W \\
 & (V\otimes W)\otimes X \arrow[r, "R_{(V\otimes W),X}"] & X\otimes(V\otimes W) \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

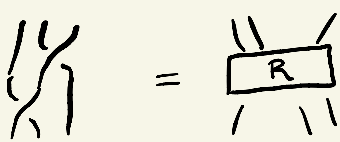
ii)
$$\begin{tikzcd}
 & (W\otimes V)\otimes X \arrow[r, "\mathrm{can}"] & W\otimes(V\otimes X) \arrow[dr, "\mathrm{id}_W\otimes R_{V,X}"] & \\
(V\otimes W)\otimes X \arrow[ur, "R_{V,W}\otimes \mathrm{id}_X"] \arrow[dr, "\mathrm{can}"'] & & & W\otimes(X\otimes V) \\
 & V\otimes(W\otimes X) \arrow[r, "R_{V,(W\otimes X)}"] & (W\otimes X)\otimes V \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

<u>Proof</u> : Postponed

Hexagon is just "coherence" condition. But does imply



top row



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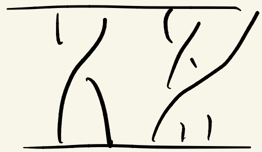
Why Hexagon?

Definition A **braid** in n strands is ...

taken up to isotopy →

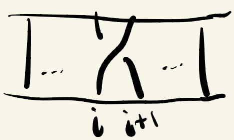


$n = 5$

The product $\beta\beta' = \begin{array}{|c|}\hline \beta \\ \hline \beta' \\ \hline \end{array}$ (stacking diagrams)

makes the set of all braids (on n strands up to isotopy) a group denoted Br_n

Theorem (Artin) $Br_n \cong \left\langle \sigma_i,\ i = 1, \dots, n-1 \;\middle|\; \begin{array}{c} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \\ |i-j| > 1 \end{array} \right\rangle$



$\longleftarrow\!\!| \ \sigma_i$

Recall (MacLane Coherence)

Fix $\lambda_V : 1 \otimes V \xrightarrow{\sim} V$

$\rho_V : V \otimes 1 \xrightarrow{\sim} V$

(functorial in U, V, W)

$\alpha_{UVW} : U \otimes (V \otimes W) \xrightarrow{\sim} (U \otimes V) \otimes W$

Pentagon axiom & Triangle axiom $\Rrightarrow$ Associativity Axiom

some diagrams w/ α's commute

some diagrams w/ λ, ρ's commute

think generators and relations

Given X_1, X_2 any two expressions obtained from $V_1 \otimes \cdots \otimes V_n$ by adding 1's and ( )'s then all isomorphisms $\varphi : X_1 \to X_2$ built out of λ, ρ, α's are equal.

(Joyal, Street Coherence Theorem)

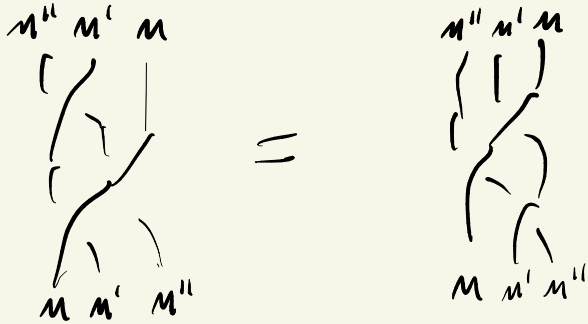
Fix $R_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ (functorial in V, W)

Hexagon, Pentagon, & Triangle $\Rrightarrow$ Given X_1, X_2 any two expressions obtained from $V_1 \otimes \cdots \otimes V_n$ by permuting factors adding 1's and adding ( )'s then all $\varphi : X_1 \xrightarrow{\sim} X_2$ built from $\alpha, \lambda, \rho, R^{\pm 1}$ depends only on the image of φ in Br_n

ex $id \otimes R_{V \otimes W} \otimes id : V_1 \otimes \cdots \otimes V_{i-1} \otimes V \otimes W \otimes V_{i+2} \otimes \cdots \otimes V_n \to V_1 \otimes \cdots \otimes V_{i-1} \otimes W \otimes V \otimes V_{i+2} \otimes \cdots \otimes V_n$

$\rightsquigarrow$ 

Prop (Yang Baxter / Reidemeister III / Braid rel'ns)
(Corollary?)



$$(R_{M',M''}\otimes \mathrm{id})\circ(\mathrm{id}\otimes R_{M,M''})\circ(R_{M,M'}\otimes \mathrm{id}) = (\mathrm{id}\otimes R_{M,M'})\circ(R_{M,M''}\otimes \mathrm{id})\circ(\mathrm{id}\otimes R_{M',M''})$$

Proof

$$\mathrm{LHS} \overset{\text{Hexagon}}{=} (R_{M',M''}\otimes \mathrm{id})\circ R_{M,M'\otimes M''}$$

$$\overset{\text{Functorality}}{=} R_{M,M''\otimes M'}\circ(\mathrm{id}\otimes R_{M',M''})$$

$$\begin{tikzcd}
M\otimes(M'\otimes M'') \arrow[r, "R_{M,M'\otimes M''}"] \arrow[d, "\mathrm{id}\otimes R_{M',M''}"'] & (M'\otimes M'')\otimes M \arrow[d, "R_{M',M''}\otimes \mathrm{id}"] \\
M\otimes(M''\otimes M') \arrow[r, "R_{M,M''\otimes M'}"] & (M''\otimes M')\otimes M
\end{tikzcd}$$

$$\overset{\text{Hexagon}}{=} \mathrm{RHS}$$

$\square$

§2 R matrix for $U_q(\mathfrak{g})$

$\Bbbk$ = field of characteristic zero

$q \in \Bbbk$ transcendental (over $\mathbb{Q}$)

- $\mathfrak{g} \rightsquigarrow \Phi, (\,,\,)$ unique W invariant bilinear form on $\mathbb{Z}\Phi$ with $(\alpha,\alpha)=2$ for short α
- choose simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ for Φ
- $\mathcal{U} = U_q(\mathfrak{g}) = \Bbbk \left\langle \begin{matrix} E_\alpha \\ F_\alpha \\ K_\alpha \end{matrix} \;\middle|\; \begin{matrix}\text{relations} \\ \text{Jantzen 4.3}\end{matrix} \right\rangle$
- $\mathcal{U}$ is Hopf algebra

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$$
$$\Delta(F_\alpha) = 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}$$
$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

(Jantzen 4.8 for S, ε)

- $\mathcal{U}^+ = \Bbbk\langle E_\alpha \rangle_{\alpha \in \Pi} \cong \Bbbk\langle E_\alpha \mid q \text{ serre relations} \rangle$

$\mathcal{U}^- = \Bbbk\langle F_\alpha \rangle_{\alpha \in \Pi} \cong \Bbbk\langle F_\alpha \mid q \text{ Serre relations} \rangle$

$\mathcal{U}^{\geq 0} = \Bbbk\langle E_\alpha, K_\alpha^{\pm 1} \rangle_{\alpha \in \Pi}$

$\mathcal{U}^{\leq 0} = \Bbbk\langle F_\alpha, K_\alpha^{\pm 1} \rangle_{\alpha \in \Pi}$

all subalgebras are $\mathbb{Z}\Phi$ graded. write A_μ for $\mu \in \mathbb{Z}\Phi$ component

- There is a bilinear pairing $(\,,\,) : \mathcal{U}^{\leq 0} \times \mathcal{U}^{\geq 0} \longrightarrow \Bbbk$ uniquely determined by

recall $(a \otimes b, c \otimes d) := (a,c)(b,d)$

- $(y, xx') = (\Delta(y), x \otimes x')$
- $(yy', x) = (y \otimes y', \Delta(x))$
- $(K_\mu, K_\nu) = q^{-(\mu,\nu)}$
- $(F_\alpha, E_\beta) = -\delta_{\alpha\beta} \dfrac{1}{q_\alpha - q_\alpha^{-1}}$
- $(K_\mu, E_\alpha) = 0$
- $(F_\alpha, K_\mu) = 0$

(Jantzen Prop 6.12)

Prop 6.18 (Jantzen) For $\mu \in \mathbb{Z}\Phi$, $\mu \geq 0$, the restriction of $(,)$ to $\mathcal{U}^-_{-\mu} \times \mathcal{U}^+_\mu$ is nondegenerate.

- Choose a basis $v_1^\mu, \ldots, v_{r(\mu)}^\mu$ for $\mathcal{U}^-_{-\mu}$ and a dual (w/r/t $(,)$) basis $u_1^\mu, \ldots, u_{r(\mu)}^\mu$ for $\mathcal{U}^+_\mu$ i.e. $(v_i^\mu, u_j^\mu) = \delta_{ij}$

- Set
$$\Theta_\mu = \sum_{i=1}^{r(\mu)} v_i^\mu \otimes u_i^\mu \in \mathcal{U} \otimes \mathcal{U}$$

  - $\mathcal{U}^+_0 = \mathbb{k} \cdot 1 = \mathcal{U}^-_0$ and $(1,1) = 1$, so $\Theta_0 = 1$

  - $\mathcal{U}^+_{n\alpha} = \mathbb{k} \cdot E_\alpha^n$ $\mathcal{U}^-_{n\alpha} = \mathbb{k} \cdot F_\alpha^n$

    so $\Theta_{n\alpha} = (F_\alpha^n, E_\alpha^n)^{-1} F_\alpha^n \otimes E_\alpha^n$

    and we computed $(F_\alpha^n, E_\alpha^n) = (-1)^n q_\alpha^{\binom{n}{2}} \frac{[n]_\alpha!}{(q_\alpha - q_\alpha^{-1})^n}$ (Jantzen 6.13(4))

  - $\mathcal{U}(\mathfrak{sl}_2)^+ \cong \mathbb{k}[E_\alpha]$, so the last two calculations recover all Θ_μ for $\mathfrak{sl}_2$ look at each term for $\Theta_{r,w}$ above.

<u>Def</u> Let $V, W \in \mathcal{U}\text{-mod}^{\text{type } 1}$. We write
$$\Theta_{V,W} := \sum_{\substack{\nu \in \mathbb{Z}\Phi \\ \nu \geq 0}} \Theta_\nu |_{V \otimes W}$$

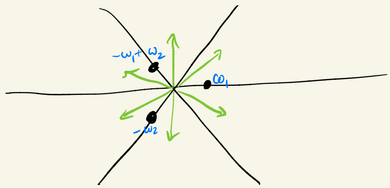
- $\Theta_\nu : V_{\lambda_1} \otimes W_{\lambda_2} \longrightarrow V_{\lambda_1 - \nu} \otimes W_{\lambda_2 + \nu}$

Since $wt(V)$, $wt(W)$ are finite sets, only finitely many $\nu \in \mathbb{Z}\Phi$ are differences of weights in V or in W.
So $\Theta_\nu(V \otimes W) = 0$ for all but finitely many ν.

Also, we can again choose bases of V, W so $\Theta_{V,W}$ is $1 +$ upper triangular nilpotent matrix

example $\mathfrak{g} = \mathfrak{sl}_3$ $\quad \Pi = \{\alpha, \beta\}$ $\quad (\,,\,)$ $\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 2 & -1 \\ \beta & -1 & 2 \end{array}$

$\mathcal{U} \circlearrowright V := L(\omega_1)$



v $(q,1)$

$E_\alpha \uparrow \downarrow F_\alpha$

$F_\alpha v$ (q^{-1}, q)

$E_\beta \uparrow \downarrow F_\beta$

$F_\beta F_\alpha v$ $(1, q^{-1})$

Weight considerations imply $\Theta_{V,V} = 1 + \Theta_\alpha + \Theta_\beta + \Theta_{\alpha+\beta}\Big|_{V \otimes V}$

$\mathcal{U}^+_{\alpha+\beta} = \mathbb{k}\{E_\alpha E_\beta, E_\beta E_\alpha\}$

$\mathcal{U}^-_{\alpha+\beta} = \mathbb{k}\{F_\alpha F_\beta, F_\beta F_\alpha\}$

maybe need PBW here?

we computed $(\,,\,) : \mathcal{U}^-_{\alpha+\beta} \times \mathcal{U}^+_{\alpha+\beta} \longrightarrow \mathbb{k}$ last time

Write $c = -(q - q^{-1})^{-1}$

- $(F_\alpha F_\beta, E_\alpha E_\beta) = (F_\alpha \otimes F_\beta, \Delta(E_\alpha E_\beta))$

recall

$\Delta(E) = E \otimes 1 + K \otimes E$

$\Delta(F) = 1 \otimes F + F \otimes K^{-1}$

$= (F_\alpha \otimes F_\beta, \cancel{E_\alpha E_\beta \otimes 1} + E_\alpha K_\beta \otimes E_\beta + \cancel{K_\alpha E_\beta \otimes E_\alpha} + \cancel{K_\alpha K_\beta \otimes E_\alpha E_\beta})$

$= (F_\alpha, E_\alpha K_\beta)(F_\beta, E_\beta)$

$= (\Delta(F_\alpha), E_\alpha \otimes K_\beta)(F_\beta, E_\beta)$

$= (\cancel{1 \otimes F_\alpha} + F_\alpha \otimes K_\alpha^{-1}, E_\alpha \otimes K_\beta)(F_\beta, E_\beta)$

$= (F_\alpha, E_\alpha)(K_\alpha^{-1}, K_\beta)(F_\beta, E_\beta)$

$= c\, q^{-(-\alpha,\beta)} c$ $\quad (c_\alpha = c_\beta)$

exercise Check

① $(\,,\,)$ $\begin{array}{c|cc} & E_\alpha E_\beta & E_\beta E_\alpha \\ \hline F_\alpha F_\beta & c^2 q^{-1} & c^2 \\ F_\beta F_\alpha & c^2 & c^2 q^{-1} \end{array}$

② $\left\{ \begin{array}{c} F_\alpha F_\beta \\ F_\beta F_\alpha \end{array} \right\} \overset{\text{dual}}{\longleftrightarrow} \left\{ \begin{array}{c} c^{-1} E_\alpha E_\beta - q c^{-1} E_\beta E_\alpha \\ -q c^{-1} E_\alpha E_\beta + c^{-1} E_\beta E_\alpha \end{array} \right\}$

Jantzen's approach to $q^{\sum h_i \otimes h_i'}$ reminder

(Jantzen suggests $d = [\Lambda : \mathbb{Z}\Phi]$ b/c $(\alpha, \lambda) \in \mathbb{Z}$)

Let Λ = weight lattice

Choose $d > 0$ so $d(\lambda, \mu) \in \mathbb{Z}$ $\forall \lambda, \mu \in \Lambda$

Assume q has a d^{th} root in $\Bbbk$

Define $f : \Lambda \times \Lambda \longrightarrow \Bbbk$

by $$f(\lambda, \mu) = \left(q^{1/d}\right)^{-d(\lambda,\mu)}$$

For $V, W \in \mathcal{U}\text{-mod}^{\text{type } 1}$ define

$$\tilde{f} : V \otimes W \longrightarrow V \otimes W$$

by $$\tilde{f}\big|_{V_\lambda \otimes W_\mu} = f(\lambda, \mu) \cdot \mathrm{id}_{V_\lambda \otimes W_\mu}.$$

<u>Claim</u>: for all $\lambda, \mu \in \Lambda$

- and $\nu \in \Lambda$
$$\left.\begin{aligned} f(\lambda, \mu+\nu) &= f(\lambda, \mu) f(\lambda, \nu) \\ f(\lambda+\mu, \nu) &= f(\lambda, \nu) f(\mu, \nu) \end{aligned}\right\} \text{needed for hexagon identity}$$

- and $\nu \in \mathbb{Z}\Phi$
$$\left.\begin{aligned} f(\lambda, \mu+\nu) &= q^{-(\nu,\lambda)} f(\lambda, \mu) \\ f(\lambda+\nu, \mu) &= q^{-(\nu,\mu)} f(\lambda, \mu) \end{aligned}\right\} \text{needed for } \Theta\Delta^{\sigma} = \Delta\Theta$$

<u>example</u> $\Phi = A_2$ i.e. $\mathfrak{g} = \mathfrak{sl}_3$

α —— β

$$\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 2 & -1 \\ \beta & -1 & 2 \end{array} \quad \longleftarrow \text{matrix for } (-,-) \text{ \& Cartan matrix in type A}$$

$$\varpi_1 = \tfrac{1}{3}(2\alpha + \beta)$$
$$\varpi_2 = \tfrac{1}{3}(\alpha + 2\beta) \quad \longleftarrow \text{invert Cartan matrix}$$

$$(\varpi_1, \varpi_1) = \tfrac{2}{3} = (\varpi_2, \varpi_2) \quad \& \quad (\varpi_1, \varpi_2) = \tfrac{1}{3} = (\varpi_2, \varpi_1)$$

$$f((a,b),(c,d)) := q^{-1/3(2ac + bc + ad + 2bd)}$$

for $\mathfrak{sl}_n$, need $q^{1/n} \in k$.

<u>exercise</u>: $V = L(\varpi_1) \circlearrowright U_q(\mathfrak{sl}_3)$

check my calculation

$$R_{V,V} = \Theta_{V,V} \circ \tilde{f} \circ P = \left(\begin{array}{c|c|c}
\begin{smallmatrix} v^2 \\ 0 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ v^{-1} \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \\ v^{-1} \end{smallmatrix} \\ \hline
\begin{smallmatrix} v^{-1} \\ 0 \\ 0 \end{smallmatrix} & \begin{smallmatrix} (cv)^{-1} & \\ 0 & v^2 \\ 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \\ v^{-1} \end{smallmatrix} \\ \hline
\begin{smallmatrix} v^{-1} \\ 0 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ v^{-1} \\ 0 \end{smallmatrix} & \begin{smallmatrix} -v^{-3}(cv)^{-1} & 0 & 0 \\ & (cv)^{-1} & 0 \\ & & v^2 \end{smallmatrix}
\end{array}\right)$$

$$v = q^{-1/3} \qquad c = \frac{-1}{q - q^{-1}}$$

Thm 1 (g) Let $V, W \in \mathcal{U}\text{-mod}^{\text{type } 1}$, then

$$R_{V,W} := \Theta_{W,V} \circ \tilde{f} \circ P : V \otimes W \to W \otimes V$$

is a $\mathcal{U}$-module isomorphism which is functorial in V and W.

Proof Three Steps:

(1) Use $\tau : \mathcal{U} \xrightarrow{\simeq} \mathcal{U}^{op}$

$E \mapsto E$
$F \mapsto F$
$K \mapsto K^{-1}$

to get new co mult.

something general for all anti autos of a Hopf algebra

$${}^{\tau}\Delta = (\tau \otimes \tau) \circ \Delta \circ \tau$$

- ${}^{\tau}\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha^{-1} \otimes E_\alpha$
- ${}^{\tau}\Delta(F_\alpha) = F_\alpha \otimes K_\alpha + 1 \otimes F_\alpha$
- ${}^{\tau}\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$

Note: ${}^{\tau}\Delta \neq \Delta^{op}$ or Δ

then argue

$$\Delta(u) \circ \Theta = \Theta \circ {}^{\tau}\Delta(u) \quad \forall u \in \mathcal{U}$$

(2) Show $\quad {}^{\tau}\Delta(u) \circ \tilde{f} = \tilde{f} \circ \Delta^{op}(u)$

exercise: use relns on f from above

③ <u>Note</u>: for $u \in \mathcal{U}$, $x \in V \otimes W$

$$P(u \cdot x) = P(\Delta(u)x) = P(\textstyle\sum_i u_i x_1 \otimes u_i' x_2) = \sum_i u_i' x_2 \otimes u_i x_1 = \Delta^{op}(u) P(x)$$

<u>Deduce</u>:

$$\begin{aligned} R_{V,W}(u \cdot x) &= \theta_{W,V} \circ \tilde{f} \circ P(\Delta(u)x) \\ &\overset{③}{=} \theta_{W,V} \circ \tilde{f} \circ \Delta^{op}(u) \circ P(x) \\ &\overset{②}{=} \theta_{W,V} \circ {}^{\tau}\Delta(u) \circ \tilde{f} \circ P(x) \\ &\overset{①}{=} \Delta(u) \circ \theta_{W,V} \circ \tilde{f} \circ P(x) \\ &= u \cdot R_{V,W}(x) \end{aligned}$$

- $R_{V,W}$ is clearly an iso. as the composition of three isomorphisms

- functorality comes from fact that $R_{V,W}$ is induced by action of element in $\mathcal{U} \otimes \mathcal{U}$ & $f \otimes g : V \otimes W \to V' \otimes W'$ is a map of $\mathcal{U} \otimes \mathcal{U}$ modules.

Proof of ① $\quad \Theta \circ {}^{\tau}\Delta(u) = \Delta(u) \circ \Theta$

- Suffices to check on generators.
- Θ is in $(U \otimes U)_0$ and $\Delta(K_\alpha) = K_\alpha \otimes K_\alpha = {}^{\tau}\Delta(K_\alpha)$

  so $\Theta \circ {}^{\tau}\Delta(K_\alpha) = \Delta(K_\alpha) \circ \Theta$

- $u = E_\alpha$ and $u = F_\alpha$ are "similar" so we focus on $u = E_\alpha$
- Since $\Theta = \sum_{\mu \geq 0} \Theta_\mu$ ① follows from

$$\underbrace{(E_\alpha \otimes 1)\Theta_\mu + (K_\alpha \otimes E_\alpha)\Theta_{\mu-\alpha}}_{\Delta(E_\alpha)} = \underbrace{\Theta_\mu (E_\alpha \otimes 1) + \Theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha)}_{{}^{\tau}\Delta(E_\alpha)}$$

- We start with

$\Theta_\mu = \sum v_i^\mu \otimes u_i^\mu$

$$(E_\alpha \otimes 1)\Theta_\mu - \Theta_\mu (E_\alpha \otimes 1) = \sum_{i=1}^{r(\mu)} (E_\alpha v_i^\mu - v_i^\mu E_\alpha) \otimes u_i^\mu$$

- Jantzen lemma 6.17 (1) says

  For $y \in U^-_{-\mu}$, $\alpha \in \Pi$, & $\mu \in \mathbb{Z}\Phi$

$$E_\alpha y - y E_\alpha = c_\alpha (K_\alpha r_\alpha(y) - r'_\alpha(y) K_\alpha^{-1})$$

recall:

- $c_\alpha = -(q_\alpha - q_\alpha^{-1})^{-1}$
- $y \in U^-_{-\mu} \rightsquigarrow \Delta(y) \in \bigoplus_{0 \leq \nu \leq \mu} U^-_{-\nu} \otimes U^-_{-(\mu-\nu)} K_\nu^{-1}$

defn of r_α, r'_α in Jantzen 6.15

- $\Delta(y) = y \otimes K_\mu^{-1} + \sum_{\alpha \in \Pi} r_\alpha(y) \otimes F_\alpha K_{\mu-\alpha}^{-1} + (\text{rest})$

so there are $r_\alpha(y), r'_\alpha(y) \in U^-_{-(\mu-\alpha)}$ s.t.

- $\Delta(y) = 1 \otimes y + \sum_{\alpha \in \Pi} F_\alpha \otimes r'_\alpha(y) K_\alpha^{-1} + (\text{rest})$

• So

$$(E_\alpha \otimes 1)\Theta_\mu - \Theta_\mu (E_\alpha \otimes 1) = -c_\alpha \sum_{i=1}^{r(\mu)} \left(K_\alpha r_\alpha(v_i^\mu) - r_\alpha'(v_i^\mu) K_\alpha\right) \otimes u_i^\mu$$

$y \in U_{-(\mu-\alpha)} \Rightarrow y = \sum_{j=1}^{r(\mu-\alpha)} (y, u_j^{\mu-\alpha}) v_j^{\mu-\alpha}$

$$= -c_\alpha \sum_{i=1}^{r(\mu)} \left(K_\alpha \sum_{j=1}^{r(\mu-\alpha)} (r_\alpha(v_i^\mu), u_j^{\mu-\alpha}) v_j^{\mu-\alpha} - \sum_{j=1}^{r(\mu-\alpha)} (r_\alpha'(v_i^\mu), u_j^{\mu-\alpha}) v_j^{\mu-\alpha} K_\alpha^{-1}\right) \otimes u_i^\mu$$

Jantzen 6.15(5)

$(r_\alpha(y), x) = \frac{1}{c_\alpha}(y, E_\alpha x)$

$(r_\alpha'(y), x) = \frac{1}{c_\alpha}(y, x E_\alpha)$

$$= \sum_i \left(-K_\alpha \sum_j (v_i^\mu, E_\alpha u_j^{\mu-\alpha}) v_j^{\mu-\alpha} + \sum_j (v_i^\mu, u_j^{\mu-\alpha} E_\alpha) v_j^{\mu-\alpha} K_\alpha^{-1}\right) \otimes u_i^\mu$$

Switch order of summation and use $y = \sum (y, u_j) v_j$ backwards to get

$$= \Theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha) - (K_\alpha \otimes E_\alpha)\Theta_{\mu-\alpha}$$

More details Jantzen lemma 7.1

$\square$

Thm 2 (g)

$R_{(-),(-)}$ satisfies the hexagon equations.

<u>Proof</u> The two identities are similar, so we focus on the first one.

Let $M, M', M'' \in \mathcal{U}\text{-mod}^{\text{type } 1}$

We want to show the diagram

$$\begin{tikzcd}
 & M\otimes(M''\otimes M') \arrow[r, "\text{can}"] & (M\otimes M'')\otimes M' \arrow[dr, "R_{M'',M}\otimes \mathrm{id}"] & \\
M\otimes(M'\otimes M'') \arrow[ur, "\mathrm{id}\otimes R_{M',M''}"] \arrow[dr, "\text{can}"'] & & & (M''\otimes M)\otimes M' \\
 & (M\otimes M')\otimes M'' \arrow[r, "R_{M\otimes M',M''}"] & M''\otimes(M\otimes M') \arrow[ur, "\text{can}"] &
\end{tikzcd}$$

commutes.

Top of the diagram

$$(\Theta\otimes 1)\circ(\tilde f\otimes 1)\circ(P\otimes 1)\circ(1\otimes\Theta)\circ(1\otimes\tilde f)\circ(1\otimes P)$$

① $\Theta_{13} = (P\otimes 1)\circ(1\otimes\Theta)\circ(P\otimes 1)$

* $\Theta = \sum_i a_i\otimes b_i \qquad \Theta_{13} = \sum a_i\otimes 1\otimes b_i$

② $\tilde f_{13} = (P\otimes 1)\circ(1\otimes\tilde f)\circ(P\otimes 1)$

* $\tilde f_{13}(\underset{M_{\lambda_1}}{m}\otimes\underset{M'_{\lambda_2}}{m'}\otimes\underset{M'_{\lambda_3}}{m''}) = f(\lambda_1,\lambda_3)\, m\otimes m'\otimes m''$

③ $\Theta'\circ(\tilde f\otimes 1) = (\tilde f\otimes 1)\circ\Theta_{13}$

* $\Theta' = \sum_u (\Theta_u)_{13}\circ(1\otimes K_u\otimes 1)$

check

$(\Theta_u)_{13}\circ(1\otimes K_u\otimes 1)\circ(\tilde f\otimes 1)$

$\parallel$

$(\tilde f\otimes 1)\circ(\Theta_u)_{13}$

using

- $\Theta_u : V_{\lambda_1}\otimes W_{\lambda_2}\to V_{\lambda_1-u}\otimes W_{\lambda_1+u}$
- $\tilde f(\lambda_1-u,\lambda_2) = q^{(u,\lambda_2)}\tilde f(\lambda_1,\lambda_2)$

① ② ③

$$\Longrightarrow\ = (\Theta\otimes 1)\circ\Theta'\circ(\tilde f\otimes 1)\ \tilde f_{13}\ (P\otimes 1)\circ(1\otimes P)$$

Bottom of diagram:

$$R_{M\otimes M', M''} = \Theta_{M'', M\otimes M'} \circ \tilde{f} \circ P_{M\otimes M', M''}$$

$$= (1\otimes\Delta)(\Theta) \circ \tilde{f} \circ (P\otimes 1) \circ (1\otimes P)$$

① $\tilde{f}(\underbrace{m''}_{\lambda_3} \otimes \underbrace{m \otimes m'}_{\lambda_1+\lambda_2}) = f(\lambda_3, \lambda_1+\lambda_2)\, m''\otimes m \otimes m'$

$$= f(\lambda_3,\lambda_1)\, f(\lambda_3,\lambda_2)\, m''\otimes m\otimes m'$$

$$= (\tilde{f}\otimes 1)\circ \tilde{f}_{13}\,(m''\otimes m\otimes m')$$

② $(1\otimes\Delta)(\Theta) = (\Theta\otimes 1)\circ \Theta'$

exercise verify $(1\otimes\Delta)(\Theta_\mu) = \sum_{0\le\nu\le\mu} (\Theta_{\mu-\nu}\otimes 1)\circ(1\otimes K_\nu\otimes 1)(\Theta_\nu)_{13}$ (Jantzen lemma 7.4)

using (*) $\Delta(x) = \sum_{0\le\nu\le\mu} \sum_{i,j} (v_i^{\mu-\nu} v_j^{\nu}, x)\, u_i^{\mu-\nu} K_\nu \otimes u_j^{\nu}$, for $x\in U_\mu^+$

To find (*) use $\Delta(x) \in \bigoplus_{0\le\nu\le\mu} U_{\mu-\nu}^+ K_\nu \otimes U_\nu^+$ to deduce

$$\Delta(x) = \sum_{\nu,i,j} c_{ij}^{\nu}\, u_i^{\mu-\nu} K_\nu \otimes u_j^{\nu}$$

then $c_{ij}^{\nu} = (v_i^{\mu-\nu}\otimes v_j, \Delta(x)) = (v_i^{\mu-\nu} v_j, x)$ ∎

§3 Hecke Algebras and q-Schur–Weyl Duality

- Schur Weyl Duality

$$\mathfrak{gl}_n \circlearrowright (\mathbb{C}^n)^{\otimes d} \circlearrowleft S_d$$

symmetric group of $1, \dots, d$

$s_i s_j = s_j s_i$ b/c holds in image of "interchange law" in $\otimes$ category

$$P_i = \mathrm{id} \otimes \underset{i,i+1}{P} \otimes \mathrm{id} \longleftarrow\!\!\!| \; s_i$$

commuting actions b/c $\Delta = \Delta^{op}$

Theorem: $\mathbb{C}[S_d] \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}_n}\left((\mathbb{C}^n)^{\otimes d}\right)$

(plus some description of the kernel...)

Def'n $\mathbb{H}_{d,t} := \mathbb{Z}[t^{\pm 1}]\mathcal{B}r \Big/ \langle \sigma_i^2 = (t^{-1} - t)\sigma_i + 1 \rangle$

Br_n = braid group

is the Hecke algebra.

- $\mathbb{H}_{d,t=1} \cong \mathbb{Z}[S_d]$
- $\mathbb{H}_{d,t}$ is free $\mathbb{Z}[t^{\pm 1}]$ module of rank $d!$
- $\mathbb{H}_{d,t=p} \cong \mathrm{End}_{GL_n(\mathbb{F}_p)}\left(\mathrm{Ind}_{B(\mathbb{F}_p)}^{GL_n(\mathbb{F}_p)}(\mathrm{triv})\right)$

need $\sqrt{p}$ $\quad \sigma_i \longmapsto -q^{-1/2} T_s$

q-Schur Weyl

$$\mathcal{U} = \mathcal{U}_q(\mathfrak{sl}_n)$$

$$V = L(\varpi_1)$$

$$R_i : V^{\otimes d} \longrightarrow V^{\otimes d}$$

$$R_i = \mathrm{id} \otimes R_{V,V} \otimes \mathrm{id}$$

$$\cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \longmapsto \cdots \otimes R_{V,V}(v_i \otimes v_{i+1}) \otimes \cdots$$

- $\mathcal{U}\text{-mod}^{\text{type } 1}$ is a braided category so we get

$$Br_d \longrightarrow \mathrm{End}_{\mathcal{U}}(V^{\otimes d})$$

example $\mathcal{U} = \mathcal{U}_q(\mathfrak{sl}_2)$

$$R = \begin{pmatrix} q^{-1/2} & & & \\ & 0 & q^{1/2} & \\ & q^{1/2} & q^{-1/2} - q^{3/2} & \\ & & & q^{-1/2} \end{pmatrix}$$

exercise

$$(R - q^{-1/2})(R + q^{3/2}) = 0$$

look at 2x2 block in middle

so $$(q^{-1/2}R - q^{-1})(q^{-1/2}R + q) = 0$$

Let $t = q$. $\mathbb{Z}[t^{\pm 1}] \rightsquigarrow \mathbb{k} \otimes \mathbb{Z}[t^{\pm 1}]$

$$\begin{tikzcd}
 & \mathbb{k}Br_d \arrow[dl] \arrow[dr] & \\
\mathbb{k} \otimes \mathbb{H}_{d,t} \arrow[rr, two heads] & & \mathrm{End}_{\mathcal{U}}(V^{\otimes d})
\end{tikzcd}$$

slightly different than above $\sigma_i \mapsto \ast R_i$

$$\bar{\sigma}_i \longmapsto q^{-1/2} R_i$$

<u>Thm</u> (Jimbo)

$$\mathbb{k} \otimes \mathbb{H}_{d,t} \twoheadrightarrow \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_n)}(V^{\otimes d})$$

(plus description of kernel)

<u>exercise</u> When $\mathfrak{g} = \mathfrak{sl}_3$, $v = q^{-1/3}$

Check $(vR)^2 = (v^3 - v^{-3})(vR) + 1$

§ 3 The quantized coordinate algebra $\mathbb{k}_q[G]$

$\mathfrak{g} \rightsquigarrow G$ connected & simply connected algebraic group $/\mathbb{k}$

- $\mathbb{k}[G]$ = ring of regular functions on G
- $\mathrm{Der}(G) = \{ X \in \mathrm{End}_{\mathbb{k}}(\mathbb{k}[G]) \mid X(fg) = X(f)g + fX(g) \}$
- $L(G) = \{ X \in \mathrm{Der}(G) \mid X \circ \ell_g = \ell_g \circ X \ \ \forall x \in G \}$
- $D_\ell(G) = \left(\text{subalgebra of } \mathrm{End}_{\mathbb{k}}(\mathbb{k}[G]) \text{ generated by } L(G) \right)$

"left invariant differential operators"

- $\mathfrak{g} \cong L(G)$ so there is algebra hom

$$\mathcal{U}(\mathfrak{g}) \longrightarrow D_\ell(\mathfrak{g})$$

<u>Thm (Cartier)</u> $\mathrm{char}(\mathbb{k}) = 0 \Rightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow D_\ell(\mathfrak{g})$ is iso.

- Get a pairing $\mathcal{U}(\mathfrak{g}) \times \mathbb{k}[G] \longrightarrow \mathbb{k}$

$$\langle X, f \rangle = X(f)(1)$$

<u>Thm</u> $\langle \, , \, \rangle$ is non degenerate (exercise)

Jantzen suggests Krull intersection thm... G connected.

So we have an embedding

$$\mathbb{k}[G] \longrightarrow \mathcal{U}(\mathfrak{g})^*$$
$$f \longmapsto \langle -, f \rangle$$

Matrix coefficients:

V = finite dimensional $\mathcal{U}(\mathfrak{g})$ mod

$v \in V \quad f \in V^*$

$c_{f,v} \in \mathcal{U}(\mathfrak{g})^*$ defined by

$$c_{f,v}(X) = f(X \cdot v)$$

Convolution on $\mathcal{U}(\mathfrak{g})^*$:

$\lambda_1, \lambda_2 \in \mathcal{U}(\mathfrak{g})^*$

$$\lambda_1 \lambda_2(u) = \sum \lambda_1(u_i) \lambda_2(u_i') \qquad \text{associative}$$

$$\Delta(u) = \sum u_i \otimes u_i'$$

Lemma ① $c_{f,v} \cdot c_{g,w} = c_{f \otimes g, v \otimes w}$

② $\underset{\text{counit}}{\varepsilon} \cdot \lambda = \lambda$ and ③ $\varepsilon = c_{1^*, 1}$ trivial module

Proof (exercise)

- <u>Main Point</u>

$$\operatorname{im}\left(\Bbbk[G] \hookrightarrow U(\mathfrak{g})^*\right)$$

is unital algebra (w/r/t convolution)

spanned by matrix coeffs

<u>Proof</u>

- $\Bbbk[G]$ spanned by matrix coeffs of G-modules (Peter Weyl)
- $\mathfrak{g}$ mods lift to G b/c G is simply connected

To define $\Bbbk_q[G]$ there is no G, but we have

<u>Defn</u>
$\Bbbk_q[G] :=$ subalgebra of $\mathcal{U}_q(\mathfrak{g})^*$ spanned by $\mathcal{U}_q(\mathfrak{g})\text{-mod}^{\text{type } 1}$ matrix coeffs

unital b/c triv is f. diml & type 1

<u>what is this algebra?</u>

Can get relations from R in $\Bbbk_q[G]$

- Let $V \in \mathcal{U}\text{-mod}^{\text{type } 1}$ basis $v_1 \dots v_n$
  dual basis $v_1^* \dots v_n^*$

set $c_{ij} := c_{v_i^*, v_j} \in \Bbbk_q[G]$

- $u \in \mathcal{U}$
$$u \cdot v_j = \sum_i c_{ij}(u) v_i$$

- $W \in \mathcal{U}\text{-mod}^{\text{type } 1}$ $w_1 \dots w_m$ basis of W
  $w_1^* \dots w_m^*$ basis of W^*

set $d_{ij} := d_{w_i^*, w_j}$

$$R_{W,V} : W \otimes V \xrightarrow{\sim} V \otimes W$$
$$w_i \otimes v_j \longmapsto \sum_{k,l} R_{ij}^{kl} v_k \otimes w_l$$

<u>lemma</u> The following relation holds in $\mathbb{k}_q[G]$

$$\sum_{h,l} R^{rs}_{hl}\, d_{li}\, c_{hj} = \sum_{h,l} R^{hl}_{ij}\, c_{rh}\, d_{sl}$$

(Jantzen lemma 7.12)

<u>Proof</u> Since

$$C_{w_l^* \otimes v_h^*,\, w_i, w_j} = C_{w_l^*, w_i}\, C_{v_h^*, w_j} = d_{li}\, c_{hj}$$

we have

$$u \cdot w_i \otimes v_j = \sum_{h,l} d_{li}\, c_{hj}(u)\, w_l \otimes v_h$$

Now, expand both sides of

$$R(u(w_i \otimes v_j)) = u(R(w_i \otimes v_j))$$

and compare coefficients $\square$

<u>example</u> $\Bbbk_q[SL_2]$ $V = W = L(\varpi_1)$

$v_1 = v$ $v_2 = Fv$

<u>exercise</u> use $R_{V,V}$ to deduce relations

$$c_{11}c_{12} = q\, c_{12}c_{11} \qquad c_{11}c_{21} = q\, c_{21}c_{11}$$

$$c_{12}c_{22} = q\, c_{22}c_{12} \qquad c_{21}c_{22} = q\, c_{22}c_{21}$$

$$c_{12}c_{21} = c_{21}c_{12}$$

$$c_{11}c_{22} - c_{22}c_{11} = (q - q^{-1})\, c_{12}c_{21}$$

<u>Facts</u>

- c_{ij}'s generate $\Bbbk_q[SL_2]$ as an algebra (all $L(n) \subseteq V^{\otimes n}$)
- adding

$$\underbrace{c_{11}c_{22} - q\, c_{12}c_{21}}_{q \text{ det}} = 1$$

gives a complete list of relations